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COMMON FIXED POINT THEOREMS FOR FINITE NUMBER OF MAPPINGS IN SYMMETRIC SPACES

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Abstract. In this paper, we prove a common fixed theorem for ten mappings on symmetric spaces. We extend our result for finite number of mappings. The mappings involved in our results are noncompatible and discontinuous. We extend and generalize several earlier results.

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1. Introduction.

Jungck [3] introduced more generalized commuting mappings called compatible mappings. This notion of compatible mappings have been frequently used to show existence of common fixed points. However, the study of the existence of common fixed points for noncompatible mappings is also interesting. Pant ([6]-[9]) initiated the study of noncompatible maps and proved some fixed point theorems for noncompatible mappings. More recently, Aamri and Moutawakil [4] defined a property (E.A) which generalizes the concept of noncompatible mappings in metric spaces and contains the class of noncompatible maps. They obtained some fixed point theorems for such mappings under strict contractive conditions. Pant and Pant [10] proved some common fixed point theorems for strict contractive noncompatible mappings in metric spaces. Recently, the results of Aamri and Moutawakil [4] and Pant and Pant [10] were extended to symmetric spaces under tight conditions by Imdad et al [5].

Wilson [12] gave two axioms (W_3) and (W_4) on a symmetric space. The axiom (W_3) was used by Imdad et al [5] to prove coincidence and common fixed point theorems on symmetric spaces. Aliouche [1] gave the axiom $(H.E)$ on symmetric spaces and used $(W_3), (W_4)$ and $(H.E)$ to prove a common fixed point theorem for noncompatible self-mappings in symmetric spaces under contractive conditions of integral type.

Cho et al [11], introduced a new axiom called $(C.C)$ which is related to the continuity of the symmetric d. They also compared the axiom (W_3) with (W_4) and $(C.C)$ with (W_3). They also gave examples to show that $(W_3) \nRightarrow (H.E)$, $(W_3) \nRightarrow (C.C), (C.C) \nRightarrow (W_4), (W_3) \nRightarrow (W_4)$. They proved some common fixed point theorems on symmetric spaces using the axioms $(W_3), (H.E)$ and $(C.C)$.

In this paper, we prove a common fixed theorem for ten mappings on symmetric spaces. We extend our result for finite number of mappings. The mappings involved in our results

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are noncompatible and discontinuous. We extend and generalize the results of Cho et al [11]. We also give an example to validate our result.

2. Preliminaries.

Definition 2.1. *A symmetric on a set X is a function $d : X \to [0, \infty)$ satisfying the following conditions:*
$d(x, y) = 0$ *if and only if* $x = y$ *for* $x, y \in X$,
$d(x, y) = d(y, x)$ *for all* $x, y \in X$.

Let d be a symmetric on X. For $x \in X$ and $\varepsilon > 0$, let $B(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$.

A topology $\tau(d)$ on X is is given by $U \in \tau(d)$ if and only if for each $x \in X$, $B(x, \varepsilon) \subset U$ for some $\varepsilon > 0$. A subset S of X is a neighborhood of $x \in X$ if there exists $U \in \tau(d)$ such that $x \in U \subset S$. A symmetric d is a semi-metric if for each $x \in X$ and for each $\varepsilon > 0$, $B(x, \varepsilon)$ is a neighborhood of x in topology $\tau(d)$.

Definition 2.2. *A symmetric (semi-metric) space X is a topological space whose topology $\tau(d)$ on X is induced by symmetric d (semi-metric d).*

Remark 2.1. *The difference of a symmetric and a metric comes from the triangle inequality. A symmetric space need not be Hausdorff.*

Definition 2.3. [3] *A pair of self-mappings (f, g) on a symmetric (semi-metric) space (X, d) is said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that*

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \in X.$$

Definition 2.4. [2] *A pair of self-mappings (f, g) on a symmetric (semi-metric) space (X, d) is said to be weakly compatible if $fx = gx$ implies $fgx = gfx$.*

Definition 2.5. [4] *A pair of self-mappings (f, g) on a symmetric (semi-metric) space (X, d) is said to enjoy property (E.A) if there exists a sequence $\{x_n\}$ such that*

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \in X.$$

In order to obtain fixed point theorems on a symmetric space, we need some axioms.

The following axioms can be found in [12]

(W_3): for a sequence $\{x_n\}$ in X and $x, y \in X$, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$ imply that $x = y$.

(W_4): for sequences $\{x_n\}, \{y_n\}$ in X and $x \in X$, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, x_n) = 0$ imply that $\lim_{n\to\infty} d(y_n, x) = 0$.

The following axiom can be found in [1].

$(H.E)$: for sequences $\{x_n\}, \{y_n\}$ in X and $x \in X$, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, x) = 0$ imply that $\lim_{n\to\infty} d(x_n, y_n) = 0$.

The following axiom can be found in [11].

$(C.C)$: for sequence $\{x_n\}$ in X and $x, y \in X$, $\lim_{n\to\infty} d(x_n, x) = 0$ implies that $\lim_{n\to\infty} d(x_n, y) = d(x, y)$.

3. Main Results.

Theorem 3.1. *Let (X, d) be a symmetric (semi-metric) space that satisfies $(H.E)$ and $(C.C)$. Let $A, B, S, T, I, J, L, U, P$ and Q be self-mappings of X such that*

(1) $P(X) \subset ABIL(X), Q(X) \subset STJU(X)$,
(2) *the pair* $(Q, ABIL)$ *(resp. the pair* $(P, STJU)$*) satisfies the property (E.A),*
(3) *for any* $x, y \in X$,

$$d(Px, Qy) < \max \left\{ \begin{array}{c} d(STJUx, ABILy), \\ \frac{k}{2}[d(Px, STJUx) + d(Qy, ABILy)], \\ \frac{1}{2}[d(Px, ABILy) + d(Qy, STJUx)] \end{array} \right\}$$

where $0 < k < 2$,
(4) *if one of* $STJU(X)$ *and* $ABIL(X)$ *is a* $d-$*closed* $(\tau(d) -$ *closed) subset of* X, *then*
(i) P *and* $STJU$ *have a coincidence point and*
(ii) Q *and* $ABIL$ *have a coincidence point.*
Further if,
(5) $LQ = QL, LI = IL, BL = LB, AL = LA, IB = BI$,
$AB = BA, AI = IA, IQ = QI, JU = UJ, SU = US, PJ = JP, PU = UP$,
$QB = BQ, TU = UT, PT = TP, JT = TJ, SJ = JS, ST = TS$,
(6) *the pairs* $(P, STJU)$ *and* $(Q, ABIL)$ *are weakly compatible, then*
(iii) $A, B, S, T, I, J, L, U, P$ *and* Q *have a unique common fixed point in* X.

Proof. Since the pair $(Q, ABIL)$ satisfies the property (E.A), therefore there exists a sequence $\{x_n\}$ in X and a point z in X such that
$\lim_{n\to\infty} d(Qx_n, z) = \lim_{n\to\infty} d(ABILx_n, z) = 0$. Since $Q(X) \subset STJU(X)$, for $x_n \in X$, there exists $y_n \in X$ such that $Qx_n = STJUy_n$. Therefore
$\lim_{n\to\infty} d(STUJy_n, z) = 0$. By $(H.E)$, $\lim_{n\to\infty} d(Qx_n, ABILx_n) = \lim_{n\to\infty} d(STUJy_n, ABILx_n) = 0$. Let $STUJ(X)$ be a $d-$closed $(\tau(d) -$ closed) subset of X. Then there exists a point $u \in X$ such that $STUJu = z$. From (3),

$$d(Pu, Qx_n) < \max \left\{ \begin{array}{c} d(STJUu, ABILx_n), \\ \frac{k}{2}[d(Pu, STJUu) + d(Qx_n, ABILu)], \\ \frac{1}{2}[d(Pu, ABILx_n) + d(Qx_n, STJUu)] \end{array} \right\}.$$

By taking $n \to \infty$, we have $\lim_{n\to\infty} d(Pu, Qx_n) = 0$. By $(C.C)$, we get
$Pu = STJUu = z$. Hence u is the coincidence point of P and $STUJ$. This proves
(i). Since $P(X) \subset ABIL(X)$, there exists a point $w \in X$ such that $Pu = ABILw$. We claim that $ABILw = Qw$. From (3),

$$\begin{aligned} d(ABILw, Qw) &= d(Pu, Qw) \\ &< \max \left\{ \begin{array}{c} d(STJUu, ABILw), \\ \frac{k}{2}[d(Pu, STJUu) + d(Qw, ABILw)], \\ \frac{1}{2}[d(Pu, ABILw) + d(Qw, STJUu)] \end{array} \right\} \\ &= \max \left\{ 0, \frac{k}{2} d(Qw, Pu), \frac{1}{2} d(Qw, Pu) \right\}. \end{aligned}$$

Hence $ABILw = Qw = Pu = z$. This shows that w is the coincidence point of Q and $ABIL$. This proves (ii).

Since the pair $(P, STJU)$ is weakly compatible, therefore P and $STUJ$ commute at their coincidence point i.e. $P(STJUu) = STJU(Pu)$ or $Pz = STJUz$.

Since the pair $(Q, ABIL)$ is weakly compatible, therefore Q and $ABIL$ commute at their coincidence point i.e. $Q(ABILw) = ABIL(Qw)$ or $Qz = ABILz$.

Now we claim that $Pu = w$. If $Pu \neq w$, then from (3), we have

$$\begin{aligned} d(Pu, PPu) &= d(PPu, Qw) \\ &< \max \left\{ \begin{array}{c} d(STJU(Pu), ABILw), \\ \frac{k}{2}[d(P(Pu), STJU(Pu)) + d(Qw, ABILw)], \\ \frac{1}{2}[d(P(Pu), ABILw) + d(Qw, STJU(Pu))] \end{array} \right\} \\ &= \max \left\{ d(PPu, Qw), 0, \frac{1}{2}[d(PPu, Qw) + d(Qw, PPu)] \right\} \\ &= d(PPu, Qw), \end{aligned}$$

which is a contradiction. Hence $Pu = w$. Similarly if $Qw \neq u$, we get a contradiction. Hence

$$Pu = w = STJUu = Qw = ABILw = u.$$

Combining the above results we have z as the common fixed point of $P, Q, ABIL$ and $STJU$ i.e.

$$Pz = STJUz = Qz = ABILz = z.$$

Now putting $x = z$ and $y = Lz$ in (3), we get

$$\begin{aligned} d(Pz, QLz) &< \max \left\{ \begin{array}{c} d(STJUz, ABILLz), \\ \frac{k}{2}[d(Pz, STJUz) + d(QLz, ABILLz)], \\ \frac{1}{2}[d(Pz, ABILLz) + d(QLz, STJUz)] \end{array} \right\} \\ &= \max \left\{ d(z, L(ABILz)), \frac{k}{2}d(LQz, L(ABILz)), \frac{1}{2}d(LQz, z) \right\} \\ &= \max \left\{ d(z, Lz), \frac{k}{2}d(Lz, Lz), \frac{1}{2}d(Lz, z) \right\} \\ &= d(z, Lz) \end{aligned}$$

i.e.

$$d(z, LQz) = d(z, Lz) < d(z, Lz),$$

which is a contradiction. Hence $Lz = z$. Since $ABILz = z$, therefore $ABIz = z$.

Now putting $x = z$ and $y = Iz$ in (3), we get

$$\begin{aligned} d(Pz, QIz) &< \max \left\{ \begin{array}{c} d(STJUz, ABILIz), \\ \frac{k}{2}[d(Pz, STJUz) + d(QIz, ABILIz)], \\ \frac{1}{2}[d(Pz, ABILIz) + d(QIz, STJUz)] \end{array} \right\} \\ &= \max \left\{ \begin{array}{c} d(z, I(ABIz)), \frac{k}{2}d(IQz, I(ABIz)), \\ \frac{1}{2}[d(z, I(ABIz)) + d(IQz, z)] \end{array} \right\} \\ &= \max \left\{ d(z, Iz), \frac{k}{2}d(Iz, Iz), \frac{1}{2}[d(z, Iz) + d(Iz, z)] \right\} \\ &= d(Iz, z), \end{aligned}$$

i.e.

$$d(z, QIz) = d(z, IQz) = d(Iz, z) < d(Iz, z),$$

which is a contradiction. Hence $Iz = z$. Therefore $ABz = z$.

Now putting $x = z$ and $y = Bz$ in (3), we get

$$\begin{aligned}
d(Pz, QBz) &< \max\left\{\begin{array}{c} d(STJUz, ABILBz), \\ \frac{k}{2}[d(Pz, STJUz) + d(QBz, ABILBz)], \\ \frac{1}{2}[d(Pz, ABILBz) + d(QBz, STJUz)] \end{array}\right\} \\
&= \max\left\{\begin{array}{c} d(z, B(ABz)), \frac{k}{2}d(BQz, B(ABz)), \\ \frac{1}{2}[d(z, B(ABz)) + d(BQz, z)] \end{array}\right\} \\
&= \max\left\{d(z, Bz), \frac{k}{2}d(Bz, Bz), \frac{1}{2}[d(z, Bz) + d(Bz, z)]\right\} \\
&= d(Bz, z),
\end{aligned}$$

i.e.

$$d(z, BQz) = d(z, Bz) < d(Bz, z),$$

which is a contradiction. Hence $Bz = z$. Consequently $Az = z$.

Now putting $x = Uz$ and $y = z$ in (3), we get

$$\begin{aligned}
d(PUz, Qz) &< \max\left\{\begin{array}{c} d(STJUUz, ABILz), \\ \frac{k}{2}[d(PUz, STJUUz) + d(Qz, ABILz)], \\ \frac{1}{2}[d(PUz, ABILz) + d(Qz, STJUUz)] \end{array}\right\} \\
&= \max\left\{\begin{array}{c} d(U(STJUz), z), \frac{k}{2}d(UPz, U(STJUz)), \\ \frac{1}{2}[d(UPz, z) + d(z, U(STJUz))] \end{array}\right\} \\
&= \max\{d(Uz, z), 0, d(z, Uz)\} \\
&= d(Uz, z),
\end{aligned}$$

i.e.

$$d(PUz, z) = d(UPz, z) = d(Uz, z) < d(Uz, z),$$

which is a contradiction. Hence $Uz = z$. Therefore $STJz = z$.

Now putting $x = Jz$ and $y = z$ in (3), we get

$$\begin{aligned}
d(PJz, Qz) &< \max\left\{\begin{array}{c} d(STJUJz, ABILz), \\ \frac{k}{2}[d(PJz, STJUJz) + d(Qz, ABILz)], \\ \frac{1}{2}[d(PJz, ABILz) + d(Qz, STJUJz)] \end{array}\right\} \\
&= \max\left\{\begin{array}{c} d(J(STJz), z), \frac{k}{2}d(JPz, J(STJz)), \\ \frac{1}{2}[d(JPz, z) + d(z, J(STJz))] \end{array}\right\} \\
&= \max\left\{d(Jz, z), 0, \frac{1}{2}[d(Jz, z) + d(z, Jz)]\right\} \\
&= d(z, Jz),
\end{aligned}$$

i.e.

$$d(JPz, z) = d(Jz, z) < d(z, Jz),$$

which is a contradiction. Hence $Jz = z$. Therefore $STz = z$.

Now putting $x = Tz$ and $y = z$ in (3), we get

$$\begin{aligned} d(PTz, Qz) &< \max\left\{ \begin{array}{c} d(STJUTz, ABILz), \\ \frac{k}{2}\left[d(PTz, STJUTz) + d(Qz, ABILz)\right], \\ \frac{1}{2}\left[d(PTz, ABILz) + d(Qz, STJUTz)\right] \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} d(T(STz), z), \frac{k}{2} d(TPz, T(STz)), \\ \frac{1}{2}\left[d(TPz, z) + d(z, T(STz))\right] \end{array} \right\} \\ &= \max\left\{ d(Tz, z), 0, \frac{1}{2}\left[d(Tz, z) + d(z, Tz)\right] \right\} \\ &= d(Tz, z), \end{aligned}$$

i.e. $d(PTz, z) = d(Tz, z) = d(Tz, z) < d(Tz, z)$, which is a contradiction. Hence $Tz = z$. Therefore $Sz = z$.

By combining the above results, we have

$$Az = Bz = Sz = Tz = Iz = Jz = Lz = Uz = Pz = Qz = z.$$

i.e. z is the common fixed point of $A, B, S, T, I, J, L, U, P$ and Q.

Let $v \in X$ be another common fixed point of $A, B, S, T, I, J, L, U, P$ and Q i.e.

$$Av = Bv = Sv = Tv = Iv = Jv = Lv = Uv = Pv = Qv = v.$$

Then by (3),

$$\begin{aligned} d(z, v) &= d(Pz, Qv) \\ &< \max\left\{ \begin{array}{c} d(STJUz, ABILv), \\ \frac{k}{2}\left[d(Pz, STJUz) + d(Qv, ABILv)\right], \\ \frac{1}{2}\left[d(Pz, ABILv) + d(Qv, STJUz)\right] \end{array} \right\} \\ &= \max\left\{ d(z, v), \frac{k}{2}\left[d(z, z) + d(v, v)\right], \frac{1}{2}\left[d(z, v) + d(v, z)\right] \right\} \\ &= d(z, v), \end{aligned}$$

which is a contradiction Hence $z = v$. This completes the proof. □

If we put $P = Q$ in the Theorem 3.1, we get the following:

Corollary 3.1. *Let (X, d) be a symmetric (semi-metric) space that satisfies $(H.E)$ and $(C.C)$. Let A, B, S, T, I, J, L, U, and P be self-mappings of X such that*

(1) $P(X) \subset ABIL(X), P(X) \subset STJU(X)$,

(2) *the pair $(P, ABIL)$ (resp. the pair $(P, STJU)$) satisfies the property (E.A),*

(3) *for any $x, y \in X$,*

$$d(Px, Py) < \max\left\{ \begin{array}{c} d(STJUx, ABILy), \\ \frac{k}{2}\left[d(Px, STJUx) + d(Py, ABILy)\right], \\ \frac{1}{2}\left[d(Px, ABILy) + d(Py, STJUx)\right] \end{array} \right\},$$

where $0 < k < 2$,
(4) *if one of* $STJU(X)$ *and* $ABIL(X)$ *is a* $d-$*closed* $(\tau(d)-$ *closed*) *subset of* X, *then*
(i) P *and* $STJU$ *have a coincidence point and*
(ii) P *and* $ABIL$ *have a coincidence point.*
Further if,
(5) $LP = PL, LI = IL, BL = LB, AL = LA, IB = BI, AB = BA, AI = IA, IP = PI, JU = UJ, SU = US, PJ = JP, PU = UP, PB = BP, TU = UT, PT = TP, JT = TJ, SJ = JS, ST = TS$,
(6) *the pairs* $(P, STJU)$ *and* $(P, ABIL)$ *are weakly compatible, then*
(iii) $A, B, S, T, I, J, L, U,$*and* P *have a unique common fixed point in* X.

If we put $L = U = I_X$ (The identity map on X) in the Theorem 3.1, then we have the following:

Corollary 3.2. *Let* (X, d) *be a symmetric (semi-metric) space that satisfies* $(H.E)$ *and* $(C.C)$. *Let* A, B, S, T, I, J, P *and* Q *be self-mappings of* X *such that*
(1) $P(X) \subset ABI(X), Q(X) \subset STJ(X)$,
(2) *the pair* (Q, ABI) *(resp. the pair* (P, STJ)*) satisfies the property (E.A),*
(3) *for any* $x, y \in X$,

$$d(Px, Qy) < \max \left\{ \begin{array}{c} d(STJx, ABIy), \\ \frac{k}{2}[d(Px, STJx) + d(Qy, ABIy)], \\ \frac{1}{2}[d(Px, ABIy) + d(Qy, STJx)] \end{array} \right\},$$

where $0 < k < 2$,
(4) *if one of* $STJ(X)$ *and* $ABI(X)$ *is a* $d-$*closed* $(\tau(d)-$ *closed*) *subset of* X, *then*
(i) P *and* STJ *have a coincidence point and*
(ii) Q *and* ABI *have a coincidence point.*
Further if,
(5) $IB = BI, AB = BA, AI = IA, IQ = QI, PJ = JP, QB = BQ, PT = TP, JT = TJ, SJ = JS, ST = TS$,
(6) *the pairs* (P, STJ) *and* (Q, ABI) *are weakly compatible, then*
(iii) A, B, S, T, I, J, P *and* Q *have a unique common fixed point in* X.

If we put $L = U = I_X$ (The identity map on X) and $P = Q$ in the Corollary 3.2, we have the following:

Corollary 3.3. *Let* (X, d) *be a symmetric (semi-metric) space that satisfies* $(H.E)$ *and* $(C.C)$. *Let* A, B, S, T, I, J *and* P *be self-mappings of* X *such that*
(1) $P(X) \subset ABI(X), P(X) \subset STJ(X)$,
(2) *the pair* (P, ABI) *(resp. the pair* (P, STJ)*) satisfies the property* $(E-A)$,
(3) *for any* $x, y \in X$,

$$d(Px, Py) < \max \left\{ \begin{array}{c} d(STJx, ABIy), \\ \frac{k}{2}[d(Px, STJx) + d(Py, ABIy)], \\ \frac{1}{2}[d(Px, ABIy) + d(Py, STJx)] \end{array} \right\},$$

where $0 < k < 2$,
(4) *if one of* $STJ(X)$ *and* $ABI(X)$ *is a* $d-$*closed* $(\tau(d)-$ *closed*) *subset of* X, *then*
(i) P *and* STJ *have a coincidence point and*
(ii) P *and* ABI *have a coincidence point.*

Further if,
(5) $IB = BI, AB = BA, AI = IA, IP = PI, PJ = JP, PB = BP, PT = TP, JT = TJ, SJ = JS, ST = TS$,
(6) *the pairs* (P, STJ) *and* (P, ABI) *are weakly compatible, then*
(iii) A, B, S, T, I, J *and* P *have a unique common fixed point in* X.

Example 3.1. *Let* $X = [0, 4]$ *and* $d(x, y) = (x - y)^2$. *Define self-mappings* A, B, S, T, I, J *and* P *of* X *by*

$$Px = \begin{cases} \frac{x}{50}, 0 \le x \le 1 \\ 0, 1 < x \le 4 \end{cases}, Ax = \begin{cases} 3x, 0 \le x \le 1 \\ 0, 1 < x \le 4 \end{cases},$$
$$Bx = \begin{cases} \frac{x}{2}, 0 \le x \le 1 \\ 0, 1 < x \le 4 \end{cases}, Ix = \begin{cases} \frac{x}{3}, 0 \le x \le 1 \\ 0, 1 < x \le 4 \end{cases},$$
$$Sx = \begin{cases} \frac{x}{2}, 0 \le x \le 1 \\ 0, 1 < x \le 4 \end{cases} \frac{x}{2}, Tx = \begin{cases} 4x, 0 \le x \le 1 \\ 0, 1 < x \le 4 \end{cases},$$
$$Jx = \begin{cases} \frac{x}{4}, 0 \le x \le 1 \\ 0, 1 < x \le 4 \end{cases}.$$

Then we have the following:
(i) (X, d) *is a symmetric space satisfying properties* $(H.E)$ *and* $(C.C)$,
(ii) $P(X) = \left[0, \frac{1}{50}\right] \subset ABI(X) = \left[0, \frac{1}{2}\right]$
and
$P(X) = \left[0, \frac{1}{50}\right] \subset STJ(X) = \left[0, \frac{1}{2}\right]$,
(iii) *The pair* (P, ABI) *satisfies the property (E.A) for the sequence* $x_n = \frac{1}{n}, n = 1, 2, 3....$
(iv) *for all* $x \neq y \in X$,

$$d(Px, Py) < \max \left\{ \begin{array}{c} d(STJx, ABIy), \frac{k}{2}[d(Px, STJx) + d(Py, ABIy)], \\ \frac{1}{2}[d(Px, ABIy) + d(Py, STJx)] \end{array} \right\}$$

(v) *the pairs* (P, STJ) *and* (P, ABI) *are* $d-$*closed* $(\tau(d) -$ *closed) subsets of* X,
(vi) *the coincidence point is* $x = 0$
(vii) $IJ = JI, IT = TI, IS = SI, IP = PI, BJ = JB, BT = TB, BS = SB, BP = PB$,
$AJ = JA, JP = PJ, AT = TA, TP = PT$,
$(viii)$ *the pairs* (P, STJ) *and* (P, ABI) *are weakly compatible.*
(ix) *Therefore all the conditions of the Corollary 4 are satisfied. The common fixed point is* $x = 0$.

If we put $B = I = T = J = I_X$ and (The identity map on X) in Corollary 3.2, we have the following:

Corollary 3.4. *Let* (X, d) *be a symmetric (semi-metric) space that satisfies* $(H.E)$ *and* $(C.C)$. *Let* A, S, P *and* Q *be self-mappings of* X *such that*
(1) $P(X) \subset A(X), Q(X) \subset S(X)$,
(2) *the pair* (Q, A) *(resp. the pair* (P, S)*) satisfies the property* $(E - A)$,

(3) *for any* $x, y \in X$,

$$d(Px, Qy) < \max\left\{\begin{array}{c} d(Sx, Ay), \frac{k}{2}[d(Px, Sx) + d(Qy, Ay)], \\ \frac{1}{2}[d(Px, Ay) + d(Qy, Sx)] \end{array}\right\},$$

where $0 < k < 2$,
(4) *if one of* $S(X)$ *and* $A(X)$ *is a* $d-$*closed* ($\tau(d)-$ *closed*) *subset of* X, *then*
(i) P *and* S *have a coincidence point and*
(ii) Q *and* A *have a coincidence point.*
Further if,
(5) *the pairs* (P, S) *and* (Q, A) *are weakly compatible, then*
(iii) A, S, P *and* Q *have a unique common fixed point in* X.

If we put $I = J = B = T = A = S = I_X$ (The identity map on X) in Corollary 3.2, we have the following:

Corollary 3.5. *Let* (X, d) *be a symmetric (semi-metric) space that satisfies* $(H.E)$ *and* $(C.C)$. *Let* P *be a self-mapping of* X *such that*
(1) *for any* $x, y \in X$,

$$d(Px, Py) < \max\left\{\begin{array}{c} d(x, y), \frac{k}{2}[d(Px, x) + d(Py, y)], \\ \frac{1}{2}[d(Px, y) + d(Py, x)] \end{array}\right\},$$

where $0 < k < 2$,
(2) $P(X)$ *is a* $d-$*closed* ($\tau(d)-$ *closed*) *subset of* X.
Then P *has a unique fixed point in* X.

Now we extend Theorem 3.1 for finite number of mappings in the following way:

Theorem 3.2. *Let* (X, d) *be a symmetric (semi-metric) space that satisfies* $(H.E)$ *and* $(C.C)$. *Let* $A_1, A_2, A_3......, A_n, S_1, S_2, S_3......, S_n$, P *and* Q *be self-mappings of* X *such that-*
(1) $P(X) \subset A_1A_2A_3......A_n(X), Q(X) \subset S_1S_2S_3......S_n(X)$,
(2) *the pair* $(Q, A_1A_2A_3......A_n)$ *(resp. the pair* $(P, S_1S_2S_3......S_n)$*) satisfies the property* $(E-A)$,
(3) *for any* $x, y \in X$,

$$d(Px, Qy) < \max\left\{\begin{array}{c} d(S_1S_2S_3......S_nx, A_1A_2A_3......A_ny), \\ \frac{k}{2}[d(Px, S_1S_2S_3......S_nx) + d(Qy, A_1A_2A_3......A_ny)], \\ \frac{1}{2}[d(Px, A_1A_2A_3......A_ny) + d(Qy, S_1S_2S_3......S_nx)] \end{array}\right\},$$

where $0 < k < 2$,
(4) *if one of* $S_1S_2S_3......S_n(X)$ *and* $A_1A_2A_3......A_n(X)$ *is a* $d-$*closed* ($\tau(d)$-*closed*) *subset of* X, *then*
(i) P *and* $S_1S_2S_3......S_n$ *have a coincidence point and*
(ii) Q *and* $A_1A_2A_3......A_n$ *have a coincidence point.*
Further if,
(5) $QA_j = A_jQ, PS_j = S_jP, j = 2, 3....n$, *and*
$A_iA_j = A_jA_i, S_iS_j = S_jS_i$ *for* $i \neq j, i = 1, 2, 3...n, j = 1, 2, 3...n$,
(6) *the pairs* $(P, S_1S_2S_3......S_n)$ *and* $(Q, A_1A_2A_3......A_n)$ *are weakly compatible, then*
(iii) $A_1, A_2, A_3......, A_n, S_1, S_2, S_3......, S_n$, P *and* Q *have a unique common fixed point in* X.

Proof. By using the method of proof of Theorem 3.1 we can see that the conclusions (i), (ii) and (iii) hold. □

4. Discussion And Auxiliary Results.

In view of the above results, it is very much clear that we extend, improve and generalize many results in metric spaces and symmetric metric spaces. We prove common fixed point theorems for finite number of mappings in symmetric metric spaces. To prove common fixed point theorems for contractive type condition with more than four mappings, some commutative conditions for mappings are always essential. How many commutative conditions are necessary? As an answer of this question we are giving the following formulas:

(i) If the number if mappings is even and finite in above theorems and corollaries, then there will be $\frac{n^2-2n-8}{4}$ commutativity conditions, where $n = 2, 4, 6....$ up to finite values. For example, if $n = 10$ then 18 commutativity conditions are required (see (5) of Theorem 3.1).

(ii) If the number if mappings is odd and finite in above theorems and corollaries, then there will be $\frac{n^2-9}{4}$ commutativity conditions, where $n = 5, 7, 9, 11....$ up to finite values. For example, if $n = 7$ then 10 commutativity conditions are required (see (5) of Corollary 3.3).

(iii) If $n = 1, 2, 3...$, the any commutativity condition is not required (see Corollary 3.4 and Corollary 3.5).

We point out that common fixed point theorems for finite number of maps can be proved without continuity of any mappings.

In all our results, we replace the completeness of the whole space with a set of alternative conditions.

Our results contain so many results in the existing literature and will be helpful for the workers in the field.

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