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# A EFFICIENT COMPUTATIONAL METHOD FOR SOLVING STOCHASTIC ITÔ-VOLTERRA INTEGRAL EQUATIONS

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Abstract. In this paper, a new stochastic operational matrix for the Legendre wavelets is presented and a general procedure for forming this matrix is given. A computational method based on this stochastic operational matrix is proposed for solving stochastic Itô-Voltera integral equations. Convergence and error analysis of the Legendre wavelets basis are investigated. To reveal the accuracy and efficiency of the proposed method some numerical examples are included.

Keywords: Legendre wavelets, Itô integral, Stochastic operational matrix, Stochastic Itô-Volterra integral equations

AMS Subject Classification: 65T60, 60H20

## 1. Introduction

Random or stochastic integrals are very important for modeling many phenomena in physics, mechanics, medical, finance, sociology, biology, etc. So many studies have been appeared in the recent literature which describe these stochastic mathematical models rather than deterministic ones. In many cases such phenomena dependent on a Gaussian white noise that mathematically are modeled as stochastic differential equations, stochastic integral equations or stochastic integro-differential equations of the Itô type [1–7].

Similar to the deterministic case, most stochastic differential and integral equation cannot be solved analytically and therefore numerical solution becomes a practical way to face this difficulty. Recently, there has been a growing interest in numerical solutions of stochastic differential and integral equations [1, 3–10].

Recently, different orthogonal basis functions, such as block pulse functions, Walsh functions, Fourier series, orthogonal polynomials and wavelets, were used to estimate solutions of functional equations. As a powerful tool, wavelets have been extensively used in signal processing, numerical analysis, and many other areas. Wavelets permit the accurate representation of a variety of functions and operators [11, 12]. Legendre wavelets have been widely applied in system analysis, system identification, optimal control and numerical solution of integral and differential equations [13–16]. In this paper, an stochastic operational matrix for Legendre wavelets is derived. Then application of this stochastic operational matrix in solving stochastic Itô-Voltera integral equation is investigated. Illustrative examples are included to demonstrate the validity and applicability of the technique.

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This paper is organized as follows: In section 2 some basic definition and preliminaries about stochastic process and Itô integral are presented. The Legendre wavelets and their properties are discussed in section 3. In section 4 stochastic operational matrix for Legendre wavelets and a general procedure for deriving this matrix are introduced. In section 5 application of this stochastic operational matrix in solving stochastic Itô-Volterra integral equations are described. Finally, a conclusion is given in section 7.

## 2. Preliminaries

In this section we review some basic definition of the stochastic calculus and the block pulse functions (BPFs).

### 2.1. **Stochastic calculus.**

**Definition 2.1.** *(Brownian motion process) A real-valued stochastic process* $B(t), t \in [0, T]$ *is called Brownian motion, if it satisfies the following properties*
*(i) The process has independent increments for* $0 \leq t_0 \leq t_1 \leq ... \leq t_n \leq T$,
*(ii) For all* $t \geq 0$, $B(t+h) - B(t)$ *has Normal distribution with mean* $0$ *and variance* $h$,
*(iii) The function* $t \to B(t)$ *is continuous functions of* $t$.

**Definition 2.2.** *Let* $\{\mathcal{N}_t\}_{t\geq 0}$ *be an increasing family of* $\sigma$*-algebras of subsets of* $\Omega$. *A process* $g(t,\omega) : [0,\infty) \times \Omega \to \mathbb{R}^n$ *is called* $\mathcal{N}_t$*-adapted if for each* $t \geq 0$ *the function* $\omega \to g(t,\omega)$ *is* $\mathcal{N}_t$*-measurable.*

**Definition 2.3.** *Let* $\mathcal{V} = \mathcal{V}(S,T)$ *be the class of functions* $f(t,\omega) : [0,\infty) \times \Omega \to R$ *such that*
*(i) The function* $(t,\omega) \to f(t,\omega)$ *is* $\mathcal{B} \times \mathcal{F}$*-measurable, where* $\mathcal{B}$ *denotes the Borel algebra on* $[0,\infty)$ *and* $\mathcal{F}$ *is the* $\sigma$ *-algebra on* $\Omega$.
*(ii)* $f$ *is adapted to* $\mathcal{F}_t$, *where* $\mathcal{F}_t$ *is the* $\sigma$ *-algebra generated by the random variables* $B(s), s \leq t$.
*(iii)*$E\left(\int_S^T f^2(t,\omega)dt\right) < \infty$.

**Definition 2.4.** *(The Itô integral) Let* $f \in \mathcal{V}(S,T)$, *then the Itô integral of* $f$ *is defined by*

$$\int_S^T f(t,\omega)dB_t(\omega) = \lim_{n\to\infty} \int_S^T \varphi_n(t,\omega)dB_t(\omega), \ (\lim in\ L^2(P))$$

*where,* $\varphi_n$ *is a sequence of elementary functions such that*

$$E\left(\int_s^T (f(t,\omega) - \varphi_n(t,\omega))^2\, dt\right) \to 0, \ as\ n \to \infty.$$

*For more details about stochastic calculus and integration please see [2].*

### 2.2. **Block pulse functions.**

BPFs have been studied by many authors and applied for solving different problems. In this section we recall definition and some properties of the block pulse functions [3, 17].

The $m$-set of BPFs are defined as

$$b_i(t) = \begin{cases} 1 & (i-1)h \leq t < ih \\ 0 & otherwise \end{cases} \tag{1}$$

in which $t \in [0,T)$, $i = 1, 2, ..., m$ and $h = \frac{T}{m}$. The set of BPFs are disjointed with each other in the interval $[0, T)$ and

$$b_i(t)b_j(t) = \delta_{ij}b_i(t), i, j = 1, 2, ..., m, \tag{2}$$

where $\delta_{ij}$ is the Kronecker delta. The set of BPFs defined in the interval $[0,T)$ are orthogonal with each other, that is

$$\int_0^T b_i(t)b_j(t)dt = h\delta_{ij},\ i,j = 1,2,...,m. \tag{3}$$

If $m \to \infty$ the set of BPFs is a complete basis for $L^2[0,T)$, so an arbitrary real bounded function $f(t)$, which is square integrable in the interval $[0,T)$, can be expanded into a block pulse series as

$$f(t) \simeq \sum_{i=1}^{m} f_i b_i(t), \tag{4}$$

where

$$f_i = \frac{1}{h}\int_0^T b_i(t)f(t)dt,\ i = 1,2,...,m. \tag{5}$$

Rewritting Eq. (37) in the vector form we have

$$f(t) \simeq \sum_{i=1}^{m} f_i b_i(t) = F^T\Phi(t) = \Phi^T(t)F, \tag{6}$$

in which

$$\Phi(t) = [b_1(t), b_2(t), ...., b_m(t)]^T,\ F = [f_1, f_2, ...., f_m]^T. \tag{7}$$

Morever, any two dimensional function $k(s,t) \in L^2([0,T_1] \times [0,T_2])$ can be expanded with respect to BPFs such as

$$k(s,t) = \Phi^T(t)K\Phi(t), \tag{8}$$

where $\Phi(t)$ is the $m$-dimensional BPFs vectors respectively, and $K$ is the $m \times m$ BPFs coefficient matrix with $(i,j)$-th element

$$k_{ij} = \frac{1}{h_1 h_2}\int_0^{T_1}\int_0^{T_2} k(s,t)b_i(t)b_j(s)dtds,\ i,j = 1,2,...,m, \tag{9}$$

and $h_1 = \frac{T_1}{m}$ and $h_2 = \frac{T_2}{m}$. Let $\Phi(t)$ be the BPFs vector, then we have

$$\Phi^T(t)\Phi(t) = 1, \tag{10}$$

and

$$\Phi(t)\Phi^T(t) = \begin{pmatrix} b_1(t) & 0 & \dots & 0 \\ 0 & b_2(t) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b_m(t) \end{pmatrix}_{m\times m}. \tag{11}$$

For an $m$-vector $F$ we have

$$\Phi(t)\Phi^T(t)F = \tilde{F}\Phi(t), \tag{12}$$

where $\tilde{F}$ is an $m \times m$ matrix, and $\tilde{F} = diag(F)$. Also, it is easy to show that for an $m \times m$ matrix $A$

$$\Phi^T(t)A\Phi(t) = \hat{A}^T\Phi(t), \tag{13}$$

where $\hat{A} = diag(A)$ is an $m$-vector.

## 3. Legendre wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function $\psi$ called the mother wavelet. When the dilation parameter $a$ and the translation parameter $b$ vary continuously, we have the following family of continuous wavelets

$$\psi_{a,b}(t) = a^{-\frac{1}{2}}\psi\left(\frac{t-a}{b}\right), \; a, b \in \mathbb{R}, \, a \neq 0. \tag{14}$$

The Legendre wavelets are defined on the interval $[0, 1)$ as

$$\psi_{mn}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}2^{\frac{k+1}{2}} p_m\left(2^{k+1}t - (2n-1)\right) & \frac{n}{2^k} \leq t < \frac{n+1}{2^k} \\ 0 & otherwise, \end{cases} \tag{15}$$

where $n = 0, 1, ..., 2^k - 1$ and $m = 0, 1, \cdots, M-1$ is the degree of the Legendre polynomials for a fixed positive integer $M$. Here $P_m(t)$ are the well-known Legendre polynomials of degree $m$ [13, 15].

Any square inegrable function $f(x)$ defined over [0, 1) can be expanded in terms of the extended Legendre wavelets as

$$f(x) \simeq \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x) = C^T\Psi(x), \tag{16}$$

where $c_{mn} = (f(t), \psi_{mn}(t))$ and $(.,.)$ denotes the inner product on $L^2[0, 1]$. If the infinite series in (16) is truncated, then it can be written as

$$f(x) \simeq \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{mn}\psi_{mn}(x) = C^T\Psi(x), \tag{17}$$

where $C$ and $\Psi(x)$ are $\hat{m} = 2^kM$ column vectors given by

$$C = \left[c_{00}, \ldots, c_{0(M-1)}|c_{10}, \ldots, c_{1(M-1)}|, \ldots, |c_{(2^k-1)0}, \ldots, c_{(2^k-1)(M-1)}\right]^T, \tag{18}$$

$$\Psi(x) = \left[\psi_{00}(x), \ldots, \psi_{0(M-1)}(x)|\psi_{10}(x), \ldots, \psi_{1(M-1)}(x)|, \ldots, |\psi_{(2^k-1)0}(x), \ldots, \psi_{(2^k-1)(M-1)}(x)\right]^T.$$

By changing indices in the vectors $\Psi(x)$ and $C$ the series (17) can be rewritten as

$$f(x) \simeq \sum_{i=1}^{\hat{m}} c_i\psi_i(x) = C^T\Psi(x), \tag{19}$$

where

$$C = [c_1, c_2, ..., c_{\hat{m}}], \; \Psi(x) = [\psi_1(x), \psi_2(x), ..., \psi_{\hat{m}}(x)], \tag{20}$$

and

$$c_i = c_{nm}, \; \psi_i(x) = \psi_{nm}(x), \; i = (n-1)M + m + 1. \tag{21}$$

Similarly, any two dimensional function $k(s, t) \in L^2([0, 1] \times [0, 1])$ can be expanded into Legendre wavelets basis as

$$k(s, t) \approx \sum_{i=1}^{\hat{m}}\sum_{j=1}^{\hat{m}} k_{ij}\psi_i(s)\psi_j(t) = \Psi^T(s)K\Psi(t), \tag{22}$$

where $K = [k_{ij}]$ and $k_{ij} = (\psi_i(s), (u(s, t), \psi_j(t)))$.

3.1. **Relation between the BPFs and Legendre wavelets.** In this section we will derive the relation between the Legendre wavelets and BPFs. It is worth mention that here we set $T = 1$ in definition of BPFs.

**Theorem 3.1.** *Let $\Psi(t)$ and $\Phi(t)$ be the $\hat{m}$-dimensional Legendre wavelets and BPFs vector respectively, the vector $\Psi(t)$ can be expanded by BPFs vector $\Phi(t)$ as*

$$\Psi(t) \simeq Q\Phi(t), \tag{23}$$

*where $Q$ is an $\hat{m} \times \hat{m}$ block matrix and*

$$Q_{ij} = \psi_i \left( \frac{2j-1}{2\hat{m}} \right), i, j = 1, 2, ..., \hat{m} \tag{24}$$

*Proof.* Let $\phi_i(t), i = 1, 2, ..., \hat{m}$ be the $i$-th element of Legendre wavelets vector. Expanding $\phi_i(t)$ into an $\hat{m}$-term vector of BPFs, we have

$$\psi_i(t) \simeq \sum_{i=1}^{\hat{m}} Q_{ij} b_j(t) = Q_i^T \Phi(t), \; i = 1, 2, ..., \hat{m}, \tag{25}$$

where $Q_i$ is the $i$-th row and $Q_{ij}$ is the $(i, j)$-th element of matrix $Q$. By using the orthogonality of BPFs we have

$$Q_{ij} = \frac{1}{h} \int_0^1 \psi_i(t) b_j(t) dt = \frac{1}{h} \int_{\frac{j-1}{\hat{m}}}^{\frac{j}{\hat{m}}} \psi_i(t) dt = \hat{m} \int_{\frac{j-1}{\hat{m}}}^{\frac{j}{\hat{m}}} \psi_i(t) dt, \tag{26}$$

by using mean value theorem for integrals in the last equation we can write

$$Q_{ij} = \hat{m} \left( \frac{j}{\hat{m}} - \frac{j-1}{\hat{m}} \right) \psi_i(\eta_i) = \psi_i(\eta_j), \; \eta_j \in \left( \frac{j-1}{\hat{m}}, \frac{j}{\hat{m}} \right), \tag{27}$$

now by choosing $\eta_j = \frac{2j-1}{2\hat{m}}$ so we have

$$Q_{ij} = \psi_i \left( \frac{2j-1}{2\hat{m}} \right), i, j = 1, 2, ..., \hat{m}. \tag{28}$$

and this prove the desired result. □

The following Remark is the consequence of relations (12), (13) and Theorem 3.1.

**Remark 3.1.** *For an $\hat{m}$-vector $F$ we have*

$$\Psi(t)\Psi^T(t)F = \tilde{F}\Psi(t), \tag{29}$$

*in which $\tilde{F}$ is an $\hat{m} \times \hat{m}$ matrix as*

$$\tilde{F} = Q\bar{F}Q^{-1}, \tag{30}$$

*where $\bar{F} = diag\left(Q^T F\right)$. Moreover, it can be easy to show that for an $\hat{m} \times \hat{m}$ matrix $A$*

$$\Psi^T(t)A\Psi(t) = \hat{A}^T \Psi(t), \tag{31}$$

*where $\hat{A}^T = UQ^{-1}$ and $U = diag(Q^T AQ)$ is a $\hat{m}$-vector.*

3.2. **Convergence and error analysis.** Here we investigate the convergence and error analysis of the Legendre wavelets basis.

**Theorem 3.2.** *Let $f(x)$ be a function defined on $[0,1)$ with bounded second derivatives, say $|f''(x)| \leq M$, and $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{mn}\psi_{mn}(x)$ be its infinite Legendre wavelets expansion, then*

$$|c_{mn}| \leq \frac{\sqrt{12}M}{(2n)^{\frac{5}{2}}\,(2m-3)^2}, \tag{32}$$

*this means the Legendre wavelets series converges uniformly to $f(x)$ and*

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x), \tag{33}$$

*Proof.* See [18]. □

**Theorem 3.3.** *Suppose $f(x)$ be a continuous function defined on $[0,1)$, with second derivatives $f''(x)$ bounded by $M$, then we have the following accuracy estimation*

$$\|e_{M,k}(t)\|_2 \leq \left(\frac{3M^2}{2}\sum_{n=0}^{\infty}\sum_{m=M}^{\infty}\frac{1}{n^5(2m-3)^4} + \frac{3M^2}{2}\sum_{n=2^k}^{\infty}\sum_{m=0}^{M-1}\frac{1}{n^5(2m-3)^4}\right)^{\frac{1}{2}}, \tag{34}$$

*where*

$$\|e_{M,k}(t)\|_2 = \left(\int_0^1\left(f(x) - \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x)\right)^2 dx\right)^{\frac{1}{2}}.$$

*Proof.* We have:

$$\sigma_{M,k}^2 = \int_0^1\left(f(x) - \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x)\right)^2 dx$$

$$= \int_0^1\left(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x) - \sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x)\right)^2 dx$$

$$= \sum_{n=0}^{\infty}\sum_{m=M}^{\infty} c_{nm}^2\int_0^1 \psi_{nm}^2(x)dx + \sum_{n=2^k}^{\infty}\sum_{m=0}^{M-1} c_{nm}^2\int_0^1 \psi_{nm}^2(x)dx = \sum_{n=0}^{\infty}\sum_{m=M}^{\infty} c_{nm}^2 + \sum_{n=2^k}^{\infty}\sum_{m=0}^{M-1} c_{nm}^2,$$

now by considering Eq. (32), the desired result is achieved. □

## 4. Stochastic operational matrix of Legendre wavelets

In this section we obtain the stochastic integration operational matrix for Legendre wavelets. For this purpose we first remind some useful results for BPFs [3, 4].

**Lemma 4.1.** *[3] Let $\Phi(t)$ be the $\hat{m}$-dimensional BPFs vector defined in (7), then integration of this vector can be derived as*

$$\int_0^t \Phi(s)ds \simeq P\Phi(t), \tag{35}$$

*where P is called the operational matrix of integration for BPFs and is given by*

$$P = \frac{h}{2}\begin{bmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ 0 & 0 & 1 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 2 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}_{\hat{m}\times\hat{m}}. \tag{36}$$

**Lemma 4.2.** *[3] Let $\Phi(t)$ be the $\hat{m}$-dimensional BPFs vector defined in (7), the Itô integral of this vector can be derived as*

$$\int_0^t \Phi(s)dB(s) \simeq P_s\Phi(t), \tag{37}$$

*where $P_s$ is called the stochastic operational matrix of BPFs and is given by*

$$P_s = \begin{bmatrix} B\left(\frac{h}{2}\right) & B(h) & B(h) & \dots & B(h) \\ 0 & B\left(\frac{3h}{2}\right) - B(h) & B(2h) - B(h) & \dots & B(2h) - B(h) \\ 0 & 0 & B\left(\frac{5h}{2}\right) - B(2h) & \dots & B(3h) - B(2h) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & B\left(\frac{(2\hat{m}-1)h}{2}\right) - B((\hat{m}-1)h) \end{bmatrix}_{\hat{m}\times\hat{m}}.$$

Now we are ready to derive a new operational matrix of stochastic integration for the Legendre wavelets basis. For this end we use BPFs and the matrix $Q$ introduced in (23).

**Theorem 4.1.** *Suppose $\Psi(t)$ be the $\hat{m}$-dimensional Legendre wavelets vector defined in (20), the integral of this vector can be derived as*

$$\int_0^t \Psi(s)ds \simeq QPQ^{-1}\Psi(t) = \Lambda\Psi(t), \tag{38}$$

*where $Q$ is introduced in (23) and $P$ is the operational matrix of integration for BPFs derived in (36).*

*Proof.* Let $\Psi(t)$ be the Legendre wavelets vector, by using Theorem 3.1 and Lemma 4.1 we have

$$\int_0^t \Psi(s)ds \simeq \int_0^t Q\Phi(s)ds = Q\int_0^t \Phi(s)ds = QP\Phi(t), \tag{39}$$

now Theorem 3.1 give

$$\int_0^t \Psi(s)ds \simeq QP\Phi(t) = QPQ^{-1}\Psi(t) = \Lambda\Psi(t), \tag{40}$$

and this complete the proof. □

**Theorem 4.2.** *Suppose $\Psi(t)$ be the $\hat{m}$-dimensional Legendre wavelets vector defined in (20), the Itô integral of this vector can be derived as*

$$\int_0^t \Psi(s)dB(s) \simeq QP_sQ^{-1}\Psi(t) = \Lambda_s\Psi(t), \tag{41}$$

*where $\Lambda_s$ is called stochastic operational matrix for Legendre wavelets, $Q$ is introduced in (23) and $P_s$ is the stochastic operational matrix of integration for BPFs derived in (38).*

*Proof.* Let $\Psi(t)$ be the Legendre wavelets vector, by using Theorem 3.1 and Lemma 4.2 we have

$$\int_0^t \Psi(s)dB(s) \simeq \int_0^t Q\Phi(s)dB(s) = Q\int_0^t \Phi(s)dB(s) = QP_s\Phi(t), \tag{42}$$

now Theorem 3.1 result

$$\int_0^t \Psi(s)dB(s) = QP_s\Phi(t) = QP_sQ^{-1}\Psi(t) = \Lambda_s\Psi(t), \tag{43}$$

and this complete the proof. □

## 5. Application in solving stochastic integral equations

In this section, we solve stochastic Itô-Volterra integral equations by using the stochastic operational matrix of the Legendre wavelets. Consider the following stochastic Itô-Volterra integral equation as

$$X(t) = f(t) + \int_0^t k_1(s,t)X(s)ds + \int_0^t k_1(s,t)X(s)dB(s),\ t \in [0,T), \tag{44}$$

where $X(t)$, $f(t)$, $k_1(s,t)$ and $k_2(s,t)$, for $s,t \in [0,T)$, are the stochastic processes defined on the same probability space $(\Omega, F, P)$, and $X(t)$ is unknown. Also $B(t)$ is a Brownian motion process and $\int_0^t k_1(s,t)X(s)dB(s)$ is the Itô integral. For solving this problem by using the stochastic operational matrix of Legendre wavelets, we approximate $X(t), f(t), k_1(s,t)$ and $k_2(s,t)$ in terms of $\hat{m}$-dimentional Legendre wavelets as follows

$$f(t) = F^T\Psi(t) = \Psi^T(t)F, \tag{45}$$

$$X(t) = X^T\Psi(t) = \Psi^T(t)X, \tag{46}$$

$$k_1(s,t) = \Psi(s)^TK_1\Psi(t) = \Psi(t)^TK_1^T\Psi(s), \tag{47}$$

$$k_2(s,t) = \Psi(s)^TK_2\Psi(t) = \Psi(t)^TK_2^T\Psi(s), \tag{48}$$

where $X$ and $F$ are Legendre wavelets coefficients vector, and $K_1$ and $K_2$ are Legendre wavelets coefficient matrices defined in Eq. (20). Substituting above approximations in Eq. (46), we have

$$X^T\Psi(t) = F^T\Psi(t) + \Psi(t)^TK_1\left(\int_0^t \Psi(s)\Psi(s)^TXds\right) + \Psi(t)^TK_2\left(\int_0^t \Psi(s)\Psi(s)^TXdB(s)\right),$$

now by using Remark 3.1 we get

$$X^T\Psi(t) = F^T\Psi(t) + \Psi^T(t)K_1\left(\int_0^t \tilde{X}\Psi(s)ds\right) + \Psi^T(t)K_2\left(\int_0^t \tilde{X}\Psi(s)dB(s)\right),$$

where $\tilde{X}$ is a linear function of vector $X$. Applying the operational matrices $\Lambda$ and $\Lambda_s$ for Legendre wavelets derived in Eqs. (38) and (41) we get

$$X^T\Psi(t) = F^T\Psi(t) + \Psi^T(t)K_1\tilde{X}\Lambda\Psi(t) + \Psi(t)^TK_2\tilde{X}\Lambda_s\Psi(t), \tag{49}$$

by setting $Y_1 = K_1\tilde{X}\Lambda, Y_2 = K_2\tilde{X}\Lambda_s$ and using Remark 3.1 we derive

$$X^T\Psi(t) - \hat{Y}_1\Psi(t) - \hat{Y}_2\Psi(t) = F^T\Psi(t), \tag{50}$$

in which where $\hat{Y}_1$ and $\hat{Y}_2$ are linear function of vectors $Y_1$ and $Y_2$. This equation is hold for all $t \in [0,1)$, so we can write

$$X^T - \hat{Y}_1 - \hat{Y}_2 = F^T. \tag{51}$$

Since $\hat{Y}_1$ and $\hat{Y}_2$ are linear functions of $X$, Eq. (51) is a linear system of equations for unknown vector $X$. After solving this linear system and determining $X$, we can approximate solution of the stochastic Itô-Volterra integral equation (44) by substituting obtained vector $X$ in Eq. (46).

## 6. Numerical examples

In this section, we demonstrate the efficiency of the proposed method in the section 5 with some illustrative examples. It will be shown that the Legendre wavelets operational matrix method is very efficient for solving stochastic Itô-Volterra integral equation. The algorithms are performed by Maple 13 with 20 digits precision.

**Example 6.1.** *Consider the following stochastic Itô-Volterra integral equation [3, 7]*

$$X(t) = 1 + \int_0^t s^2 X(s)ds + \int_0^t sX(s)dB(s), \ \ s,t \in [0,1], \tag{52}$$

*where $X(t)$ is an unknown stochastic process defined on the probability space $(\Omega, \mathcal{F}, P)$, and $B(t)$ is a Brownian motion process. The exact solution of this stochastic Itô-Volterra integral equation is*

$$X(t) = \exp\left(\frac{t^3}{6} + \int_0^t sdB(s)\right). \tag{53}$$

*The stochastic operational matrix of Legendre wavelets and the presented method in section 5 are employed for deriving a numerical solution of this Itô-Volterra integral equation. The approximate solution computed by the presented method and exact solution are represented in Fig. 6.1 for $\hat{m} = 128$. The absolute error of the numerical results are shown in the Table 6.1 for different values of $\hat{m}$.*

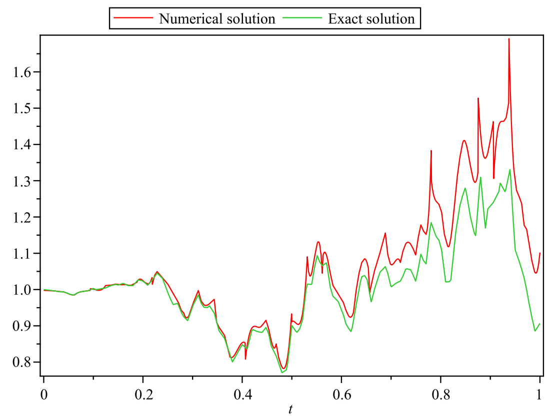


Figure 1. The approximate solution and exact solution for $\hat{m} = 128$.

**Example 6.2.** *Let us consider the following stochastic Itô-Volterra integral equation [3, 7]*

$$X(t) = \frac{1}{12} + \int_0^t \cos(s)X(s)ds + \int_0^t \sin(s)X(s)dB(s), \ \ s,t \in [0,1], \tag{54}$$

*where $X(t)$ is an unknown stochastic process defined on the probability space $(\Omega, \mathcal{F}, P)$, and $B(t)$ is a Brownian motion process. The exact solution of this stochastic Volterra*

TABLE 1. The absolute error of the numerical results for different values of $\hat{m}$.

| $t$ | $\hat{m} = 32$ | $\hat{m} = 64$ | $\hat{m} = 128$ |
|---|---|---|---|
| 0.1 | 0.00319130 | 0.00057210 | 0.00222540 |
| 0.3 | 0.00371524 | 0.00917624 | 0.00402876 |
| 0.5 | 0.44932060 | 0.86719460 | 0.95334270 |
| 0.7 | 0.05396580 | 0.06228580 | 0.06238580 |
| 0.9 | 0.12015880 | 0.12185880 | 0.13135880 |

*integral equation is*

$$X(t) = \frac{1}{12} \exp\left( \frac{-t}{4} + \sin(t) + \frac{\sin(2t)}{8} + \int_0^t \sin(s) dB(s) \right). \tag{55}$$

*This stochastic Itô-Volterra integral equation is solved by using the stochastic operational matrix of Legendre wavelets and the proposed method in section 5. In Fig. 6.2 the approximate solution computed by the presented method and exact solution are shown for* $\hat{m} = 128$. *The absolute error of the numerical results for different values of* $\hat{m}$ *are shown in the Table 6.2.*

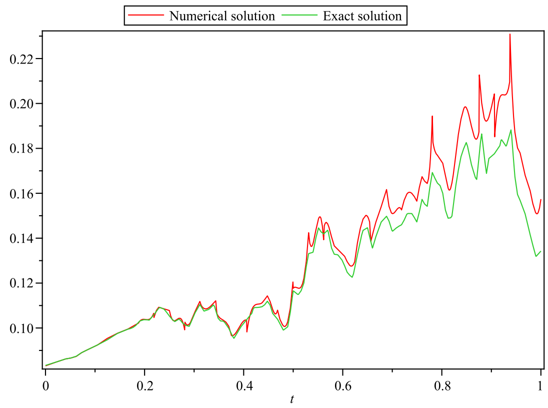


FIGURE 2. The approximate solution and exact solution for $\hat{m} = 128$.

TABLE 2. The absolute error of the numerical results for different values of $\hat{m}$.

| $t$ | $\hat{m} = 32$ | $\hat{m} = 64$ | $\hat{m} = 128$ |
|---|---|---|---|
| 0.1 | 0.00027709 | 0.00004562 | 0.00020525 |
| 0.3 | 0.00030467 | 0.00097477 | 0.00045423 |
| 0.5 | 0.06034923 | 0.11215321 | 0.12302135 |
| 0.7 | 0.00676411 | 0.00789211 | 0.00795211 |
| 0.9 | 0.01402822 | 0.01434822 | 0.01538822 |

**Example 6.3.** *Consider the following stochastic Itô-Volterra integral equation [6]*

$$X(t) = \frac{1}{3} + \int_0^t \ln(s+1) X(s) ds + \int_0^t \sqrt{\ln(s+1)} X(s) dB(s), \;\; s, t \in [0, 1], \tag{56}$$

*where* $X(t)$ *is an unknown stochastic process defined on the probability space* $(\Omega, \mathcal{F}, P)$, *and* $B(t)$ *is a Brownian motion process. The exact solution of this stochastic Volterra* *integral equation is*

$$X(t) = \frac{1}{3}\exp\left(\frac{-t}{2} + \frac{1}{2}t\ln(t+1) + \frac{1}{2}\ln(t+1) + \int_0^t \sqrt{\ln(s+1)}dB(s)\right). \tag{57}$$

*The Legendre wavelets stochastic operational matrix and the proposed method in section 5 are used for solving this stochastic Itô-Volterra integral equation. The exact solution and approximate solution computed by the presented method for $\hat{m} = 128$ are shown in Fig. 6.3. The absolute error of the numerical results are shown in the Table 6.3 for different values of $\hat{m}$.*

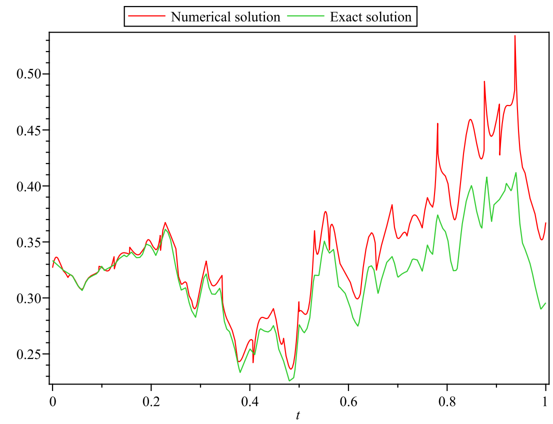


FIGURE 3. The approximate solution and exact solution for $\hat{m} = 128$.

TABLE 3. The absolute error of the numerical results for different values of $\hat{m}$.

| $t$ | $\hat{m} = 32$ | $\hat{m} = 64$ | $\hat{m} = 128$ |
|---|---|---|---|
| 0.1 | 0.00102914 | 0.00106840 | 0.00142866 |
| 0.3 | 0.00344068 | 0.00001788 | 0.00625188 |
| 0.5 | 0.11389535 | 0.28145811 | 0.31242841 |
| 0.7 | 0.03226066 | 0.03587166 | 0.03487166 |
| 0.9 | 0.05533170 | 0.05679170 | 0.05930170 |

## 7. Conclusion

A new stochastic operational matrix for the Legendre wavelets is derived. The BPFs and their relation with Legendre wavelets are used to derive this stochastic operational matrix. An efficient computational method based on this stochastic operational matrix is introduced for solving stochastic Itô-Voltera integral equations. Convergence and error analysis of the Legendre wavelets basis are considerd. Efficiency of the proposed method is confirmed by some numerical examples.

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