

MONOMIAL GEOMETRIC PROGRAMMING WITH FUZZY RELATION EQUATION CONSTRAINTS REGARDING MAX-BOUNDED DIFFERENCE COMPOSITION OPERATOR

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ABSTRACT. In this paper, an optimization model with an objective function as monomial subject to a system of the fuzzy relation equations with max-bounded difference (max-BD) composition operator is presented. We firstly determine its feasible solution set. Then some special characteristics of its feasible domain and the optimal solutions are studied. Some procedures for reducing and decomposing the problem into several sub-problems with smaller dimensions are proposed. Finally, an algorithm is designed to optimize the objective function of each sub-problem.

Keywords: Geometric optimization, Fuzzy relation equations, Max-bounded difference composition operator, Non-convex programming, Fuzzy optimization.

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1. INTRODUCTION

Let $I = \{1, 2, \dots, m\}$, $J = \{1, 2, \dots, n\}$, $A = (a_{ij})_{m \times n}$, $0 \leq a_{ij} \leq 1$, be a fuzzy matrix and $b = (b_1, b_2, \dots, b_n)$, $0 \leq b_j \leq 1$, be an n -dimensional vector. Then the fuzzy relation equations are introduced as follows:

$$x \underset{BD}{o} A = b, \quad (1)$$

where “ $\underset{BD}{o}$ ” is the max-BD composition operator [18]. In this paper, we intend to find the solution vector $x = (x_1, x_2, \dots, x_m)$, $0 \leq x_i \leq 1$, such that

$$\max_{i \in I} (\max \{0, x_i + a_{ij} - 1\}) = b_j, j \in J. \quad (2)$$

The fuzzy relation equations topic is one of the interesting subjects and is investigated by many researchers [1–4, 7–9, 13–16]. Also, fuzzy relation equations as the feasible region for optimization problems is an interesting and on-going research topic [6, 10–12]. On the other hand, Zeneret et al firstly studied the geometric programming theory in 1961, [19, 20]. Its applications can be considered in business administration, economic analysis, resource allocation and environmental engineering [21]. The fuzzy geometric programming problem was proposed by Cao [22]. Yang and Cao [17, 23] proposed a fuzzy relation geometric

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programming problem to show the importance of geometric programming and the fuzzy relation equation in theory and applications.

In this paper, we will give one kind of such problems. Let $c, \alpha_i \in R$ and $c > 0$. We want to study the problem below:

$$\begin{aligned} \min z(x) &= c \prod_{i=1}^m x_i^{\alpha_i} \\ \text{s.t. } x \underset{BD}{o} A &= b, \\ 0 \leq x_i &\leq 1 \quad \forall i \in I. \end{aligned} \tag{3}$$

According to [3,8], in a general case, the non-empty solution set of the fuzzy relation equations with the max-min composition is a non-convex set, and can be expressed in terms of the maximum solution and the finite number of minimal solutions. We show that these facts are true for fuzzy relation equations regarding the max-BD composition, as well. Since the feasible solution set is non-convex, traditional programming methods become useless.

In this paper we study the solution set of system (1) and solve problem (3). In order to determine the feasible domain of problem (3), we need to find the minimal solutions of its feasible domain. But the generating all the minimal solutions is a huge work and a NP-hard problem [5]. In this paper, using the special structure of the problem, we can find an optimal solution without explicitly generating every minimal solution. In this paper, we explore the feasible domain of a system of fuzzy relation equations with the max-BD composition operator. Then we show that problem (3) can be divided into two sub-problems; one with non-negative exponents and the other with negative exponents. Also, some procedures for reducing the original problem are proposed. Some considerations for decomposing the reduced problem into several sub-problems are presented. Finally, we design an algorithm to solve these sub-problems that it reduces computations, considerable. The algorithm is outlined and illustrated by an example. Then, the conclusion is derived.

2. CHARACTERIZATION OF FEASIBLE SOLUTIONS SET

We first express two the following definitions:

Definition 1. For each ${}^1x, {}^2x \in X[A, b] : {}^1x \leq {}^2x$ iff ${}^1x_i \leq {}^2x_i$ for $\forall i \in I$, where $X[A, b]$ denotes the feasible solutions set of problem (3).

Definition 2. $\hat{x} \in X[A, b]$ is the maximum solution if $x \leq \hat{x}$ for $\forall x \in X[A, b]$. Similarly, $\check{x} \in X[A, b]$ is the minimal solution if $x \leq \check{x}$ implies $x = \check{x}$ for $\forall x \in X[A, b]$.

To determine $X[A, b]$, we take apart (1) into the following equation:

$$x \underset{BD}{o} a_j = b_j, \quad \forall j \in J, \tag{4}$$

where a_j is the j th column of matrix A .

Now, we consider feasibility conditions of (4). If x is a feasible solution in (4), for a fixed $j \in J$, then we will have:

$$\begin{aligned} (a) \quad & \forall i \in I \quad \max \{0, x_i + a_{ij} - 1\} \leq b_j \\ (b) \quad & \exists i \in I \quad \max \{0, x_i + a_{ij} - 1\} = b_j \end{aligned} \quad (5)$$

Remark 1. If $b_j = 0$ then conditions (a) and (b) can be converted to (a) $\forall i \in I : 0 \leq x_i \leq 1 - a_{ij}$; (b) $\exists i \in I$ s.t. $0 \leq x_i \leq 1 - a_{ij}$. We keep this obvious case out of our consideration and suppose $b_j > 0 \quad \forall j \in J$.

We can simplify (5), with regard to Remark 1 and $b_j > 0$, as follows:

$$\begin{aligned} (a) \quad & \forall i \in I \quad x_i + a_{ij} - 1 \leq b_j \\ (b) \quad & \exists i \in I \quad x_i + a_{ij} - 1 = b_j \end{aligned} \quad (6)$$

Now, we introduce the following notations:

$${}^1I_j = \{i \in I : a_{ij} < b_j\}, {}^2I_j = \{i \in I : a_{ij} = b_j\}, {}^3I_j = \{i \in I : a_{ij} > b_j\}$$

For any x in (4) the condition (a) from (6) is always true for each $i \in {}^1I_j \cup {}^2I_j$ and the condition (b) from (6) is not true for each $i \in {}^1I_j$. Hence, we can easily simplify (6) as follows:

$$\begin{aligned} (a) \quad & \forall i \in {}^3I_j \quad x_i \leq b_j - a_{ij} + 1 \\ (b) \quad & \exists i \in {}^2I_j \cup {}^3I_j \quad x_i = b_j - a_{ij} + 1 \end{aligned} \quad (7)$$

Now, we define an m -dimensional vector ${}^j\hat{x} = ({}^j\hat{x}_1, {}^j\hat{x}_2, \dots, {}^j\hat{x}_m)$ such that

$${}^j\hat{x}_i = \begin{cases} b_j - a_{ij} + 1 & \text{if } i \in {}^3I_j \\ 1 & \text{otherwise} \end{cases} \quad (8)$$

Also, we let ${}^j\check{x}(i) = ({}^j\check{x}(i)_1, {}^j\check{x}(i)_2, \dots, {}^j\check{x}(i)_m)$ for each $i \in {}^2I_j \cup {}^3I_j$ such that

$${}^j\check{x}(i)_k = \begin{cases} b_j - a_{ij} + 1 & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases} \quad (9)$$

Remark 2. If $b_j = 0$ then for each $0 \leq x_i \leq 1 - a_{ij}$, we have $(x_i + a_{ij} - 1) \vee 0 = b_j$, $\hat{x}_i = 1 - a_{ij}$, and $\check{x}_i = 0$. We keep this case of obvious, out of our consideration and suppose $b_j > 0 \quad \forall j \in J$.

In the following lemma, we attempt to show vectors ${}^j\hat{x}$ and ${}^j\check{x}(i)$ are feasible and also they are the maximum and minimal solution, respectively, and then find their relation with $X[A, b]$.

Lemma 1. (a) ${}^j\hat{x}$ is the maximum solution of Eq. (4).

(b) ${}^j\check{x}(i)$, for each $i \in {}^2I_j \cup {}^3I_j$, is the minimal solution of Eq. (4).

(c) ${}^jX[A, b] = \bigcup_{i \in {}^2I_j \cup {}^3I_j} [{}^j\check{x}(i), {}^j\hat{x}]$.

Where ${}^jX [A, b]$ is the set of feasible solutions of (4), for a $j \in J$ fixed.

Proof. (a) Firstly, the definition of ${}^j\hat{x}_i$ clearly give us feasibility conditions (7). It remain to show for each $x \in {}^jX [A, b]$, $x \leq {}^j\hat{x}$. To carry out this, suppose it is not valid, that is, there exists a feasible solution “say y ” such that $y \geq {}^j\hat{x}$ and $y \neq {}^j\hat{x}$. Now, define $q = \{i \in I : y_i > {}^j\hat{x}_i\}$. Choose $i \in q$, if $i \in {}^1I_j \cup {}^2I_j$ then $y_i > 1$ that it clearly is not possible. Now, suppose that $i \in {}^3I_j$. Then $y_i > b_j - a_{ij} + 1$ that it results $\max\{0, y_i + a_{ij} - 1\} > b_j$ that it is a contradiction with feasibility of y . Therefore, we have: $y \leq {}^j\hat{x} \quad \forall y \in {}^jX [A, b]$.

(b) Once $i \in {}^2I_j \cup {}^3I_j$, according to (9), we have: ${}^j\check{x}(i)_k = 0 \leq b_j - a_{ij} + 1$ for $k \neq i$, and ${}^j\check{x}(i)_k = b_j - a_{ij} + 1$ for $k = i$.

As a result, conditions (7) are satisfied. So, ${}^j\check{x}(i)$, for $i \in {}^2I_j \cup {}^3I_j$, is a feasible point. Now, let $\exists y \in {}^jX [A, b] \quad s.t. \quad y < {}^j\check{x}(i)$. If $k \neq i$, then $y_k < 0$ that it clearly is not possible. If $k = i$ then $y_k < b_j - a_{kj} + 1$, that it is a contradiction with feasibility y . Hence, ${}^j\check{x}(i)$ is the minimal vector for $i \in {}^2I_j \cup {}^3I_j$.

(c) Let $P = \bigcup_{i \in {}^2I_j \cup {}^3I_j} [{}^j\check{x}(i), {}^j\hat{x}]$. If $x \in P$, then $\exists i \in {}^2I_j \cup {}^3I_j$ s.t. ${}^j\check{x}(i) \leq x \leq {}^j\hat{x}$. If $i \in {}^2I_j \cup {}^3I_j$, then $x_i = b_j - a_{ij} + 1$. Therefore, the conditions (a) and (b) from (7) are satisfied. Hence, $x \in {}^jX [A, b]$. Conversely, assume $x \in {}^jX [A, b]$ and x is fulfilled in conditions (a) and (b) from (7). Then $\forall i \in {}^2I_j \cup {}^3I_j \quad 0 \leq x_i \leq b_j - a_{ij} + 1$. We conclude that ${}^j\check{x}(i) \leq x \leq {}^j\hat{x}$ for $i \in {}^2I_j \cup {}^3I_j$.

It is noticeable that once ${}^j\hat{x} = o$ then the equation (4), for a $j \in J$ fixed, obviously has only solution ${}^j\check{x} = o$.

Now, let $I_j(x) = \{i \in I \mid x_i + a_{ij} - 1 = b_j\}$ and $b_j > 0, \forall j \in J$, and $I(x) = I_1(x) \times I_2(x) \times \dots \times I_n(x)$.

Remark 3. If $x \in X [A, b]$ then from conditions (7), $I_j(x) \neq \emptyset$ for $\forall j \in J$ and hence, $I(x) \neq \emptyset$.

Definition 3. Suppose ${}^j\hat{x}$ be the maximum solution of equation (4), for $j \in J$. We define $\hat{x}_i = \min_{j \in J} {}^j\hat{x}_i$ for each $i \in I$.

With regard to (8) and Lemma 1, it can easily be shown that $\hat{x} = (\hat{x}_1, \dots, \hat{x}_m)$ is maximum solution of Eq. (1) (see proof of Theorem 1 on p.68 in [8]).

Definition 4. For each $x \in X [A, b]$, $f = (f(1), \dots, f(n)) \in I(x)$. Let $f[x] = (f[x]_1, f[x]_2, \dots, f[x]_m)$ such that:

$$f[x]_i = \begin{cases} \max_{j \in J_f^{(i)}} \{b_j - a_{ij} + 1\}, & J_f(i) \neq \emptyset, \\ 0, & J_f(i) = \emptyset, \end{cases} \quad \forall i \in I. \tag{10}$$

Where $J_f(i) = \{j \in J : f(j) = i\}$ and let $F(x) = \{f[x] : f \in I(x)\}$.

Now, we express the following lemma.

Lemma 2. Suppose $x \in X[A, b]$, $f \in I(x)$, and $j, j' \in J_f(i)$ then $b_j - a_{ij} + 1 = b_{j'} - a_{ij'} + 1 = x_i$.

Proof. Since $j, j' \in J_f(i)$, $x_i + a_{ij} - 1 = b_j$ and $x_i + a_{ij'} - 1 = b_{j'}$. Hence, $b_j - a_{ij} + 1 = b_{j'} - a_{ij'} + 1 = x_i$ for any $j, j' \in J_f(i)$.

Theorem 1. Suppose $x \in X[A, b]$ and $f \in I(x)$ then $f[x] \leq x$, and $f[x] \in X[A, b]$.

Proof. From Definition 4 and Lemma 2, $f[x]_i = \max_{j \in J_f(i)} \{b_j - a_{ij} + 1\} = x_i$, for $\forall i \in I$, when $J_f(i) \neq \emptyset$. If $J_f(i) = \emptyset$, then $f[x]_i = 0 \leq x_i$. Hence, $f[x] \leq x$. Since $x \in X[A, b]$ and $f[x] \leq x$, $\max_{i \in I} \{f[x]_i + a_{ij} - 1\} \leq \max_{i \in I} \{x_i + a_{ij} - 1\} = b_j$ for $\forall j \in J$. Hence, $f[x]$ satisfies condition (a) in (6). Furthermore, since $x \in X[A, b]$, then $I_j(x) \neq \emptyset$, $\forall j \in J$. Hence, for each $j \in J$, $\exists i \in I$ such that $f(j) = i \in I_j(x)$. Since $f[x]_i = \max_{k \in J_f(i)} \{b_k - a_{ik} + 1\} = b_j - a_{ij} + 1$ from Lemma 2, $f[x]_i + a_{ij} - 1 = b_j$ that it means $f[x]$ satisfies condition (b) in (6). Hence, $f[x]$ satisfies in (6) and the proof is completed.

Theorem 2. Suppose that $x^1, x^2 \in X[A, b]$ such that $x^1 \leq x^2$. If $f \in I(x^1)$, then $f \in I(x^2)$ and $f[x^1] = f[x^2]$.

Proof. Since $f \in I(x^1)$ then $x^1_{f(j)} = b_j - a_{f(j)j} + 1, \forall j \in J$. On the other hand, since $x^1 \leq x^2$, we have:

$$b_j = x^1_{f(j)} + a_{f(j)j} - 1 \leq x^2_{f(j)} + a_{f(j)j} - 1 \leq b_j.$$

The second inequality of right-hand side is set up because $x^2 \in X[A, b]$. So, $x^2_{f(j)} + a_{f(j)j} - 1 = b_j, \forall j \in J$. Then $f \in I(x^2)$. Now, according to definition of $f[x]$, we have: $f[x^1] = f[x^2]$.

Theorem 3. Assume that $x^1, x^2 \in X[A, b]$ such that $x^1 \leq x^2$. If $f \in I(x^1)$, then for each $j \in J$, we have: $x^1_{f(j)} = x^2_{f(j)}$.

Proof. With regard to Theorem 2 and $f \in I(x^1)$, we conclude that $x^1_{f(j)} = b_j - a_{f(j)j} + 1$. Hence, $f \in I(x^2)$. Thus, $x^2_{f(j)} = b_j - a_{f(j)j} + 1$. As a result $x^1_{f(j)} = x^2_{f(j)}$.

Theorem 4. Let $x \in X[A, b]$. $x \in X_0[A, b]$ if and only if for each $f \in I(x)$, $f[x] = x$ ($X_0[A, b]$ denotes the minimal solution set of $X[A, b]$).

Proof. Proof of this theorem is similar to that of Lemma 8 in [8].

Theorem 5. $X_0 [A, b] \subseteq F(\hat{x}) \subseteq X [A, b]$.

Proof. Assume that $x \in X_0 [A, b]$. According to Theorem 4, $f [x] = x$ and $f \in I(x)$. For each $f \in I(\hat{x})$ and also, according to Theorem 2, we have $f [\hat{x}] = f [x]$. Therefore, $x = f [x] = f [\hat{x}] \in F(\hat{x})$ and thus $x \in F(\hat{x})$. With regard to definition of $F(\hat{x})$, it is obvious $F(\hat{x}) \subseteq X [A, b]$.

Corollary 1. $X [A, b] = \bigcup_{f \in I(\hat{x})} \{x \in X \mid f [\hat{x}] \leq x \leq \hat{x}\}$, where

$$X = \{x \in R^m \mid 0 \leq x_i \leq 1, \forall i \in I\}.$$

From the Theorem 5, we can find the minimal solutions in the set $F(\hat{x})$ by pairwise comparison.

3. OPTIMIZATION PROCESS

Next theorem presents optimal solution of problem (3) where all the exponents are non-positive.

Theorem 6. Given problem (3). If $\alpha_i \leq 0, \forall i \in I$, then \hat{x} is optimal solution.

Proof. If $\alpha_i \leq 0, \forall i \in I$, then the objective function of problem (3) becomes a decreasing function. Hence, \hat{x} is the optimal solution of the problem.

The case in which $\alpha_i > 0, \forall i \in I$, is considered as follows:

Theorem 7. Given problem (3). If $\alpha_i > 0, \forall i \in I$, then one of the minimal solutions of feasible solution set of problem (3) is optimal solution.

Proof. If $\alpha_i > 0$, for each $i \in I$, then the objective function of problem (3) is increasing. Now, assume that $y \in X[A, b]$ is an arbitrary element. Then there is an $x \in X_0[A, b]$ such that $y \geq x$. Since $\prod_{j \in I} x_j^{\alpha_j}$ is an increasing function. With respect to x , then $z(y) \geq z(x)$.

Thus, one of the elements of $X_0[A, b]$ is the optimal solution of problem (3).

We now define two sets R^+ and R^- as follows:

$$R^+ = \{i \in I \mid \alpha_i \geq 0\} \text{ and } R^- = \{i \in I \mid \alpha_i < 0\}. \tag{11}$$

Consequently, problem (3) is decomposed into two sub-problems:

$$\begin{aligned} P1 : \quad & \min \prod_{j \in R^+} x_j^{\alpha_j} & P2 : \quad & \min \prod_{j \in R^-} x_j^{\alpha_j} \\ \text{s.t.} \quad & x \underset{BD}{o} A = b & \text{and} & \text{s.t.} \quad x \underset{BD}{o} A = b \\ & 0 \leq x_i \leq 1 & & 0 \leq x_i \leq 1 \end{aligned} \tag{12}$$

The optimal solutions of sub-problems P2 and P1 are resulted from Theorem 6 and 7, respectively. Next corollary presents the optimal solution of general problem (3) with combining these two optimal solutions.

Corollary 2. If \hat{x} and $\check{x}^* \in X_0[A, b]$ are optimal solution of sub-problems P2 and P1, respectively, then x^* is optimal solution of problem (3), where $x^* = (x_1^*, x_2^*, \dots, x_m^*)$ with

$$x_i^* = \begin{cases} \hat{x}_i & i \in R^-, \\ \check{x}^* & i \in R^+. \end{cases}$$

Proof. With regard to relations (11) and Theorems 6 and 7, we have:

$$\prod_{i \in I} x_i^{\alpha_i} = \left(\prod_{i \in R^-} x_i^{\alpha_i} \right) \times \left(\prod_{i \in R^+} x_i^{\alpha_i} \right) \geq \left(\prod_{i \in R^-} (\hat{x}_i)^{\alpha_i} \right) \times \left(\prod_{i \in R^+} (\check{x}_i^*)^{\alpha_i} \right) = \prod_{i \in I} (x_i^*)^{\alpha_i}.$$

This completes the proof.

As we discussed, the optimal solution of sub-problem P2 is easily obtained. But it is necessary to search minimal solutions of feasible region, i.e., $X[A, b]$, for finding optimal solution of sub-problem P1. Theorem 5 proposes the set $F(\hat{x})$ for this purpose. We present an algorithm later to give such an optimal solution without finding all the minimal solutions of feasible region. Furthermore, since $x \leq \hat{x}$ for each $x \in X[A, b]$ then $x_{f(j)} = \hat{x}_{f(j)}$ for each $j \in J$ and $\forall f \in I(x)$ according to Theorem 3.

Specially, if $\check{x} \in X_0[A, b]$ then $\check{x}_{f(j)} = \hat{x}_{f(j)}$ for each $j \in J$, and $\forall f \in I(\check{x})$. This fact makes structure of our algorithm.

Definition 5. (i) For each $f \in I(\hat{x})$, let vector $x(f)$ be such that $x(f)_{f(j)} = \hat{x}_{f(j)}$, $\forall j \in J$ and $x(f)_i = 0$ when $J_f(i) = \emptyset$. (ii) Let $S = \{x(f) : f \in I(\hat{x})\}$.

In order to optimize the process of minimal solutions search, we propose the next corollary.

Corollary 3. $X_0[A, b] \subseteq S$.

Proof. Assume $\check{x} \in X_0[A, b]$ and $f \in I(\check{x})$. Then $\check{x} = x(f)$ according to Theorem 3 and thus $\check{x} \in S$.

We primarily find $x(f)$ for each $f \in I(\hat{x})$ in order to search optimal solution of sub-problem P1 in set S . In fact, Corollary 3 reduces the search region to find the set of $X_0[A, b]$.

Now, we can find optimal solution from generated $x(f)$'s in order to optimize sub-problem P1. We will present an algorithm that it finds a vector f such that vector $x(f)$ optimizes objective function of sub-problem P1. Details of this algorithm are presented below.

4. PROBLEM REDUCTION AND AN ALGORITHM

In this section, in order to minimize process of solving the problem, we present some properties to reduce the size of original problem. The main idea behind the reduction is that some of the x_i 's can be determined immediately without solving the problem but just by identifying the special characteristic of the problem. Now, we provide the following lemma that is required in continuation of section:

Lemma 3. If $|I_j(\hat{x})| = 1$, then $\hat{x}_i = \bigvee x_i = b_j - a_{ij} + 1$ for $i \in I_j(\hat{x})$.

Proof: In order to satisfy constraint j , x_i has to be equal to $b_j - a_{ij} + 1$. Since x_i is the only variable that can satisfy constraint j , it can take on only one value, i.e. $b_j - a_{ij} + 1$. Therefore, $\hat{x}_i = \bigvee x_i = b_j - a_{ij} + 1$.

Special cases which we can eliminate from consideration are as follows:

Case 1: $c_i \leq 0$.

According to Theorem 6 and Corollary 2, we know that $x_i^* = \hat{x}_i$, if $\alpha_i < 0$. Hence, we can take these parts that are related to these \hat{x}_i 's out of consideration. Here, we define:

$$\hat{I} = \{i \in I \mid \alpha_i < 0\}, \hat{J} = \left\{j \in J \mid \hat{x}_i + a_{ij} - 1 = b_j, \forall i \in \hat{I}\right\},$$

In fact, \hat{J} is a set of indices of constraints which can be satisfied by a set of \hat{x}_i 's for $i \in \hat{I}$. Now, we eliminate row i , $i \in \hat{I}$, and column j , $j \in \hat{J}$, from matrix A as well as the j th element, $j \in \hat{J}$, from vector b . Suppose that A' and b' be the updated fuzzy matrix and fuzzy vector, respectively. Define $J' = J - \hat{J}$, and $I' = I - \hat{I}$. J' represents a reduced set of constraints. Also, update $I_j(\hat{x})$'s and $I(\hat{x})$, with respect to variations of I, J , and call them by $I'_j(\hat{x})$'s and $I'(\hat{x})$, respectively. Also, omit $I'_j(\hat{x})$'s that they are empty.

Case 2: $I_j(\hat{x})$ has only one element.

Consider constraint $j \in J'$. If $I_j(\hat{x})$ contains only one element, it means that only one x_i , $i \in I_j(\hat{x})$, can satisfy the j th constraint. From Lemma 3, we have $x_i = b_j - a_{ij} + 1$. Define:

$$\bar{I} = \{i \in I_j(\hat{x}) \mid |I_j(\hat{x})| = 1, j \in J'\}, \bar{J} = \{j \in J' \mid x_i + a_{ij} - 1 = b_j; i \in \bar{I}\}$$

Again, we can eliminate row i , $i \in \bar{I}$, and column j , $j \in \bar{J}$, from the updated fuzzy matrix A' as well as the j th element, $j \in \bar{J}$, from the updated vector b' . Let A'' and b'' be the reduced fuzzy matrix and fuzzy vector corresponding to A' and b' , respectively. We also need to update $I(\hat{x})$. Define $J'' = J' - \bar{J}$, and $I'' = I' - \bar{I}$. The updated $I'_j(\hat{x})$'s and $I'(\hat{x})$, with respect to variations of I', J' , and call them by $I''_j(\hat{x})$'s and $I''(\hat{x})$, respectively. Also, omit $I''_j(\hat{x})$'s that they are empty. The search process will be performed on the set of S resulted from A'' and b'' . If b'' is empty, then all constraints have been taken care of. Therefore, in order to minimize the objective value, since we are now left with positive c_i 's, we can assign the minimum value, i.e. zero, to all x_i 's whose values have not been assigned yet. When b'' is not empty, we need to proceed further. Details will be discussed below.

5. DECOMPOSITION OF THE PROBLEM

In order to identify whether the problem is decomposable, consider a set of constraints, say B , which can be satisfied by a certain set of variables, say X_B . If the decision to choose which variable in the set X_B to satisfy a constraint in B , does not impact the decision on the rest of the problem, then we can extract this part from the whole problem. Let k be the number of sub-problems, $1 \leq k \leq |J''|$. Define:

$$\begin{aligned} \pi &= \{I''_j(\hat{x}) \mid I''_j(\hat{x}) \neq \emptyset \wedge j \in J''\}; \\ \pi_l &= \left\{ I''_j(\hat{x}) \in \pi \mid \bigcap_{j \in J''} I''_j(\hat{x}) \neq \emptyset \right\} \quad \text{Where } l = 1, \dots, k; \\ \pi_l \cap \pi_{l'} &= \emptyset \text{ for } l \neq l'; \quad \pi = \pi_1 \cup \pi_2 \cup \dots \cup \pi_k; \quad \Lambda_l(\hat{x}) = \prod_{I''_j(\hat{x}) \in \pi_l} I''_j(\hat{x}); \end{aligned} \quad (13)$$

$$I^{(l)} = \{i \mid i \in I''_j(\hat{x}), I''_j(\hat{x}) \in \pi_l\}; J^{(l)} = \{j \mid I''_j(\hat{x}) \in \pi_l\}$$

In this way, π_l contains sets of $I''_j(\hat{x})$'s which have some element(s) in common and we can decompose the original problem into k sub-problems. $I^{(l)}$ and $J^{(l)}$ correspond to sets of indices of variables and constraints, respectively.

5.1. An Algorithm. Based on the concepts and method discussed in preceding section, we present an algorithm for finding an optimal solution of problem (3).

Algorithm 1:

(1) Find the maximum solution \hat{x} by the Definition 3. If $\hat{x} \underset{BD}{o} A \neq b$, then stop. $X[A, b] = \emptyset$. Otherwise, go to next step.

(2) For each row $i, i \in I$, if $\exists j \in J, \hat{x}_i + a_{ij} - 1 = b_j$, and $b_j = 0$, then $x_i^* = 0$ and the optimal objective value is zero. If $\alpha_i < 0$, then let $x_i^* = \hat{x}_i$. Let i in I_0 and j in J_0 .

(2-1) Remove row $i, i \in I_0$, and column $j, j \in J_0$, from matrix A to obtain A_1 .

(2-2) Remove the j th element, $j \in J_0$, from vector b to obtain b_1 .

(3) Let $I_1 = I - I_0$, and $J_1 = J - J_0$.

(4) for each row $i, i \in I_1$, if there is not $j \in J_1, \hat{x}_i + a_{ij} - 1 = b_j$, let $x_i^* = 1$ if $\alpha_i < 0$, and $x_i^* = 0$ if $\alpha_i > 0$. Let i in I'_1 .

(4-1) Remove row $i, i \in I'_1$, from matrix A_1 to obtain A_2 .

(5) Let $I_2 = I_1 - I'_1$ and $J_2 = J_1$.

(6) Calculate $I_j(\hat{x}) = \{i \in I_2 \mid \hat{x}_i + a_{ij} - 1 = b_j\}$, for $\forall j \in J_2$.

(7) Obtain R^- and R^+ by (11).

(8) Reduce problem as follows:

Obtain $\hat{I} = \{i \in I_2 \mid \alpha_i < 0\}$ and $\hat{J} = \{j \in J_2 \mid \hat{x}_i + a_{ij} - 1 = b_j; i \in \hat{I}\}$.

(8-1) Remove row $i, i \in \hat{I}$, and column $j, j \in \hat{J}$, from matrix A_2 to obtain A'_2 .

(8-2) Remove the j th element, $j \in \hat{J}$, from vector b_1 to obtain b'_1 .

(8-3) Assign an optimal value $x_i^* = \hat{x}_i$, for $i \in \hat{I}$.

(8-4) If $b'_1 = \emptyset$, assign zero to unassigned x_i^* and go to step 12. Otherwise, calculate $J' = J_2 - \hat{J}$ and $I' = I_2 - \hat{I}$. Update $I_j(\hat{x})$'s and $I(\hat{x})$ and obtain $I'_j(\hat{x})$'s and $I'(\hat{x})$, and omit $I'_j(\hat{x})$'s that they are empty. Proceed to the next step.

(9) Calculate $\bar{I} = \{i \in I_j(\hat{x}) \mid |I_j(\hat{x})| = 1, j \in J'\}$, $\bar{J} = \{j \in J' \mid x_i + a_{ij} - 1 = b_j; i \in \bar{I}\}$.

(9-1) Remove row $i, i \in \bar{I}$, and column $j, j \in \bar{J}$, from matrix A'_2 to obtain A''_2 .

(9-2) Remove the j th element, $j \in \bar{J}$, from vector b'_1 to obtain b''_1 .

(9-3) Assign $x_i^* = b_j - a_{ij} + 1$, for $i \in \bar{I}$ and $i \in I_j(\hat{x})$.

(9-4) If $b''_1 = \emptyset$ assign zero to unassigned x_i^* and go to step 12. Otherwise, calculate $J'' = J' - \bar{J}$ and $I'' = I' - \bar{I}$. Update $I'_j(\hat{x})$'s and $I'(\hat{x})$ and obtain $I''_j(\hat{x})$'s and $I''(\hat{x})$, and omit $I''_j(\hat{x})$'s that they are empty. Proceed to the next step.

(10) Decompose the problem by Relations of (13). For each sub-problem l , define problem P1 from (12).

(11) Generate optimal solution of sub-problem l , using pairwise comparison of elements of set S in objective function of sub-problem l .

(12) Generate an optimal solution for original problem via combining obtained solutions from steps of (2), (4), (8), (9), and (11). Also, assign zero to unassigned x_i^* , in during implementation of algorithm.

(13) End.

Now, we illustrate steps of Algorithm 1 by following example.

Example 1. Consider the following problem:

$$\begin{aligned} \min \quad & z = x_1 \times x_2^{-2} \times x_3^{-3} \times x_4^7 \times x_5^5 \times x_6^4 \times x_7 \times x_8 \times x_9^2 \times x_{10}, \\ \text{s.t.} \quad & x_{BD} A = b \\ & 0 \leq x_i \leq 1 \quad i = 1, 2, \dots, 10 \end{aligned}$$

Where A and b are as follows:

$$A = \begin{bmatrix} 0.7 & 0.3 & 1 & 0.25 & 0 & 0.81 \\ 0.11 & 0.5 & 0 & 0.91 & 0 & 0.2 \\ 0.25 & 0.2 & 0.76 & 0.58 & 0.9 & 0.33 \\ 0.1 & 0.75 & 0.45 & 0.83 & 0.8 & 0.41 \\ 0.22 & 0.85 & 0.15 & 0.13 & 0 & 0 \\ 0.6 & 0.35 & 0.2 & 0.28 & 0 & 0.1 \\ 0.55 & 0.55 & 0 & 0.8 & 0 & 0.9 \\ 0.5 & 0.44 & 0 & 0.75 & 0 & 0.75 \\ 0.05 & 0.36 & 0 & 0.35 & 0 & 0.95 \\ 0.18 & 0.22 & 0 & 0.3 & 0 & 1 \end{bmatrix}$$

and

$$b = [0.25 \ 0.6 \ 0 \ 0.5 \ 0 \ 0.9]^t.$$

Step 1. We obtain $\hat{x} = [0, 0.59, 0.1, 0.2, 0.75, 0.65, 0.7, 0.75, 0.95, 0.9]$ from the Definition 3. Since $\hat{x} \underset{BD}{\circ} A = b$. Hence, $X[A, b] \neq \emptyset$.

Step 2. 1- Since $\hat{x}_1 + a_{13} - 1 = b_3$, $b_3 = 0$, and $\alpha_1 > 0$ then $x_1^* = 0$ and $z^* = 0$. We continue to find the optimal values of other variables. 2- Since $\hat{x}_3 + a_{35} - 1 = b_5$, $b_5 = 0$, and $\alpha_3 < 0$ then $x_3^* = \hat{x}_3 = 0.1$. 3- Since $\hat{x}_4 + a_{45} - 1 = b_5$, $b_5 = 0$, and $\alpha_4 > 0$, then $x_4^* = 0$. $I_0 = \{1, 3, 4\}$, and $J_0 = \{3, 5\}$.

(2-1)

$$A_1 = \begin{bmatrix} 0.11 & 0.5 & 0.91 & 0.2 \\ 0.22 & 0.85 & 0.13 & 0 \\ 0.6 & 0.35 & 0.28 & 0.1 \\ 0.55 & 0.55 & 0.8 & 0.9 \\ 0.5 & 0.44 & 0.75 & 0.75 \\ 0.05 & 0.36 & 0.35 & 0.95 \\ 0.18 & 0.22 & 0.3 & 1 \end{bmatrix}, \quad (2-2) \quad b_1 = [0.25 \ 0.6 \ 0.5 \ 0.9]$$

Step 3. $I_1 = \{2, 5, 6, 7, 8, 9, 10\}$, and $J_1 = \{1, 2, 4, 6\}$.

Step 4. There is not such $i, i \in I_1$, satisfying in step 4. Therefore, $I_1' = \emptyset$.

Step 5. $I_2 = I_1 = \{2, 5, 6, 7, 8, 9, 10\}$, and $J_2 = J_1 = \{1, 2, 4, 6\}$.

Step 6. $I_1(\hat{x}) = \{6, 7, 8\}$, $I_2(\hat{x}) = \{5\}$, $I_4(\hat{x}) = \{2, 7, 8\}$, and $I_6(\hat{x}) = \{9, 10\}$.

Step 7. $R^+ = \{1, 4, 5, 6, 7, 8, 9, 10\}$ and $R^- = \{2, 3\}$.

Step 8. $\hat{I} = \{2\}$, and $\hat{J} = \{4\}$.

$$(8-1) \quad A_2' = \begin{bmatrix} 0.22 & 0.85 & 0 \\ 0.6 & 0.35 & 0.1 \\ 0.55 & 0.55 & 0.9 \\ 0.5 & 0.44 & 0.75 \\ 0.05 & 0.36 & 0.95 \\ 0.18 & 0.22 & 1 \end{bmatrix},$$

$$(8-2) \quad b_1' = [0.25 \ 0.6 \ 0.9],$$

$$(8-3) \quad x_2^* = \hat{x}_2 = 0.59,$$

(8-4) $J' = J_2 - \hat{J} = \{1, 2, 6\}$, $I' = I_2 - \hat{I} = \{5, 6, 7, 8, 9, 10\}$. $I'_j(\hat{x}) = I_j(\hat{x})$ for $j = 1, 2, 6$.

Step 9. $\bar{I} = \{5\}$, and $\bar{J} = \{2\}$.

$$(9-1) A''_2 = \begin{bmatrix} 0.6 & 0.1 \\ 0.55 & 0.9 \\ 0.5 & 0.75 \\ 0.05 & 0.95 \\ 0.18 & 1 \end{bmatrix},$$

(9-2) $b''_1 = [0.25 \ 0.9]$, (9-3) $x_5^* = b_2 - a_{52} + 1 = 0.75$,

(9-4) $J'' = J' - \bar{J} = \{1, 6\}$, and $I'' = I' - \bar{I} = \{6, 7, 8, 9, 10\}$. $I''_j(\hat{x}) = I'_j(\hat{x})$ for $j = 1, 6$.

Step 10. $\pi = \{I''_1(\hat{x}), I''_6(\hat{x})\}$; $\pi_1 = \{I''_1(\hat{x})\}$, $\pi_2 = \{I''_6(\hat{x})\}$; $\pi_1 \cap \pi_2 = \emptyset$;
 $\pi = \pi_1 \cup \pi_2$; $\Lambda_1(\hat{x}) = I''_1(\hat{x})$, $\Lambda_2(\hat{x}) = I''_6(\hat{x})$; $I^{(1)} = \{6, 7, 8\}$, and $I^{(2)} = \{9, 10\}$;
 $J^{(1)} = \{1\}$, and $J^{(2)} = \{6\}$.

Problem 1:

Problem 2:

$$\begin{aligned} \min \quad & x_6^4 \times x_7 \times x_8, & \min \quad & x_9^2 \times x_{10}, \\ \text{s.t.} \quad & [x_6 \ x_7 \ x_8] \underset{BD}{o} \begin{bmatrix} 0.6 \\ 0.55 \\ 0.5 \end{bmatrix} = [0.25], & \text{s.t.} \quad & [x_9 \ x_{10}] \underset{BD}{o} \begin{bmatrix} 0.95 \\ 1 \end{bmatrix} = [0.9], \\ & 0 \leq x_i \leq 1 \text{ for } i = 6, 7, 8. & & 0 \leq x_i \leq 1 \text{ for } i = 9, 10. \end{aligned}$$

Step 11. Now, we create the sets of S related to Problem 1 and Problem 2. At first, we create the set of S related to Problem 1 as: $S^1 = \{(0.65, 0, 0), (0, 0.7, 0), (0, 0, 0.75)\}$. With pairwise comparison, we conclude that the optimal solution is not unique. Each element of S^1 can be an optimal solution of Problem 1.

Similarly, for Problem 2, we have $S^2 = \{(0.95, 0), (0, 0.9)\}$. Similarly, each element of set S^2 can be an optimal solution of Problem 2.

Step 12. We generate optimal solution for the original problem via combining obtained solutions from steps of (2), (4), (7), (8), and (10). Hence, one of the optimal solutions is as follows:

$$x^* = [0 \ 0.59 \ 0.1 \ 0 \ 0.75 \ 0 \ 0.7 \ 0 \ 0 \ 0.9] \text{ with } z^* = 0.$$

6. CONCLUSIONS

In this paper, we have studied a monomial optimization problem with fuzzy relation equation constraints regarding max-bounded difference. Due to the non-convexity nature of its feasible domain, we tend to believe that there is no polynomial-time algorithm for this problem. The best we can do this is that, after analyzing the properties of its feasible domain, we presented an algorithm for solving problem (3). Also, in order to minimize process of solving the problem, we presented some properties to reduce the size of original problem and decomposed it (if possible) into several sub-problems with smaller dimensions.

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