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THE GENERALIZED HAHN POLYNOMIALS

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Abstract. In this paper, we represent the generalized Hahn polynomials $\varphi_n^{(a)}(x,y)$ by the Cauchy operator for deriving its identities: generating function, Mehler's formula, Rogers formula (with some of its applications), Rogers-type formula, extended generating function, extended Mehler's formula, extended Rogers formula and another extended identities. Also, the Rogers-type formula for the bivariate (generalized) (classical) Rogers-Szegö polynomials will be given by two methods. Then we give the q-integral representation for the generalized Hahn polynomials, bivariate Rogers-Szegö polynomials, and the generalized Rogers-Szegö polynomials.

Keywords: Hahn polynomials, Cauchy operator, generating function, extended Mehler's formula, q-integral.

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1. Introduction

Chen and Liu [13] developed a method of deriving hypergeometric identities by parameter augmentation, this method has more realizations as in [1, 2, 3, 4, 5, 11, 12, 14, 21, 22]. A new realization will be given in this paper, where we will represent the generalized Hahn polynomials by the Cauchy operator to derive their basic and extended identities.

Let us review some common notation and terminology for basic hypergeometric series in [18]. Assume that $|q| < 1$. The q-shifted factorial is defined by:

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \qquad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k). \tag{1}$$

Where:

$$\begin{aligned} (a;q)_n &= (a;q)_\infty/(aq^n;q)_\infty\,, \\ (a;q)_{n+k} &= (a;q)_k\,(aq^k;q)_n\,. \end{aligned}$$

We also adopt the following notation for multiple q-shifted factorial:

$$\begin{aligned} (a_1,a_2,\cdots,a_m;q)_n &= (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n\,, \\ (a_1,a_2,\cdots,a_m;q)_\infty &= (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty\,. \end{aligned}$$

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The q-binomial coefficient is defined by:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

The basic hypergeometric series ${}_{r+1}\phi_r$ is defined by:

$${}_{r+1}\phi_r \left(\begin{matrix} a_1, \ldots, a_{r+1} \\ b_1, \ldots, b_r \end{matrix} ; q, x \right) = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_{r+1}; q)_n}{(q, b_1, \ldots, b_r; q)_n} x^n.$$

The inverse pair is defined for any sequences a_n and b_n as follows:

$$a_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} b_k \iff b_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} a_{n-k}. \quad (2)$$

The Cauchy identity is defined as:

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} x^k = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x| < 1. \quad (3)$$

Putting $a = 0$, (3) becomes Euler's identity:

$$\sum_{k=0}^{\infty} \frac{x^k}{(q;q)_k} = \frac{1}{(x;q)_\infty}, \quad |x| < 1, \quad (4)$$

and its inverse relation:

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} x^k}{(q;q)_k} = (x;q)_\infty. \quad (5)$$

The Cauchy polynomials is defined by:

$$p_n(x,y) = (x-y)(x-qy)\cdots(x-q^{n-1}y) = (y/x;q)_n \, x^n, \quad (6)$$

The generalized Hahn polynomials [7, 8, 9, 10] or the bivariate form of Al-Salam-Carlitz polynomials is defined as:

$$\varphi_n^{(a)}(x,y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a;q)_k \, x^k \, y^{n-k}. \quad (7)$$

The classical Rogers-Szegö polynomials [4, 6, 12, 14, 21, 22] is a $(a = 0, y = 1)$ case of the generalized Hahn polynomials (7), is defined in 1926 by Szegö, as:

$$h_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k. \quad (8)$$

The generalized Rogers-Szegö polynomials [1, 16, 17, 21] is a $(a = 0)$ case of the generalized Hahn polynomials (7), is defined as:

$$r_n(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k \, y^{n-k}, \quad (9)$$

which has the following generating function [1, 16, 17, 21]:

$$\sum_{n=0}^{\infty} r_n(x,y) \frac{t^n}{(q;q)_n} = \frac{1}{(xt, yt; q)_\infty}, \quad (10)$$

where $max\{|xt|, |yt|\} < 1$.

The bivariate Rogers-Szegö polynomials [22] is a $(x = 1, y = x, a = y)$ case of the generalized Hahn polynomials (7), is defined as:

$$h_n(x, y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (y; q)_k \, x^{n-k}. \tag{11}$$

F.H. Jackson [19] defined the q-integral as follows:

$$\int_0^d f(t) d_q t = d(1-q) \sum_{n=0}^{\infty} f(dq^n) q^n, \tag{12}$$

and

$$\int_c^d f(t) d_q t = \int_0^d f(t) d_q t - \int_0^c f(t) d_q t. \tag{13}$$

M. Wang [23] derived the following integral identity:

$$\int_c^d \frac{(qt/c, qt/d; q)_\infty t^n}{(at, bt; q)_\infty} d_q t = \frac{d(1-q)(q, dq/c, c/d, abcd; q)_\infty}{(ac, ad, bc, bd; q)_\infty} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(ad, bd; q)_k}{(abcd; q)_k} c^k d^{n-k}. \tag{14}$$

The q-differential operator [12] is defined by:

$$D_q f(a) = \frac{f(a) - f(aq)}{a} ,$$

where [12]:

$$D_q^k \{x^n\} \quad = \quad \frac{(q; q)_n}{(q; q)_{n-k}} \, x^{n-k}, \tag{15}$$

In 2008, Chen and Gu [11] introduced the following Cauchy augmentation operator for basic hypergeometric series as a general form for the q-exponential operator $T(bD_q)$ when $a = 0$.

$$T(a, b; D_q) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} (bD_q)^n. \tag{16}$$

They derived three important results for this operator:

Proposition 1.1. [11].

$$T(a, b; D_q) \left\{ \frac{1}{(ct; q)_\infty} \right\} \quad = \quad \frac{(abt; q)_\infty}{(bt, ct; q)_\infty}; \quad |bt| < 1. \tag{17}$$

$$T(a, b; D_q) \left\{ \frac{1}{(cs, ct; q)_\infty} \right\} \quad = \quad \frac{(abt; q)_\infty}{(bt, cs, ct; q)_\infty} \, {}_2\phi_1 \left(\begin{matrix} a, ct \\ abt \end{matrix} ; q, bs \right), \tag{18}$$

where $max\{|bs|, |bt|\} < 1$.

$$T(a, b; D_q) \left\{ \frac{(cv; q)_\infty}{(cs, ct; q)_\infty} \right\} \quad = \quad \frac{(abs, cv; q)_\infty}{(bs, cs, ct; q)_\infty} \, {}_3\phi_2 \left(\begin{matrix} a, cs, v/t \\ abs, cv \end{matrix} ; q, bt \right). \tag{19}$$

where $max\{|bs|, |bt|\} < 1$.

Notice that when we set $s = 0$ in (19), we get the following operator identity (see [20]), which will be used later to derive the Rogers-type formula for the generalized Hahn polynomials and for the bivariate Rogers-Szegö polynomials.

$$T(a, b; D_q) \left\{ \frac{(cv; q)_\infty}{(ct; q)_\infty} \right\} = \frac{(cv; q)_\infty}{(ct; q)_\infty} \, {}_2\phi_1 \left(\begin{matrix} a, v/t \\ cv \end{matrix} ; q, bt \right); \quad |bt| < 1. \tag{20}$$

So that, (20) reduce to the following new identity for the q-exponential operator, when we set $a=0$ which will be used later to derive the Rogers-type formula for the classical and generalized Rogers-Szegö polynomials.

$$T(bD_q)\left\{\frac{(cv;q)_\infty}{(ct;q)_\infty}\right\}=\frac{(cv;q)_\infty}{(ct;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}0,v/t\\ cv\end{matrix};q,bt\right);\quad |bt|<1. \tag{21}$$

In 2009, the author [1] introduced the following Cauchy operator results:

Proposition 1.2. [1].

$$T(a,b;D_q)\left\{\frac{c^n}{(ct;q)_\infty}\right\} = \frac{(abt;q)_\infty}{(bt,ct;q)_\infty}\sum_{j=0}^{n}\begin{bmatrix}n\\ j\end{bmatrix}\frac{(a,ct;q)_j}{(abt;q)_j}\,b^j\,c^{n-j}\quad ;|bt|<1 \tag{22}$$

$$T(a,b;D_q)\left\{\frac{c^n}{(cs,ct;q)_\infty}\right\} = \frac{(abt;q)_\infty}{(bt,ct,cs;q)_\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{n}\begin{bmatrix}n\\ j\end{bmatrix}\frac{(a,ct;q)_{j+l}(cs;q)_j}{(abt;q)_{j+l}(q;q)_l}\,b^{j+l}c^{n-j}s^l, \tag{23}$$

where $max\{|bs|,|bt|\}<1$.

$$T(a,b;D_q)\left\{\frac{c^n(cv;q)_\infty}{(cs,ct;q)_\infty}\right\} = \frac{(abs,cv;q)_\infty}{(bs,cs,ct;q)_\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{n}\begin{bmatrix}n\\ j\end{bmatrix}\frac{(a,cs;q)_{j+l}(ct;q)_j(v/t;q)_l}{(abs,cv;q)_{j+l}(q;q)_l}\,b^{j+l}c^{n-j}t^l,$$

where $max\{|bs|,|bt|\}<1$. (24)

$$T(a,b;D_q)\left\{\frac{1}{(cs,ct,cv;q)_\infty}\right\} = \frac{1}{(cv,bt;q)_\infty}\sum_{k=0}^{\infty}\frac{(a;q)_k(bv)^k(abtq^k;q)_\infty}{(q;q)_k(csq^k,ctq^k;q)_\infty}\,{}_2\phi_1\left(\begin{matrix}aq^k,ctq^k\\ abtq^k\end{matrix};q,bs\right),$$

where $max\{|bs|,|bt|\}<1$. (25)

2. The Basic Identities

In this section, we recall the definition of generalized Hahn polynomials $\varphi_n^{(a)}(x,y)$ and give a corresponding definition for it. We represent this polynomials by the Cauchy operator, then we derive its generating function, Mehler's formula, Rogers-type formula, Rogers formula, linearization formula, and another some identities.

Definition 2.1. *The generalized Hahn polynomials* [7, 8, 9, 10] *is defined as:*

$$\varphi_n^{(a)}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}(a;q)_k\,x^k\,y^{n-k}.$$

Notice that, the bivariate Rogers-Szegö polynomials (11) is a ($y=1, x=y$ and $a=x$) case of the generalized Hahn polynomials, also the generalized Rogers-Szegö polynomials (9) is a ($a=0$) case of the generalized Hahn polynomials therefore when we setting $y=1, x=y$ and $a=x$ in all identities of $\varphi_n^{(a)}(x,y)$ (which are given in this paper) we will get the corresponding identities for $h_n(x,y|q)$, so that we will get the corresponding identities for $r_n(x,y)$ when we setting $a=0$.

Now, we can represent the polynomials $\varphi_n^{(a)}(x,y)$ by the following case of the Cauchy operator:

Proposition 2.1.

$$T(a,x;D_q)\{y^n\}=\varphi_n^{(a)}(x,y)\,. \tag{26}$$

Proof. By definition of the Cauchy operator (16) and identity (15). □

Depending on our representation (26) for the generalized Hahn polynomials $\varphi_n^{(a)}(x,y)$, we can give the following proofs of the generating function, Mehler's formula, Rogers formula and another Rogers-type formula by using the roles of the Cauchy operator. Firstly, we derive the generating function by using (26) and identity (17) as follows:

Theorem 2.1. (The generating function for $\varphi_n^{(a)}(x,y)$). *We have:*

$$\sum_{n=0}^{\infty}\varphi_n^{(a)}(x,y)\,\frac{t^n}{(q;q)_n}=\frac{(axt;q)_\infty}{(xt,yt;q)_\infty}, \tag{27}$$

where $\max\{|xt|,|yt|\}<1$.

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty}\varphi_n^{(a)}(x,y)\,\frac{t^n}{(q;q)_n} &= \sum_{n=0}^{\infty}T(a,x;D_q)\{y^n\}\,\frac{t^n}{(q;q)_n}\\ &= T(a,x;D_q)\left\{\sum_{n=0}^{\infty}\frac{(yt)^n}{(q;q)_n}\right\}\\ &= T(a,x;D_q)\left\{\frac{1}{(yt;q)_\infty}\right\}\\ &= \frac{(axt;q)_\infty}{(xt,yt;q)_\infty}.\end{aligned}$$

□

Notice that, we can give another definition to the generalized Hahn polynomials by using the above generating function (27), Cauchy identity (3), Euler identity (4), and Cauchy polynomial (6) as follows:

Lemma 2.1. *We have:*

$$\varphi_n^{(a)}(x,y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}p_k(y,ax)\,x^{n-k}. \tag{28}$$

Proof. Rewrite the generating function (27) as the form:

$$\begin{aligned}\sum_{n=0}^{\infty}\varphi_n^{(a)}(x,y)\,\frac{t^n}{(q;q)_n} &= \frac{(axt;q)_\infty}{(xt,yt;q)_\infty}\\ &= \sum_{n=0}^{\infty}\frac{(ax/y;q)_n\,(yt)^n}{(q;q)_n}\sum_{k=0}^{\infty}\frac{(xt)^k}{(q;q)_k}\\ &= \sum_{n=0}^{\infty}\frac{(ax/y;q)_n\,y^n}{(q;q)_n}\sum_{k=0}^{\infty}\frac{x^k}{(q;q)_k}\,t^{n+k}.\end{aligned}$$

Set $n\to n-k$ and replace the summations, then compare the coefficients of t^n in the two sides, we get:

$$\begin{aligned}\varphi_n^{(a)}(x,y) &= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(ax/y;q)_{n-k}\,y^{n-k}x^k\\ &= \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}p_k(y,ax)\,x^{n-k}.\end{aligned}$$

□

By using the inverse pair (2), we can give the inverse relation of (28) as:

$$p_n(y, ax) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} \varphi_{n-k}^{(a)}(x, y)\, x^{-k}. \tag{29}$$

Mehler's formula for the generalized Hahn polynomials will be derived by applying our representation (26) and the Cauchy operator identity (19) as follows:

Theorem 2.2. (Mehler's formula for $\varphi_n^{(a)}(x, y)$) . *We have:*

$$\sum_{n=0}^{\infty} \varphi_n^{(a)}(x, y)\, \varphi_n^{(b)}(z, w)\, \frac{t^n}{(q; q)_n} = \frac{(axzt, bzyt; q)_\infty}{(xzt, zyt, wyt; q)_\infty}\, {}_3\phi_2 \left(\begin{matrix} a, zyt, bz/w \\ axzt, bzyt \end{matrix} ; q, xwt \right), \tag{30}$$

where $\max\{|xzt|, |xwt|, |zyt|, |wyt|\} < 1$.

Proof.

$$\begin{aligned} \sum_{n=0}^{\infty} \varphi_n^{(a)}(x, y)\, \varphi_n^{(b)}(z, w)\, \frac{t^n}{(q; q)_n} &= \sum_{n=0}^{\infty} T(a, x; D_q)\{y^n\}\, \varphi_n^{(b)}(z, w)\, \frac{t^n}{(q; q)_n} \\ &= T(a, x; D_q) \left\{ \sum_{n=0}^{\infty} \varphi_n^{(b)}(z, w)\, \frac{(yt)^n}{(q; q)_n} \right\} \\ &= T(a, x; D_q) \left\{ \frac{(bzyt; q)_\infty}{(zyt, wyt; q)_\infty} \right\} \\ &= \frac{(axzt, bzyt; q)_\infty}{(xzt, zyt, wyt; q)_\infty}\, {}_3\phi_2 \left(\begin{matrix} a, zyt, bz/w \\ axzt, bzyt \end{matrix} ; q, xwt \right). \end{aligned}$$

□

In the following theorem, we give the Rogers-type formula for $\varphi_n^{(a)}(x, y)$ by using identity (20) of the Cauchy operator and our representation (26).

Theorem 2.3. (Rogers-type formula for $\varphi_n^{(a)}(x, y)$) . *We have:*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \varphi_{n+m}^{(a)}(x, y)\, \frac{t^n}{(q; q)_n}\, \frac{(-1)^m q^{\binom{m}{2}} s^m}{(q; q)_m} = \frac{(ys; q)_\infty}{(yt; q)_\infty}\, {}_2\phi_1 \left(\begin{matrix} a, s/t \\ ys \end{matrix} ; q, xt \right), \tag{31}$$

where $\max\{|xt|, |yt|\} < 1$.

Proof.

$$\begin{aligned} &\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \varphi_{n+m}^{(a)}(x, y)\, \frac{t^n}{(q; q)_n}\, \frac{(-1)^m q^{\binom{m}{2}} s^m}{(q; q)_m} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} T(a, x; D_q) \left\{ y^{n+m} \right\} \frac{t^n}{(q; q)_n}\, \frac{(-1)^m q^{\binom{m}{2}} s^m}{(q; q)_m} \\ &= T(a, x; D_q) \left\{ \sum_{n=0}^{\infty} \frac{(yt)^n}{(q; q)_n} \sum_{m=0}^{\infty} \frac{(-1)^m q^{\binom{m}{2}} (ys)^m}{(q; q)_m} \right\} \\ &= T(a, x; D_q) \left\{ \frac{(ys; q)_\infty}{(yt; q)_\infty} \right\} \\ &= \frac{(ys; q)_\infty}{(yt; q)_\infty}\, {}_2\phi_1 \left(\begin{matrix} a, s/t \\ ys \end{matrix} ; q, xt \right). \end{aligned}$$

□

Notice that, when we set $x = 1, y = x$, and $a = y$ in the above theorem, we will get the Rogers-type formula for the bivariate Rogers-Szegö polynomials, which can be proved by using the identity (20) of the Cauchy operator and the author representation $T(y, 1; D_q)\{x^n\} = h_n(x, y|q)$ due to [1] as follows:

Corollary 2.1. (Rogers-type formula for $h_n(x, y|q)$) . *We have:*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x,y|q)\,\frac{t^n}{(q;q)_n}\,\frac{(-1)^m q^{\binom{m}{2}} s^m}{(q;q)_m} = \frac{(xs;q)_\infty}{(xt;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} y, s/t \\ xs \end{matrix}\ ; q, t\right), \tag{32}$$

where $\max\{|t|, |xt|\} < 1$.

Proof.

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x,y|q)\,\frac{t^n}{(q;q)_n}\,\frac{(-1)^m q^{\binom{m}{2}} s^m}{(q;q)_m} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} T(y,1;D_q)\left\{x^{n+m}\right\}\,\frac{t^n}{(q;q)_n}\,\frac{(-1)^m q^{\binom{m}{2}} s^m}{(q;q)_m} \\
&= T(y,1;D_q)\left\{\sum_{n=0}^{\infty}\frac{(xt)^n}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(-1)^m q^{\binom{m}{2}} (xs)^m}{(q;q)_m}\right\} \\
&= T(y,1;D_q)\left\{\frac{(xs;q)_\infty}{(xt;q)_\infty}\right\} \\
&= \frac{(xs;q)_\infty}{(xt;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} y, s/t \\ xs \end{matrix}\ ; q, t\right).
\end{aligned}$$

□

Also, when we set $a = 0$ in (31), we will introduce the following Rogers-type formula for the generalized Rogers-Szegö polynomials, which can be derived by using the identity (21) of the q-exponential operator and the representation $T(xD_q)\{y^n\} = r_n(x, y)$ according to the symmetry property $r_n(x, y) = r_n(y, x)$ due to [21].

Corollary 2.2. (Rogers-type formula for $r_n(x, y)$) . *We have:*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} r_{n+m}(x,y)\,\frac{t^n}{(q;q)_n}\,\frac{(-1)^m q^{\binom{m}{2}} s^m}{(q;q)_m} = \frac{(ys;q)_\infty}{(yt;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} 0, s/t \\ ys \end{matrix}\ ; q, xt\right), \tag{33}$$

where $\max\{|xt|, |yt|\} < 1$.

Proof.

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} r_{n+m}(x,y)\,\frac{t^n}{(q;q)_n}\,\frac{(-1)^m q^{\binom{m}{2}} s^m}{(q;q)_m} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} T(xD_q)\left\{y^{n+m}\right\}\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m}
\end{aligned}$$

$$= T(xD_q)\left\{\sum_{n=0}^{\infty}\frac{(yt)^n}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(-1)^m q^{\binom{m}{2}}(ys)^m}{(q;q)_m}\right\}$$
$$= T(xD_q)\left\{\frac{(ys;q)_\infty}{(yt;q)_\infty}\right\}$$
$$= \frac{(ys;q)_\infty}{(yt;q)_\infty}\, {}_2\phi_1\left(\begin{matrix}0, s/t\\ ys\end{matrix};q,xt\right).$$

□

In the same technique, we can give the Rogers-type formula for the classical Rogers-Szegö polynomials $h_n(x|q)$ which can be proved by using the identity (21) of the q-exponential operator and the representation $T(D_q)\{x^n\} = h_n(x|q)$ due to [12], or by replace x with y in (33) according to the symmetry property of the generalized Rogers-Szegö polynomials and then set $x = 1$, or by setting $y = 0$ directly in (32)

Corollary 2.3. (Rogers-type formula for $h_n(x|q)$) . *We have:*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} h_{n+m}(x|q)\,\frac{t^n}{(q;q)_n}\,\frac{(-1)^m q^{\binom{m}{2}} s^m}{(q;q)_m} = \frac{(xs;q)_\infty}{(xt;q)_\infty}\, {}_2\phi_1\left(\begin{matrix}0, s/t\\ xs\end{matrix};q,t\right), \tag{34}$$

where $\max\{|t|, |xt|\} < 1$.

Now, we are deriving the Rogers formula for the generalized Hahn polynomials depending on our representation (26) and the Cauchy operator identity (18) as follows:

Theorem 2.4. (The Rogers formula for $\varphi_n^{(a)}(x,y)$) . *We have:*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \varphi_{n+m}^{(a)}(x,y)\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m} = \frac{(axs;q)_\infty}{(yt, ys, xs;q)_\infty}\, {}_2\phi_1\left(\begin{matrix}a, ys\\ axs\end{matrix};q,xt\right), \tag{35}$$

where $\max\{|xs|, |xt|, |ys|, |yt|\} < 1$.

Proof.

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \varphi_{n+m}^{(a)}(x,y)\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} T(a,x;D_q)\left\{y^{n+m}\right\}\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m}$$
$$= T(a,x;D_q)\left\{\sum_{n=0}^{\infty}\frac{(yt)^n}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(ys)^m}{(q;q)_m}\right\}$$
$$= T(a,x;D_q)\left\{\frac{1}{(yt, ys;q)_\infty}\right\}$$
$$= \frac{(axs;q)_\infty}{(yt, ys, xs;q)_\infty}\, {}_2\phi_1\left(\begin{matrix}a, ys\\ axs\end{matrix};q,xt\right).$$

□

Here, we introduce some applications of the Rogers formula (35) such as the linearization formula and some relations between the generalized Hahn polynomials and the generalized Rogers-Szegö polynomials.

Corollary 2.4. (The Linearization formula for polynomial $\varphi_n^{(a)}(x,y)$). *We have:*

$$\sum_{k=0}^{n}\sum_{l=0}^{m}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m\\l\end{bmatrix}(a;q)_k\,(ax/y;q)_l\,x^k\,y^l\,\varphi_{n+m-k-l}^{(a)}(x,y)$$
$$=\sum_{k=0}^{n}\sum_{l=0}^{m}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}m\\l\end{bmatrix}(a;q)_k\,(ax/y;q)_l\,x^k\,(yq^l)^k\,\varphi_{n-k}^{(a)}(x,y)\,\varphi_{m-l}^{(a)}(x,y)\,, \qquad (36)$$

where $\max\{|xt|,|ys|\}<1$.

Proof. Rewrite the Rogers formula (35) in the following form:

$$\frac{(axt,axs;q)_\infty}{(xt,ys;q)_\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\varphi_{n+m}^{(a)}(x,y)\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m}$$
$$=\sum_{k=0}^{\infty}\frac{(a;q)_k\,(axsq^k;q)_\infty}{(q;q)_k\,(ysq^k;q)_\infty}\,(xt)^k\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\varphi_n^{(a)}(x,y)\,\varphi_m^{(a)}(x,y)\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m}\,. \qquad (37)$$

Expanding $(axt;q)_\infty/(xt;q)_\infty$, $(axs;q)_\infty/(ys;q)_\infty$, and $(axsq^k;q)_\infty/(ysq^k;q)_\infty$ by the Cauchy identity (3) as follows:

$$\frac{(axt;q)_\infty}{(xt;q)_\infty} = \sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}\,(xt)^k;\quad |xt|<1,$$
$$\frac{(axs;q)_\infty}{(ys;q)_\infty} = \sum_{l=0}^{\infty}\frac{(ax/y;q)_l}{(q;q)_l}\,(ys)^l;\quad |ys|<1,$$
$$\frac{(axsq^k;q)_\infty}{(ysq^k;q)_\infty} = \sum_{l=0}^{\infty}\frac{(ax/y;q)_l}{(q;q)_l}\,(ysq^k)^l;\quad |ys|<1.$$

Now substitute these expansions in (37):

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(a;q)_k\,x^k}{(q;q)_k}\,\frac{(ax/y;q)_l\,y^l}{(q;q)_l}\,\varphi_{n+m}^{(a)}(x,y)\,\frac{t^{n+k}}{(q;q)_n}\,\frac{s^{m+l}}{(q;q)_m}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(a;q)_k\,x^k}{(q;q)_k}\,\frac{(ax/y;q)_l}{(q;q)_l}\,(yq^k)^l\,\varphi_n^{(a)}(x,y)\,\varphi_m^{(a)}(x,y)\,\frac{t^{n+k}}{(q;q)_n}\,\frac{s^{m+l}}{(q;q)_m}.$$

Set $n\to n-k$ and $m\to m-l$:

$$\sum_{n=k}^{\infty}\sum_{m=l}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(a;q)_k\,x^k}{(q;q)_k}\,\frac{(ax/y;q)_l\,y^l}{(q;q)_l}\,\varphi_{n+m-k-l}^{(a)}(x,y)\,\frac{t^n}{(q;q)_{n-k}}\,\frac{s^m}{(q;q)_{m-l}}$$
$$=\sum_{n=k}^{\infty}\sum_{m=l}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(a;q)_k\,x^k}{(q;q)_k}\,\frac{(ax/y;q)_l}{(q;q)_l}\,(yq^k)^l\,\varphi_{n-k}^{(a)}(x,y)\,\varphi_{m-l}^{(a)}(x,y)\,\frac{t^n}{(q;q)_{n-k}}\,\frac{s^m}{(q;q)_{m-l}}.$$

By equating the coefficients of $t^n s^m$, we get the required identity. □

Corollary 2.5. *For $n, m \geq 0$, we have:*

$$\sum_{k=0}^{\min\{n,m\}} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} (q;q)_k \, (xy)^k \, r_{n+m-2k}(x,y) = \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (ax)^k \, \varphi_{n-k}^{(a)}(x,y) \right) \left(\sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix} (ax)^j \, \varphi_{m-j}^{(a)}(x,y) \right), \quad (38)$$

where $\max\{|xt|, |xs|, |yt|, |ys|, |axt|, |axs|, |xyst|\} < 1$.

Proof. Let $a = 0$ in the Rogers formula (35), we get:

$$\begin{aligned} &\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} r_{n+m}(x,y) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m} \\ &= \frac{1}{(xs, ys, yt; q)_\infty} \sum_{k=0}^{\infty} \frac{(ys;q)_k}{(q;q)_k} (xt)^k \\ &= \frac{(xyst;q)_\infty}{(xs, xt, ys, yt; q)_\infty} \\ &= \frac{(xyst;q)_\infty}{(axt, axs; q)_\infty} \frac{(axt;q)_\infty}{(xt, yt; q)_\infty} \frac{(axs;q)_\infty}{(xs, ys; q)_\infty} \\ &= \frac{(xyst;q)_\infty}{(axt, axs; q)_\infty} \sum_{n=0}^{\infty} \varphi_n^{(a)}(x,y) \frac{t^n}{(q;q)_n} \sum_{m=0}^{\infty} \varphi_m^{(a)}(x,y) \frac{s^m}{(q;q)_m}. \end{aligned}$$

Then:

$$\frac{1}{(xyst;q)_\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} r_{n+m}(x,y) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m} = \frac{1}{(axt, axs; q)_\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \varphi_n^{(a)}(x,y) \varphi_m^{(a)}(x,y) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m}. \quad (39)$$

By applying Euler identity (4) on the terms: $1/(xyst;q)_\infty, 1/(axt;q)_\infty$ and $1/(axs;q)_\infty$:

$$\begin{aligned} &\sum_{k=0}^{\infty} \frac{(xyst)^k}{(q;q)_k} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} r_{n+m}(x,y) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m} \\ &= \sum_{k=0}^{\infty} \frac{(axt)^k}{(q;q)_k} \sum_{j=0}^{\infty} \frac{(axs)^j}{(q;q)_j} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \varphi_n^{(a)}(x,y) \varphi_m^{(a)}(x,y) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m} \\ \Rightarrow &\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(xy)^k}{(q;q)_k} r_{n+m}(x,y) \frac{t^{n+k}}{(q;q)_n} \frac{s^{m+k}}{(q;q)_m} \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(ax)^k}{(q;q)_k} \frac{(ax)^j}{(q;q)_j} \varphi_n^{(a)}(x,y) \varphi_m^{(a)}(x,y) \frac{t^{n+k}}{(q;q)_n} \frac{s^{m+j}}{(q;q)_m}. \end{aligned}$$

Set $n \to n-k$ and $m \to m-k$:

$$\Rightarrow \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\sum_{m=k}^{\infty} \frac{(xy)^k}{(q;q)_k} r_{n+m-2k}(x,y) \frac{t^n}{(q;q)_{n-k}} \frac{s^m}{(q;q)_{m-k}}$$
$$= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{n=k}^{\infty}\sum_{m=j}^{\infty} \frac{(ax)^k}{(q;q)_k} \frac{(ax)^j}{(q;q)_j} \varphi_{n-k}^{(a)}(x,y) \varphi_{m-j}^{(a)}(x,y) \frac{t^n}{(q;q)_{n-k}} \frac{s^m}{(q;q)_{m-j}}$$
$$\Rightarrow \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\min\{n,m\}} \frac{(xy)^k}{(q;q)_k} r_{n+m-2k}(x,y) \frac{t^n}{(q;q)_{n-k}} \frac{s^m}{(q;q)_{m-k}}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{m} \frac{(ax)^k}{(q;q)_k} \frac{(ax)^j}{(q;q)_j} \varphi_{n-k}^{(a)}(x,y) \varphi_{m-j}^{(a)}(x,y) \frac{t^n}{(q;q)_{n-k}} \frac{s^m}{(q;q)_{m-j}}.$$

The proof will be completed after comparing the coefficients of $t^n s^m$. □

Notice that when we set $a = 0$ in (38), we get the linearization formula of the generalized Rogers-Szegö polynomials: (see [16, 17]):

$$r_n(x,y)\, r_m(x,y) = \sum_{k=0}^{\min\{n,m\}} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} (q;q)_k\, (xy)^k\, r_{n+m-2k}(x,y). \tag{40}$$

So that, when we put $m = 0$ in (38), we will get the following relation between the generalized Rogers-Szegö polynomials $r_n(x,y)$ and the generalized Hahn polynomials $\varphi_n^{(a)}(x,y)$:

$$r_n(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (ax)^k\, \varphi_{n-k}^{(a)}(x,y). \tag{41}$$

Where (41) has the inverse relation according to (2) as follows:

$$\varphi_n^{(a)}(x,y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} (ax)^k\, r_{n-k}(x,y). \tag{42}$$

The following proof for identity (41) due to the generating function of the generalized Rogers-Szegö polynomials (10):

Proof. Rewrite identity (10) as follows:

$$\sum_{n=0}^{\infty} r_n(x,y) \frac{t^n}{(q;q)_n}$$
$$= \frac{1}{(xt,yt;q)_\infty}$$
$$= \frac{(axt;q)_\infty}{(xt,yt;q)_\infty} \frac{1}{(axt;q)_\infty}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \varphi_n^{(a)}(x,y) \frac{t^n}{(q;q)_n} \sum_{k=0}^{\infty} \frac{(axt)^k}{(q;q)_k} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \varphi_n^{(a)}(x,y) \frac{(ax)^k}{(q;q)_n (q;q)_k}\, t^{n+k}; \qquad \textit{set } n = n-k \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \varphi_{n-k}^{(a)}(x,y) \frac{(ax)^k}{(q;q)_{n-k} (q;q)_k}\, t^{n}.
\end{aligned}
$$

By comparing the coefficient of t^n in two sides, we get identity (41), and the proof will be finished. □

Corollary 2.6. *For $n, m \geq 0$, we have:*

$$
\begin{aligned}
&\sum_{k=0}^{n} \sum_{j=0}^{m} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ j \end{bmatrix} q^{\binom{k}{2}+\binom{j}{2}} (-ax)^{k+j}\, r_{n+m-k-j}(x,y) \\
&\qquad = \sum_{k=0}^{\min\{n,m\}} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} (q;q)_k\, q^{\binom{k}{2}} (-xy)^k\, \varphi_{n-k}^{(a)}(x,y) \varphi_{m-k}^{(a)}(x,y). \qquad (43)
\end{aligned}
$$

Proof. Rewrite identity (39) as follows:

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} r_{n+m}(x,y) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m} \\
&\qquad = \frac{(xyst;q)_\infty}{(axt, axs;q)_\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \varphi_n^{(a)}(x,y) \varphi_m^{(a)}(x,y) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m}.
\end{aligned}
$$

Multiply it by $(axt, axs;q)_\infty$:

$$
\begin{aligned}
&(axt, axs;q)_\infty \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} r_{n+m}(x,y) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m} \\
&\qquad = (xyst;q)_\infty \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \varphi_n^{(a)}(x,y) \varphi_m^{(a)}(x,y) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m}.
\end{aligned}
$$

Now expand $(xyst;q)_\infty, (axt;q)_\infty$ and $(axs;q)_\infty$ by Euler's identity (5) as follows:

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (axt)^k}{(q;q)_k} \sum_{j=0}^{\infty} \frac{(-1)^j q^{\binom{j}{2}} (axs)^j}{(q;q)_j} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} r_{n+m}(x,y) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m} \\
&\qquad = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (xyst)^k}{(q;q)_k} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \varphi_n^{(a)}(x,y) \varphi_m^{(a)}(x,y) \frac{t^n}{(q;q)_n} \frac{s^m}{(q;q)_m} \\
&\Rightarrow \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{k+j} q^{\binom{k}{2}+\binom{j}{2}} (ax)^{k+j}}{(q;q)_k (q;q)_j}\, r_{n+m}(x,y) \frac{t^{n+k}}{(q,q)_n} \frac{s^{m+j}}{(q,q)_m} \\
&\qquad = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k\, q^{\binom{k}{2}} (xy)^k}{(q;q)_k}\, \varphi_n^{(a)}(x,y) \varphi_m^{(a)}(x,y) \frac{t^{n+k}}{(q,q)_n} \frac{s^{m+k}}{(q,q)_m}.
\end{aligned}
$$

Setting $n \to n-k$ and $m \to m-j$:

$$\Rightarrow \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{n=k}^{\infty}\sum_{m=j}^{\infty} \frac{(-1)^{k+j} q^{\binom{k}{2}+\binom{j}{2}} (ax)^{k+j}}{(q;q)_k (q;q)_j} r_{n+m-k-j}(x,y) \frac{t^n}{(q,q)_{n-k}} \frac{s^m}{(q,q)_{m-j}}$$

$$= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\sum_{m=k}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (xy)^k}{(q;q)_k} \varphi_{n-k}^{(a)}(x,y)\varphi_{m-k}^{(a)}(x,y) \frac{t^n}{(q,q)_{n-k}} \frac{s^m}{(q,q)_{m-k}}.$$

By comparing the coefficients of $t^n s^m$ after replacing the summations, we get the required identity. □

When we setting $a=0$, (43) reduce to the inverse lineariztion formula of the generalized Rogers-Szegö polynomials (see [16]):

$$r_{m+n}(x,y) = \sum_{k=0}^{\min\{m,n\}} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ k \end{bmatrix} (q;q)_k \, q^{\binom{k}{2}} (-1)^k x^k y^k \, r_{n-k}(x,y) \, r_{m-k}(x,y). \tag{44}$$

3. An Extended Identities

In this section, we introduce an extended generating function, extended Mehler's formula and extended Rogers formula for the generalized Hahn polynomials $\varphi_n^{(a)}(x,y)$ by using our representation (26) and the roles (22), (24) and (23) of the Cauchy operator respectively, so that we give another extended identities.

Theorem 3.1. (Extended generating function for $\varphi_n^{(a)}(x,y)$). *We have:*

$$\sum_{n=0}^{\infty} \varphi_{n+k}^{(a)}(x,y) \frac{t^n}{(q;q)_n} = \frac{(axt;q)_\infty}{(xt,yt;q)_\infty} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(a,yt;q)_j}{(axt;q)_j} x^j y^{k-j}, \tag{45}$$

where $\max\{|xt|,|yt|\} < 1$.

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} \varphi_{n+k}^{(a)}(x,y) \frac{t^n}{(q;q)_n} &= \sum_{n=0}^{\infty} T(a,x;D_q)\left\{y^{n+k}\right\} \frac{t^n}{(q;q)_n} \\
&= T(a,x;D_q)\left\{y^k \sum_{n=0}^{\infty} \frac{(yt)^n}{(q;q)_n}\right\} \\
&= T(a,x;D_q)\left\{\frac{y^k}{(yt;q)_\infty}\right\} \\
&= \frac{(axt;q)_\infty}{(xt,yt;q)_\infty} \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \frac{(a,yt;q)_j}{(axt;q)_j} x^j y^{k-j}.
\end{aligned}$$

□

Setting $k=0$ in the above theorem, we get the generating function (27) for the generalized Hahn polynomials $\varphi_n^{(a)}(x,y)$.

Theorem 3.2. (Extended Mehler's formula for $\varphi_n^{(a)}(x,y)$). *We have:*

$$\sum_{n=0}^{\infty}\varphi_{n+k}^{(a)}(x,y)\varphi_n^{(b)}(z,w)\frac{t^n}{(q;q)_n}=\frac{(axzt,bzyt;q)_\infty}{(xzt,zyt,wyt;q)_\infty}$$

$$\times\sum_{l=0}^{\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{(a,zyt;q)_{j+l}(wyt;q)_j(bz/w;q)_l}{(axzt,bzyt;q)_{j+l}(q;q)_l}x^{j+l}y^{k-j}(wt)^l, \qquad (46)$$

where $\max\{|xzt|,|xwt|,|yzt|,|ywt|\}<1$.

Proof.

$$\sum_{n=0}^{\infty}\varphi_{n+k}^{(a)}(x,y)\,\varphi_n^{(b)}(z,w)\,\frac{t^n}{(q;q)_n}$$

$$=\sum_{n=0}^{\infty}T(a,x;D_q)\left\{y^{n+k}\right\}\,\varphi_n^{(b)}(z,w)\frac{t^n}{(q;q)_n}$$

$$=T(a,x;D_q)\left\{y^k\sum_{n=0}^{\infty}\varphi_n^{(b)}(z,w)\frac{(yt)^n}{(q;q)_n}\right\}$$

$$=T(a,x;D_q)\left\{y^k\,\frac{(bzyt;q)_\infty}{(zyt,wyt;q)_\infty}\right\}.$$

Which was lead us to the required identity after substitute $b\mapsto x, v\mapsto bzt, s\mapsto zt, t\mapsto wt$ and $c\mapsto y$ in the Cauchy operator role (24). □

Setting $k=0$ in the above theorem, we get Mehler's formula (30) for the generalized Hahn polynomials $\varphi_n^{(a)}(x,y)$.

Theorem 3.3. (Extended Rogers formula for $\varphi_n^{(a)}(x,y)$). *We have:*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\varphi_{n+m+k}^{(a)}(x,y)\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m}=\frac{(axs;q)_\infty}{(xs,yt,ys;q)_\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{(a,ys;q)_{j+l}(yt;q)_j}{(axs;q)_{j+l}(q;q)_l}x^{j+l}t^ly^{k-j}, \quad (47)$$

where $\max\{|xs|,|xt|,|ys|,|yt|\}<1$.

Proof.

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\varphi_{n+m+k}^{(a)}(x,y)\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}T(a,x;D_q)\left\{y^{n+m+k}\right\}\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m}$$

$$=T(a,x;D_q)\left\{y^k\sum_{n=0}^{\infty}\frac{(yt)^n}{(q;q)_n}\,\sum_{m=0}^{\infty}\frac{(ys)^m}{(q;q)_m}\right\}$$

$$=T(a,x;D_q)\left\{\frac{y^k}{(ys,yt;q)_\infty}\right\}$$

$$=\frac{(axs;q)_\infty}{(xs,yt,ys;q)_\infty}\sum_{l=0}^{\infty}\sum_{j=0}^{k}\begin{bmatrix}k\\j\end{bmatrix}\frac{(a,ys;q)_{j+l}\,(yt;q)_j}{(axs;q)_{j+l}\,(q;q)_l}\,x^{j+l}\,t^l\,y^{k-j}.$$

□

Setting $k=0$ in the above theorem, we get the Rogers formula (35) for the generalized Hahn polynomials $\varphi_n^{(a)}(x,y)$.

Theorem 3.4. *We have*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\varphi_{n+m+k}^{(a)}(x,y)\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m}\,\frac{(-1)^k q^{\binom{k}{2}} v^k}{(q;q)_k} = \frac{(axs,yv;q)_\infty}{(xs,ys,yt;q)_\infty}\;{}_3\phi_2\left(\begin{matrix} a,ys,v/t \\ axs,yv\end{matrix};q,xt\right), \quad (48)$$

where $\max\{|xs|,|xt|,|ys|,|yt|\}<1$.

Proof.

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\varphi_{n+m+k}^{(a)}(x,y)\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m}\,\frac{(-1)^k q^{\binom{k}{2}} v^k}{(q;q)_k}\\
&= T(a,x;D_q)\left\{\sum_{n=0}^{\infty}\frac{(yt)^n}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(ys)^m}{(q;q)_m}\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(yv)^k}{(q;q)_k}\right\}\\
&= T(a,x;D_q)\left\{\frac{(yv;q)_\infty}{(ys,yt;q)_\infty}\right\}\\
&= \frac{(axs,yv;q)_\infty}{(xs,ys,yt;q)_\infty}\;{}_3\phi_2\left(\begin{matrix} a,ys,v/t \\ axs,yv\end{matrix};q,xt\right).
\end{aligned}$$

□

The following extended identity for the generalized Hahn polynomials will be derived based on (25) and the operator representation (26).

Theorem 3.5. *We have*

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\varphi_{n+m+k}^{(a)}(x,y)\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m}\,\frac{v^k}{(q;q)_k}\\
&\quad = \frac{1}{(yv,xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(a;q)_k\,(xv)^k}{(q;q)_k}\frac{(axtq^k;q)_\infty}{(ysq^k,ytq^k;q)_\infty}\;{}_2\phi_1\left(\begin{matrix} aq^k,ytq^k \\ axtq^k\end{matrix};q,xs\right), \quad (49)
\end{aligned}$$

where $\max\{|xt|,|xs|,|yt|,|ys|,|yv|\}<1$.

Proof.

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\varphi_{n+m+k}^{(a)}(x,y)\,\frac{t^n}{(q;q)_n}\,\frac{s^m}{(q;q)_m}\,\frac{v^k}{(q;q)_k}\\
&= T(a,x;D_q)\left\{\sum_{n=0}^{\infty}\frac{(yt)^n}{(q;q)_n}\sum_{m=0}^{\infty}\frac{(ys)^m}{(q;q)_m}\sum_{k=0}^{\infty}\frac{(yv)^k}{(q;q)_k}\right\}\\
&= T(a,x;D_q)\left\{\frac{1}{(ys,yt,yv;q)_\infty}\right\}\\
&= \frac{1}{(yv,xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(a;q)_k\,(xv)^k}{(q;q)_k}\frac{(axtq^k;q)_\infty}{(ysq^k,ytq^k;q)_\infty}\;{}_2\phi_1\left(\begin{matrix} aq^k,ytq^k \\ axtq^k\end{matrix};q,xs\right).
\end{aligned}$$

□

4. q-Integral Representation

In this section, we give some q-integral representation for some q-polynomials depending on special cases of the q-integral identity (14).

The q-integral representation of the generalized Hahn polynomials or the bivariate form of the Al-Salam-Carlitz polynomials, is given in the following theorem:

Theorem 4.1. *We have:*

$$\varphi_n^{(a)}(x,y|q) = \frac{(a, ax/y;q)_\infty}{y(1-q)(q, qy/x, x/y;q)_\infty}\int_x^y \frac{(qt/x, qt/y;q)_\infty\, t^n}{(at/y;q)_\infty}\, d_qt, \tag{50}$$

where no zero factors occur in the denominator.

Proof. By setting $b=0, c=x, d=y$, and $a=a/y$ in (14). □

Notice that when we set $y=1$, (50) reduce to the q-integral representation of Hahn polynomials or the univariate form of the Al-Salam-Carlitz polynomials due to [23]:

$$\varphi_n^{(a)}(x|q) = \frac{(a, ax;q)_\infty}{(1-q)(q, q/x, x;q)_\infty}\int_x^1 \frac{(qt/x, qt;q)_\infty\, t^n}{(at;q)_\infty}\, d_qt, \tag{51}$$

Here, we give the q-integral representation of the bivariate Rogers-Szegö polynomials as follows:

Theorem 4.2. *We have:*

$$h_n(x,y|q) = \frac{(y, y/x;q)_\infty}{x(1-q)(q, qx, 1/x;q)_\infty}\int_1^y \frac{(qt, qt/x;q)_\infty\, t^n}{(yt/x;q)_\infty}\, d_qt, \tag{52}$$

where no zero factors occur in the denominator.

Proof. By setting $x=1, y=x$, and $a=y$ in (14). □

So that, the q-integral representation of the generalized Rogers-Szegö polynomials is introduced as:

Theorem 4.3. *We have:*

$$r_n(x,y) = \frac{1}{y(1-q)(q, qy/x, x/y;q)_\infty}\int_x^y (qt/x, qt/y;q)_\infty\, t^n\, d_qt, \tag{53}$$

where no zero factors occur in the denominator.

Proof. By setting $a=0, b=0, c=x$, and $d=y$ in (14). □

Notice that when we setting $y=1$ in (53), we will get the q-integral representation for the classical Rogers-Szegö polynomials $h_n(x|q)$ as in [24]:

$$h_n(x|q) = \frac{1}{(1-q)(q, x, q/x;q)_\infty}\int_x^1 (qt, qt/x;q)_\infty\, t^n\, d_qt. \tag{54}$$

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References

[1] Abdlhusein,M. A., (2009), The q-Operators and Rogers-Szegö polynomials, M.SC. Thesis, University of Basrah, Basrah, Iraq.

[2] Abdlhusein,M. A., (2012), The basic and extended identities for certain q-polynomials, J. College of education for pure sciences-Thiqar university, 2, pp. 11-23.

[3] Abdlhusein,M. A., (2013), Representation of some q-series by the q-exponential operator $R(bD_q)$, J. Missan researches, 18, pp. 355-362.

[4] Abdlhusein,M. A., (2014), The Euler operator for basic hypergeometric series, Int. J. Adv. Appl. Math. and Mech.,2, pp. 42 - 52.

[5] Abdlhusein,M. A., (2015), The new application of the Cauchy operator, J. Zankoy Sulaimani (JZS-A), 17, pp. 193-204.

[6] Askey,R. and Ismail,M. E. H., (1983), A generalization of ultraspherical polynomials, In: "Studies in Pure Mathematics", P. Erdös, Ed., Birkhäuser, Boston, MA, pp. 55-78.

[7] Carlitz,L. and Al-Salam,W. A., (1965), Some orthogonal q-polynomials, Math. Nachr.,30, pp. 47-61.

[8] Carlitz,L., (1972), Generating functions for certain q-orthogonal polynomials, Collectanea Math.,23, pp. 91-104.

[9] Cao,J., (2012), Generalizations of certain Carlitz's trilinear and Srivastava-Agarwal type generating functions, J. Math. Anal. Appl.,396, pp. 351-362.

[10] Cao,J., (2012), On Carlitz's trilinear generating functions, Applied Mathematics and Computation,218, pp. 9839-9847.

[11] Chen,V.Y.B. and Gu,N.S.S., (2008), The Cauchy operator for basic hypergeometric series, Adv. Appl. Math.,41, pp. 177-196.

[12] Chen,W.Y.C. and Liu,Z.G., (1997), Parameter augmenting for basic hypergeometric series, II, J. Combin. Theory, Ser. A,80, pp. 175-195.

[13] Chen,W.Y.C. and Liu,Z.G., (1998), Parameter augmentation for basic hypergeometric series, I, Mathematical Essays in Honor of Gian-Carlo Rota, Eds., B. E. Sagan and R. P. Stanley, Birkhäuser, Boston, pp. 111-129.

[14] Chen,W. Y. C., Saad,H. L. and Sun,L. H., (2007), The bivariate Rogers-Szegö polynomials, J. Phys. A: Math. Theor.,40, pp. 6071-6084.

[15] Chen,W. Y. C., Saad,H. L. and Sun,L. H., (2010), An operator approach to the Al-Salam-Carlitz polynomials, J. Math. Phys.,51, pp. 1-13.

[16] Cigler,J., (1982), Elementare q-identitäten, Publication de L'institute de recherche Mathématique avancée, pp. 23-57.

[17] Désarménien,J., (1982), Les q-analogues des polynômes d'Hermite, Sém, Lothar. Combin.,B06b, pp. 12.

[18] Gasper,G. and Rahman,M., (2004), Basic Hypergeometric Series, 2nd Ed., Cambridge University Press, Cambridge, MA.

[19] Jackson,F. H., (1910), On q-definite integrals, Quart. J. Pure Appl. Math.,50, pp. 101-112.

[20] Lu,D.-q., (2009), q-Difference equation and the Cauchy operator identities, J. Math. Anal. Appl.,359, pp. 265-274.

[21] Saad,H. L. and Abdlhusein,M. A., (2014), The q-exponential operator and generalized Rogers-Szegö polynomials, J. Advances in Mathematics,8, pp. 1440-1455.

[22] Saad,H. L. and Sukhi,A. A., (2010), Another homogeneous q-difference operator, J. Applied Mathematics and Computation,215, pp. 332-4339.

[23] Wang,M., (2009), q-Integral Representation for the Al-Salam-Carlitz Polynomials, J. Applied Mathematics Letters,22, pp. 943-945.

[24] Wang,M., (2012), An Identity from the Al-Salam-Carlitz Polynomials, J. Mathematica Aeterna,2, pp. 185-187.

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