

## ON CURVES OF CONSTANT BREADTH IN $\mathbb{G}_3^1$

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ABSTRACT. In this work, differential equations characterizing curves of constant breadth have been given in pseudo-Galilean space  $\mathbb{G}_3^1$ . The special cases related to these differential equations have been studied in  $\mathbb{G}_3^1$ .

Keywords: Pseudo-Galilean space, pseudo-Galilean norm, curves of constant breadth.

AMS Subject Classification: 53A35, 53Z05

### 1. INTRODUCTION

The works concerning shapes of constant breadth are based on the paper "De curvis triangularibus" by Euler in 1870 [6]. In [11], a method was given to obtain some curves of constant breadth to use their applications in the kinematics of machinery by Reuleaux. In [2, 3, 6], some properties of the plane curves of constant breadth were given. The curves of constant breadth on the sphere were given by Blaschke in [3]. Fujivara had put forward a problem based on determining whether there exist "space curves of constant breadth" or not, and as a solution of the problem, the "breadth" concept for space curves was defined and these curves were shown on a surface of constant breadth in [7].

Some geometric properties of the curves of constant breadth were given in the plane in [8]. Then these properties were extended to the Euclidean 3-space  $\mathbb{E}^3$  in [9]. Also, these kind of curves were studied in four dimensional Euclidean space  $\mathbb{E}^4$  in [10]. Also the curves of constant breadth were studied in Euclidean  $n$ -space in [1]. Furthermore, the results of the curves of constant breadth were also obtained in Galilean space in [12].

In this paper, we give the differential equations characterizing curves of constant breadth in pseudo-Galilean space  $\mathbb{G}_3^1$ .

### 2. PRELIMINARIES

Let  $\mathbb{G}_3^1$  be the pseudo-Galilean 3-space, that is,  $\mathbb{G}_3^1$  is a Cayley–Klein space equipped with the projective metric of signature  $(0, 0, +, -)$ . The absolute figure of the pseudo-Galilean geometry consists of an ordered triple  $\{\omega, f, I\}$ , where  $\omega$  is the real (absolute)

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plane,  $f$  the real line (absolute line) in  $\omega$  and  $I$  the fixed hyperbolic involution of points of  $f$ . In [4], the hyperbolic involution is given by

$$(0 : 0 : x_2 : x_3) \rightarrow (0 : 0 : x_3 : x_2) \tag{2.1}$$

at homogeneous coordinates.

There are two types of plane in the pseudo-Galilean space. Pseudo-Euclidean planes are in the following form  $x = k, k \in \mathbb{R}$ . Other planes are isotropic. A vector  $u = (u_1, u_2, u_3)$  is said to be non-isotropic if  $u_1 \neq 0$ . All unit non-isotropic vectors are of the form  $u = (1, u_2, u_3)$ . For isotropic vectors,  $u_1$  vanishes [4].

Let  $a = (x, y, z)$  and  $b = (x_1, y_1, z_1)$  be vectors in the pseudo-Galilean space. The scalar product is defined by

$$\langle a, b \rangle = x_1x. \tag{2.2}$$

The norm of  $a$  is defined by  $\|a\| = |x|$ , and  $a$  is called a unit vector if  $\|a\| = 1$ . The scalar product of two isotropic vectors  $p = (0, y, z)$  and  $q = (0, y_1, z_1)$  is defined by

$$\langle p, q \rangle_1 = yy_1 - zz_1. \tag{2.3}$$

The norm of  $p$  is defined by  $\|p\|_1 = \sqrt{|y^2 - z^2|}$ . An isotropic vector  $p = (0, y, z)$  is said to be spacelike, timelike or lightlike if  $y^2 - z^2 > 0, y^2 - z^2 < 0$  or  $y = \pm z$ , respectively [4].

The cross product in the pseudo-Galilean space can be defined analogously to the Minkowski case. The cross product of  $u = (u_1, u_2, u_3)$  and  $v = (v_1, v_2, v_3)$  is defined as follows:

$$u \wedge v = \begin{vmatrix} 0 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (0, u_3v_1 - u_1v_3, u_2v_1 - u_1v_2). \tag{2.4}$$

Given an admissible curve  $r(u) = (u, y(u), z(u))$ , the associated invariant moving trihedron is given by

$$\begin{aligned} T &= (1, y'(u), z'(u)), \\ N &= \frac{1}{\kappa}(0, y''(u), z''(u)), \\ B &= \frac{1}{\kappa}(0, \varepsilon z''(u), \varepsilon y''(u)) \end{aligned} \tag{2.5}$$

where  $\kappa = \sqrt{|y''(u)^2 - z''(u)^2|}$  is the curvature and  $\tau = \frac{1}{\kappa^2} \det[r'(u), r''(u), r'''(u)]$  is the torsion [5].

A curve is said to be timelike or spacelike if the principal normal vector  $N$  is spacelike or timelike vector, respectively. The principal normal vector or simply the normal is spacelike if  $\varepsilon = +1$  and time-like if  $\varepsilon = -1$ .

For derivatives of the tangent (vector)  $t$ , the normal  $n$  and the binormal  $b$ , respectively, the following Serret-Frenet formulas hold

$$T' = \kappa N, \quad N' = \tau B, \quad B' = \tau N. \tag{2.6}$$

From (6), the following important relation

$$r(x) = \kappa(x)N(x) + \kappa(x)\tau(x)B(x)$$

is obtained as in [5].

### 3. CURVES OF CONSTANT BREADTH IN $\mathbb{G}_3^1$

Let  $\varphi = \varphi(s)$  and  $\varphi^* = \varphi^*(s)$  be simple closed curves of constant breadth in pseudo-Galilean space  $\mathbb{G}_3^1$ . These curves will be denoted by  $C$ . The normal plane at every point  $P$  on the curve meets the curve in the class  $\Gamma$  as in [7] having parallel tangents  $T$  and  $T^*$  in opposite directions at the opposite points  $\varphi$  and  $\varphi^*$  of the curve. A simple closed curve of constant breadth having parallel tangents in opposite directions at opposite points can be represented with respect to Frenet frame by the equation

$$\varphi^* = \varphi + m_1T + m_2N + m_3B, \quad (3.1)$$

where  $m_i(s)$ ,  $1 \leq i \leq 3$  arbitrary functions of  $s$  and  $\varphi$  and  $\varphi^*$  are opposite points. The vector  $d = \varphi^* - \varphi$  is called "the distance vector" of  $C$ . Differentiating both sides of (3.1) and considering Frenet equations, we have

$$\begin{aligned} \frac{d\varphi^*}{ds} = T^* \frac{ds^*}{ds} &= \left(1 + \frac{dm_1}{ds}\right)T + \left(m_1\kappa + \frac{dm_2}{ds} + m_3\tau\right)N \\ &+ \left(\frac{dm_3}{ds} + m_2\tau\right)B. \end{aligned} \quad (3.2)$$

Thus,  $T^* = -T$ . Rewriting (3.2) we obtain following system of equations,

$$\begin{aligned} \frac{dm_1}{ds} + 1 + \frac{ds^*}{ds} &= 0, \\ \frac{dm_2}{ds} + m_1\kappa + m_3\tau &= 0, \\ \frac{dm_3}{ds} + m_2\tau &= 0. \end{aligned} \quad (3.3)$$

If we call  $\theta$  as the angle between the tangent of the curve  $C$  at point  $\varphi$  with a given fixed direction and  $s$  as arc length parameter of  $\varphi(s)$ , consider  $\frac{d\theta}{ds} = \kappa$ , we have (3.3) as following;

$$\begin{aligned} \frac{dm_1}{d\theta} &= -f(\theta), \\ \frac{dm_2}{d\theta} &= -m_1 - m_3\rho\tau, \\ \frac{dm_3}{d\theta} &= -m_2\rho\tau, \end{aligned} \quad (3.4)$$

where  $f(\theta) = \rho + \rho^*$ ;  $\rho = \frac{1}{\kappa}$  and  $\rho^* = \frac{1}{\kappa^*}$  denote the radius of curvature at  $\varphi$  and  $\varphi^*$ , respectively. If  $m_1$ ,  $m_3$  and their derivatives are eliminated in equations (3.4), we obtain the following equation with respect to  $m_2$ :

$$\begin{aligned} \frac{d^3m_2}{d\theta^3} + \frac{d}{d\theta} \left[ \ln\left(\frac{\tau}{\kappa}\right) \right] \frac{d^2m_2}{d\theta^2} + \left[ \left(\frac{\tau}{\kappa}\right)^2 + \frac{d^2}{d\theta^2} \left( \ln\left(\frac{\tau}{\kappa}\right) \right) \right] \frac{dm_2}{d\theta} \\ + 2\left(\frac{\tau}{\kappa}\right) \frac{d}{d\theta} \left(\frac{\tau}{\kappa}\right) m_2 + \frac{df}{d\theta} - \frac{d}{d\theta} \left[ \ln\left(\frac{\tau}{\kappa}\right) \right] f - f \frac{d\theta}{d\theta} \frac{d^2}{d\theta^2} \left[ \ln\left(\frac{\tau}{\kappa}\right) \right] = 0. \end{aligned} \quad (3.5)$$

The equation (3.5) is the characterization of the curve  $\varphi^*$ . If the distance between opposite points of  $C$  and  $C^*$  is constant, then we can write that

$$\|\varphi^* - \varphi\|^2 = \left\{ \begin{array}{l} m_2^2 - m_3^2, \quad m_1 = 0 \\ m_1^2, \quad m_1 \neq 0 \end{array} \right\} \quad (3.6)$$

According to the conditions in (3.6), we shall study the following cases:

**Case 1.** If  $m_1 = 0$ , then we write that

$$m_2^2 - m_3^2 = k^2 = \text{const.}, \quad (3.7)$$

hence, by differentiating (3.7) we have

$$m_2 \frac{dm_2}{d\theta} - m_3 \frac{dm_3}{d\theta} = 0, \quad (3.8)$$

considering system (3.4) in (3.8), we obtain

$$m_1 m_2 = 0, \quad (3.9)$$

therefore, we have

$$m_2 = 0 \text{ or } m_2 \neq 0. \quad (3.10)$$

Also, as a result of case 1, we find

$$f = 0$$

from (3.4), so the equation (3.5) turns to

$$\begin{aligned} \frac{d^3 m_2}{d\theta^3} + \frac{d}{d\theta} \left[ \ln\left(\frac{\tau}{\kappa}\right) \right] \frac{d^2 m_2}{d\theta^2} + \left[ \left(\frac{\tau}{\kappa}\right)^2 + \frac{d^2}{d\theta^2} \left( \ln\left(\frac{\tau}{\kappa}\right) \right) \right] \frac{dm_2}{d\theta} \\ + 2\left(\frac{\tau}{\kappa}\right) \frac{d}{d\theta} \left(\frac{\tau}{\kappa}\right) m_2 = 0. \end{aligned} \quad (3.11)$$

Based on the equation (3.9), we shall study the subcases:

*Case 1.1.* If  $m_2 = 0$ , then

$$m_3 = 0$$

from the equations (3.4). Therefore the curves  $\varphi^*$  and  $\varphi$  coincide.

*Case 1.2:* If  $m_2 \neq 0$  and  $\frac{\tau}{\kappa} = \text{const.}$ , then we have

$$m_2 = m_3 = e^{-\frac{\tau}{\kappa}\theta}$$

from the equations (3.4). Thus again the curves  $\varphi^*$  and  $\varphi$  coincide.

**Case 2.** If  $m_1 \neq 0$ , then from the equation (3.6), we have

$$m_1^2 = k^2 \quad (3.12)$$

where  $k \in \mathbb{R}$ . Hence, differentiating (3.12) we obtain

$$m_1 \frac{dm_1}{d\theta} = 0. \quad (3.13)$$

Then  $m_1 = k \in \mathbb{R}$ , from the equation (3.5), we obtain the following differential equation of third order

$$\begin{aligned} \frac{d^3 m_2}{d\theta^3} + \frac{d}{d\theta} \left[ \ln\left(\frac{\tau}{\kappa}\right) \right] \frac{d^2 m_2}{d\theta^2} + \left[ \left(\frac{\tau}{\kappa}\right)^2 + \frac{d^2}{d\theta^2} \left( \ln\left(\frac{\tau}{\kappa}\right) \right) \right] \frac{dm_2}{d\theta} \\ + 2\left(\frac{\tau}{\kappa}\right) \frac{d}{d\theta} \left(\frac{\tau}{\kappa}\right) m_2 - \frac{d}{d\theta} \left[ \ln\left(\frac{\tau}{\kappa}\right) \right] f - \int f d\theta \frac{d^2}{d\theta^2} \left[ \left( \ln\left(\frac{\tau}{\kappa}\right) \right) \right] = 0. \end{aligned} \quad (3.14)$$

## 4. CONCLUSION

In this work, differential equations characterizing curves of constant breadth have been given in pseudo-Galilean space  $\mathbb{G}_3^1$ . The special cases related to these differential equations have been studied in  $\mathbb{G}_3^1$ . It is an open problem to get the results of curves of constant breadth in higher dimensions of both Galilean and pseudo-Galilean spaces.

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