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STUDY OF THE FIRST BOUNDARY VALUE PROBLEM FOR A FOURTH ORDER PARABOLIC EQUATION IN A NONREGULAR DOMAIN OF $\mathbb{R}^{N+1}$

AREZKI KHELOUFI[1], §

Abstract. This paper is concerned with the extension of solvability results obtained for a fourth order parabolic equation, set in a nonregular domain of $\mathbb{R}^3$ obtained in [1], to the case where the domain is cylindrical, not with respect to the time variable, but with respect to N space variables, $N > 1$. More precisely, we determine optimal conditions on the shape of the boundary of a $(N+1)$-dimensional domain, $N > 1$, under which the solution is regular.

Keywords: Fourth order parabolic equations, Nonregular domains, Anisotropic weighted Sobolev spaces.

AMS Subject Classification: 35K05, 35K55

1. Introduction

Let Ω be an open set of $\mathbb{R}^2$ defined by

$$\Omega = \left\{(t, x_1) \in \mathbb{R}^2 : 0 < t < T; \varphi_1(t) < x_1 < \varphi_2(t)\right\}$$

where T is a finite positive number, while φ_1 and φ_2 are continuous real-valued functions defined on $[0, T]$, Lipschitz continuous on $[0, T]$, and such that

$$\varphi_2(t) - \varphi_1(t) > 0, \text{ for } t \in]0, T]$$

and

$$\varphi_2(0) = \varphi_1(0) = 0.$$

The lateral boundary of Ω is defined by

$$\Gamma_i = \left\{(t, \varphi_i(t)) \in \mathbb{R}^2 : 0 < t < T\right\}, \; i = 1, 2.$$

For fixed positive numbers $b_i, i = 1, ..., N-1$, with $N > 1$, let Q be the $(N+1)$-dimensional domain defined by

$$Q = \Omega \times \prod_{i=1}^{N-1}]0, b_i[\,.$$

[1] Laboratoire de Mathématiques Appliquées, Faculté des Sciences Exactes, Université de Bejaia, 6000 Bejaia, Algérie. Lab. E.D.P.N.L. and Hist. of Maths, Ecole Normale Supérieure, 16050-Kouba, Algiers, Algeria.
e-mail: arezkinet2000@yahoo.fr;

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In this work, we study the existence and the regularity of the solution of the fourth order parabolic equation with Cauchy-Dirichlet boundary conditions

$$\begin{cases} \partial_t u + \sum_{k=1}^{N} \partial_{x_k}^4 u = f \text{ in } Q, \\ u|_{t=0} = 0, \\ u|_{\Sigma_i} = \partial_{x_1} u|_{\Sigma_i} = 0, \ i = 1,\, 2, \\ u|_{\Sigma_0 \cup \Sigma_b} = \partial_{x_2} u|_{\Sigma_0 \cup \Sigma_b} = ... = \partial_{x_N} u|_{\Sigma_0 \cup \Sigma_b} = 0, \end{cases} \tag{1}$$

where $\Sigma_i = \Gamma_i \times \prod_{k=1}^{N-1}]0, b_k[$, $i = 1,\, 2$, Σ_0 is the part of the boundary of Q where $x_k = 0, k = 2, ..., N$ and Σ_b is the part of the boundary of Q where $x_k = b_{k-1}, k = 2, ..., N$. The right-hand side term f of the equation lies in $L^2_\omega(Q)$ the space of square-integrable functions on Q with the measure $\omega dt dx_1 ... dx_N$. Here the weight ω is a real-valued differentiable function on $[0, T]$.

We are especially interested in the question of what sufficient conditions, as weak as possible, the functions φ_1, φ_2 and ω must verify in order that Problem (1) has a solution with optimal regularity, that is a solution u belonging to the anisotropic weighted Sobolev space

$$H^{1,4}_{0,\omega}(Q) = \left\{ u \in H^{1,4}_\omega(Q) : u|_{\partial_p Q} = 0 \right\}$$

with

$$H^{1,4}_\omega(Q) = \left\{ u \in L^2_\omega(Q) : \partial_t u, \partial_{x_1}^{i_1} \partial_{x_2}^{i_2} ... \partial_{x_N}^{i_N} u \in L^2_\omega(Q), 1 \le i_1 + ... + i_N \le 4 \right\}$$

and $u|_{\partial_p Q} = 0$ means that

$$u|_{t=0} = u|_{\Sigma_i} = \partial_{x_1} u|_{\Sigma_i} = u|_{\Sigma_0 \cup \Sigma_b} = \partial_{x_2} u|_{\Sigma_0 \cup \Sigma_b} = ... = \partial_{x_N} u|_{\Sigma_0 \cup \Sigma_b} = 0,\ i = 1,\, 2.$$

Observe that the domain Q considered here is nonstandard since it shrinks at $t = 0$, $\varphi_2(0) = \varphi_1(0)$. This prevents the nonregular domain Q to be transformed into a usual cylindrical domain by means of a smooth transformation. On the other hand, the semi group generating the solution cannot be defined since the initial condition is defined on a set measure zero.

In Sadallah [2] a similar result has been obtained for a 2m-parabolic operator in the case of one space variable. The solvability of boundary value problems for a 2m-th order parabolic equation in Hölder spaces for noncylindrical domains (of the same kind but which cannot include our domain) with a nonsmooth (in t) lateral boundary was established in [3], [4] and [5]. Further references on the analysis of parabolic problems in noncylindrical domains are: Galaktionov [6], Baderko [7], Mikhailov [8], Savaré [9], Hoffmann and Lewis [10], Labbas, Medeghri and Sadallah [11], [12] and Kheloufi et al. [13], [14], [15], [16] and [17].

The organization of this paper is as follows. In Section 2, we prove that Problem (1) admits a (unique) solution in the case of a truncated domain. In Section 3 we approximate Q by a sequence (Q_n) of such domains and we establish (for T small enough) a uniform estimate of the type

$$\|u_n\|_{H^{1,4}_\omega(Q_n)} \le K \|f\|_{L^2_\omega(Q_n)},$$

where u_n is the solution of Problem (1) in Q_n and K is a constant independent of n. Finally, in Section 4 we prove the two main results of this paper.

The main assumptions on the functions φ_1, φ_2 and ω are

$$\varphi_i'(t)(\varphi_2 - \varphi_1)^2(t) \quad \to \quad 0 \qquad \text{as } t \to 0, \quad i = 1,\, 2, \tag{2}$$

$$\forall t \in [0, T] : \omega(t) > 0, \tag{3}$$

and

$$\omega \text{ is a decreasing function on }]0, T]. \tag{4}$$

Note that this work may be extended at least in the following directions:

1. The function f on the right-hand side of the equation of Problem (1), may be taken in $L^p_\omega(Q)$, $p \in]1,\infty[$. The domain decomposition method used here does not seem to be appropriate for the space $L^p_\omega(Q)$ when $p \neq 2$.

2. The nonregular domain Q may be replaced by a noncylindrical conical type domain, such as, for example, the following domain

$$Q = \left\{ (t, x_1, x_2, ..., x_N) \in \mathbb{R}^{N+1} : 0 \leq \sqrt{x_1^2 + x_2^2 + ... + x_N^2} < \varphi(t), 0 < t < T \right\}$$

where φ is similar to $\varphi_i, i = 1, 2$. These questions will be developed in forthcoming works.

2. Resolution of Problem (1) in a truncated domain Q_n

In this section, we replace Q by $Q_n, n \in \mathbb{N}^*$ and $\frac{1}{n} < T$:

$$Q_n = \left\{ (t, x_1, ..., x_N) \in Q : \frac{1}{n} < t < T \right\}.$$

Theorem 2.1. *For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, the problem*

$$\begin{cases} \partial_t u_n + \sum_{k=1}^N \partial^4_{x_k} u_n = f_n \in L^2_\omega(Q_n), \\ u_n|_{t=\frac{1}{n}} = u_n|_{\Sigma_{i,n}} = \partial_{x_1} u_n|_{\Sigma_{i,n}} = 0, \ i = 1, 2, \\ u_n|_{\Sigma_{0,n} \cup \Sigma_{b,n}} = \partial_{x_2} u_n|_{\Sigma_{0,n} \cup \Sigma_{b,n}} = ... = \partial_{x_N} u_n|_{\Sigma_{0,n} \cup \Sigma_{b,n}} = 0, \end{cases} \tag{5}$$

admits a (unique) solution $u_n \in H^{1,4}_\omega(Q_n)$. Here,
$\Sigma_{i,n} = \left\{ (t, \varphi_i(t)) \in \mathbb{R}^2 : \frac{1}{n} < t < T \right\} \times \prod_{k=1}^{N-1}]0, b_k[, i = 1, 2$, $\Sigma_{0,n}$ is the part of the boundary of Q_n where $x_k = 0$, $k = 2, ..., N$ and $\Sigma_{b,n}$ is the part of the boundary of Q_n where $x_k = b_{k-1}$, $k = 2, ..., N$.

Proof of Theorem 2.1: The change of variables

$$(t, x_1, x_2, ..., x_N) \longmapsto (t, y_1, y_2, ..., y_N) = \left(t, \frac{x_1 - \varphi_1(t)}{\varphi_2(t) - \varphi_1(t)}, x_2, ..., x_N \right),$$

transforms Q_n into the cylindrical domain $P_n = \left] \frac{1}{n}, T \right[\times]0, 1[\times \prod_{i=1}^{N-1}]0, b_i[$. Putting

$$v_n(t, y_1, y_2, ..., y_N) = u_n(t, x_1, x_2, ..., x_N)$$

and

$$g_n(t, y_1, y_2, ..., y_N) = f_n(t, x_1, x_2, ..., x_N),$$

then Problem (5) becomes

$$\begin{cases} \partial_t v_n + a(t, y_1) \partial_{y_1} v_n + c(t) \partial^4_{y_1} v_n + \sum_{k=2}^N \partial^4_{y_k} v_n = g_n \in L^2_\omega(P_n) \\ v_n|_{t=\frac{1}{n}} = v_n|_{\Sigma_{i,P_n}} = \partial_{y_1} v_n|_{\Sigma_{i,P_n}} = 0, \ i = 1, 2, \\ v_n|_{\Sigma_{0,P_n} \cup \Sigma_{b,P_n}} = \partial_{y_2} v_n|_{\Sigma_{0,P_n} \cup \Sigma_{b,P_n}} = ... = \partial_{y_N} v_n|_{\Sigma_{0,P_n} \cup \Sigma_{b,P_n}} = 0, \end{cases}$$

where $\Sigma_{1,P_n} = \left] \frac{1}{n}, T \right[\times \{0\} \times \prod_{i=1}^{N-1}]0, b_i[$, $\Sigma_{2,P_n} = \left] \frac{1}{n}, T \right[\times \{1\} \times \prod_{i=1}^{N-1}]0, b_i[$, Σ_{0,P_n} is the part of the boundary of P_n where $x_k = 0, k = 2, ..., N$, Σ_{b,P_n} is the part of the boundary of P_n where $x_k = b_{k-1}, k = 2, ..., N$, $c(t) = \frac{1}{[\varphi_2(t) - \varphi_1(t)]^4}$ and $a(t, y_1) = -\frac{y_1(\varphi_2'(t) - \varphi_1'(t)) + \varphi_1'(t)}{\varphi_2(t) - \varphi_1(t)}$.

Since the functions a, c and $(\varphi_2 - \varphi_1)$ are bounded when $t \in \left] \frac{1}{n}, T \right[$, then the above change of variable which is $(N+1)$-Lipschitz preserves the spaces L^2_ω and $H^{1,4}_\omega$. In other words

$$f_n \in L^2_\omega(Q_n) \Longleftrightarrow g_n \in L^2_\omega(P_n), \ u_n \in H^{1,4}_\omega(Q_n) \Longleftrightarrow v_n \in H^{1,4}_\omega(P_n).$$

Proposition 2.1. *For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, the following operator is compact*

$$a(t, y_1)\,\partial_{y_1} : H^{1,4}_{0,\omega}(P_n) \longrightarrow L^2_\omega(P_n).$$

Proof. P_n has the "horn property" of Besov [19], so

$$\partial_{y_1} : H^{1,4}_{0,\omega}(P_n) \longrightarrow H^{\frac{3}{4},3}_\omega(P_n),\ v_n \longmapsto \partial_{y_1} v_n,$$

is continuous. Since P_n is bounded, the canonical injection is compact from $H^{\frac{3}{4},3}_\omega(P_n)$ into $L^2_\omega(P_n)$, where

$$H^{\frac{3}{4},3}(P_n) = L^2\left(\frac{1}{n}, T; H^3\left(]0,1[\times\prod_{i=1}^{N-1}]0,b_i[\right)\right) \cap H^{\frac{3}{4}}\left(\frac{1}{n}, T; L^2\left(]0,1[\times\prod_{i=1}^{N-1}]0,b_i[\right)\right).$$

For the complete definitions of the $H^{r,s}$ Hilbertian Sobolev spaces see for instance [20]. Consider the composition

$$\partial_{y_1} : H^{1,4}_{0,\omega}(P_n) \to H^{\frac{3}{4},3}_\omega(P_n) \to L^2_\omega(P_n),\ v_n \mapsto \partial_{y_1} v_n \mapsto \partial_{y_1} v_n,$$

then ∂_{y_1} is a compact operator from $H^{1,4}_{0,\omega}(P_n)$ into $L^2_\omega(P_n)$. Since $a(.,.)$ is a bounded function for $\frac{1}{n} < t < T$, the operator $a\partial_{y_1}$ is also compact from $H^{1,4}_{0,\omega}(P_n)$ into $L^2_\omega(P_n)$. □

So, thanks to Proposition 2.1, to complete the proof of Theorem 2.1, it is sufficient to show that the operator

$$\partial_t + c(t)\,\partial^4_{y_1} + \sum_{k=2}^{N} \partial^4_{y_k}$$

is an isomorphism from $H^{1,4}_{0,\omega}(P_n)$ into $L^2_\omega(P_n)$.

Lemma 2.1. *For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, the operator*

$$\partial_t + c(t)\,\partial^4_{y_1} + \sum_{k=2}^{N} \partial^4_{y_k}$$

is an isomorphism from $H^{1,4}_{0,\omega}(P_n)$ into $L^2_\omega(P_n)$.

Proof. Since the coefficient $\frac{1}{[\varphi_2(t)-\varphi_1(t)]^4}$ is continuous in $\overline{P_n}$, the optimal regularity is given by Ladyzhenskaya-Solonnikov-Ural'tseva [18]. □

We shall need the following result in order to justify some calculations in the next section, see [1].

Lemma 2.2. *For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, the space*

$$\left\{u_n \in H^4(P_n);\ \ u_n|_{\partial P_n - \Gamma_T} = 0\right\}$$

is dense in the space

$$\left\{u_n \in H^{1,4}(P_n);\ \ u_n|_{\partial P_n - \Gamma_T} = 0\right\}.$$

Here Γ_T be the part of the boundary of P_n where $t = T$.

Remark 2.1. *In Lemma 2.2, we can replace P_n by Q_n with the help of the change of variable defined above.*

3. An "energy" type estimate

For each $n \in \mathbb{N}^*$ such that $\frac{1}{n} < T$, we denote by $u_n \in H_\omega^{1,4}(Q_n)$ the solution of Problem (5) corresponding to the right-hand side $f_n = f|_{Q_n} \in L_\omega^2(Q_n)$. Such a solution exists by Theorem 2.1.

Proposition 3.1. *Assume that φ_1 and φ_2 fulfil condition (2) and the weight function ω verifies assumptions (3) and (4). Then, for T small enough, there exists a constant M independent of n such that*

$$\|u_n\|_{H_\omega^{1,4}(Q_n)} \leq M \|f_n\|_{L_\omega^2(Q_n)} \leq M \|f\|_{L_\omega^2(Q)},$$

where

$$\|u_n\|_{H_\omega^{1,4}(Q_n)} = \left(\|u_n\|^2_{L_\omega^2(Q_n)} + \|\partial_t u_n\|^2_{L_\omega^2(Q_n)} + \sum_{\substack{i_1,i_2,\dots,i_N=0 \\ 1\leq i_1+i_2+\dots+i_N\leq 4}}^{4} \left\|\partial_{x_1}^{i_1}\partial_{x_2}^{i_2}\dots\partial_{x_N}^{i_N} u_n\right\|^2_{L_\omega^2(Q_n)} \right)^{1/2}.$$

Remark 3.1. *Let $\epsilon > 0$ be a real which we will choose small enough. The hypothesis (2) implies the existence of a real number $T > 0$ small enough such that*

$$\forall t \in (0,T), \left|\varphi_i'(t)(\varphi_2 - \varphi_1)^2(t)\right| \leq \epsilon,\ i = 1,\ 2. \tag{6}$$

To derive the basic inequality of Proposition (3.1), we need the following lemmas.

Lemma 3.1. *Let $]\gamma,\delta[\subset \mathbb{R}$. There exists a positive constant K_2 (independent of γ and δ) such that for each $v \in H^4(]\gamma,\delta[) \cap H_0^2(]\gamma,\delta[)$*

$$\left\|v^{(l)}\right\|^2_{L^2(]\gamma,\delta[)} \leq (\delta-\gamma)^{2(4-l)} K_2 \left\|v^{(4)}\right\|^2_{L^2(]\gamma,\delta[)},\ l = 0,1,2,3.$$

The proof of the previous Lemma can be found in [1].

Lemma 3.2. *For every $\epsilon > 0$, chosen such that $(\varphi_2(t) - \varphi_1(t)) \leq \epsilon$, there exists a constant C_1 independent of n such that*

$$\left\|\partial_{x_1}^l u_n\right\|^2_{L_\omega^2(Q_n)} \leq C_1 \epsilon^{2(4-l)} \left\|\partial_{x_1}^4 u_n\right\|^2_{L_\omega^2(Q_n)},\ l = 0,1,2,3.$$

Proof. Replacing in Lemma 3.1 v by u_n and $]\gamma,\delta[$ by $]\varphi_1(t), \varphi_2(t)[$, for a fixed t, we obtain

$$\begin{aligned} \int_{\varphi_1(t)}^{\varphi_2(t)} \left(\partial_{x_1}^l u_n\right)^2 dx_1 &\leq K_2 (\varphi_2(t) - \varphi_1(t))^{2(4-l)} \int_{\varphi_1(t)}^{\varphi_2(t)} \left(\partial_{x_1}^4 u_n\right)^2 dx_1 \\ &\leq K_2 \epsilon^{2(4-l)} \int_{\varphi_1(t)}^{\varphi_2(t)} \left(\partial_{x_1}^4 u_n\right)^2 dx_1. \end{aligned}$$

Multiplying the previous inequality by $\omega(t)$ (which is positive) and integrating with respect to t, then with respect to x_2, x_3,..., x_N, we get the desired result with $C_1 = K_2$. □

Lemma 3.3. *Let us denote the inner product in $L_\omega^2(Q_n)$ by $\langle .,. \rangle$. Under the assumptions of Proposition (3.1), we have*

i) $2\langle \partial_t u_n, \partial_{x_1}^4 u_n \rangle \geq -K\epsilon \left\|\partial_{x_1}^4 u_n\right\|_{L_\omega^2(Q_n)}$ *(for T small enough).*

ii) $2\langle \partial_t u_n, \partial_{x_k}^4 u_n \rangle \geq 0,\ k = 2, ..., N.$

iii) $2\langle \partial_{x_j}^4 u_n, \partial_{x_k}^4 u_n \rangle = 2 \left\|\partial_{x_j}^2 \partial_{x_k}^2 u_n\right\|^2_{L_\omega^2(Q_n)},\ j = 1, ..., N-1,\ k = j+1, ..., N.$

Proof. **1) Estimation of** $2\langle \partial_t u_n, \partial_{x_1}^4 u_n \rangle$: We have

$$\partial_t u_n.\partial_{x_1}^4 u_n = \partial_{x_1}\left(\partial_t u_n.\partial_{x_1}^3 u_n\right) - \partial_{x_1}\left(\partial_{x_1}\partial_t u_n.\partial_{x_1}^2 u_n\right) + \tfrac{1}{2}\partial_t \left(\partial_{x_1}^2 u_n\right)^2.$$

Then

$$\begin{aligned} 2\langle \partial_t u_n, \partial_{x_1}^4 u_n\rangle &= 2\int_{Q_n} \partial_t u_n.\partial_{x_1}^4 u_n.\omega(t)\ dtdx_1...dx_N \\ &= \int_{\partial Q_n} \left[\left(\partial_{x_1}^2 u_n\right)^2 \nu_t + 2\left(\partial_t u_n.\partial_{x_1}^3 u_n - \partial_{x_1}\partial_t u_n.\partial_{x_1}^2 u_n\right)\nu_{x_1}\right].\omega(t)\, d\sigma \\ &\quad - \int_{Q_n} \left(\partial_{x_1}^2 u_n\right)^2 .\omega'(t)\ dtdx_1...dx_N. \end{aligned}$$

We shall rewrite the boundary integral making use of the boundary conditions. On the parts of the boundary of Q_n where $t = \frac{1}{n}$, $x_k = 0,\ k = 2,...,N$ and $x_k = b_{k-1},\ k = 2,...,N$ we have $\partial_{x_1} u_n = 0$ and consequently $\partial_{x_1}^2 u_n = \partial_{x_1}^3 u_n = 0$. The corresponding boundary integral vanishes. On the part of the boundary where $t = T$, we have $\nu_{x_1} = 0$ and $\nu_t = 1$. Accordingly the corresponding boundary integral

$$\int_0^{b_{N-1}} ... \int_0^{b_1} \int_{\varphi_1(T)}^{\varphi_2(T)} \left[\partial_{x_1}^2 u_n(T, x_1, ..., x_N)\right]^2 \omega(T)\, dx_1...dx_N$$

is nonnegative. On the part of the boundary where $x_1 = \varphi_i(t)$, $i = 1, 2$, we have $\nu_{x_1} = \frac{(-1)^i}{\sqrt{1+(\varphi_i')^2(t)}}$, $\nu_t = \frac{(-1)^{i+1}\varphi_i'(t)}{\sqrt{1+(\varphi_i')^2(t)}}$ and $u = \partial_{x_1} u_n = 0$. Differentiating with respect to t we obtain

$$\partial_t u_n(t, \varphi_i(t), ..., x_N) = -\varphi_i'(t)\,\partial_{x_1} u_n(t, \varphi_i(t), ..., x_N),$$
$$\partial_t\partial_{x_1} u_n(t, \varphi_i(t), ..., x_N) = -\varphi_i'(t)\,\partial_{x_1}^2 u_n(t, \varphi_i(t), ..., x_N).$$

Consequently, the corresponding boundary integrals I_1 and I_2 are the following:

$$\begin{aligned} I_1 &= -\int_0^{b_{N-1}} ... \int_0^{b_1} \int_{\frac{1}{n}}^T \varphi_1'(t)\left[\partial_{x_1}^2 u_n(t, \varphi_1(t), ..., x_N)\right]^2\ \omega(t)\, dtdx_2...dx_N \\ I_2 &= \int_0^{b_{N-1}} ... \int_0^{b_1} \int_{\frac{1}{n}}^T \varphi_2'(t)\left[\partial_{x_1}^2 u_n(t, \varphi_2(t), ..., x_N)\right]^2\ \omega(t)\, dtdx_2...dx_N. \end{aligned}$$

In virtue of (3) and (4), we have

$$2\langle \partial_t u_n, \partial_{x_1}^4 u_n\rangle \geq -|I_1| - |I_2|. \tag{7}$$

□

Lemma 3.4. *There exists a constant K_3 independent of n such that*

$$|I_i| \leq K_3\epsilon \left\|\partial_{x_1}^4 u_n\right\|_{L^2_\omega(Q_n)}^2, \quad i = 1, 2.$$

Proof. We convert the boundary integral I_1 into a surface integral by setting

$$\begin{aligned} \left[\partial_{x_1}^2 u_n(t, \varphi_1(t), x_2, ..., x_N)\right]^2 &= -\frac{\varphi_2(t)-x_1}{\varphi_2(t)-\varphi_1(t)}\left[\partial_{x_1}^2 u_n(t, x_1, x_2, ..., x_N)\right]^2\Big|_{x_1=\varphi_1(t)}^{x_1=\varphi_2(t)} \\ &= -\int_{\varphi_1(t)}^{\varphi_2(t)} \partial_{x_1}\left\{\frac{\varphi_2(t)-x_1}{\varphi_2(t)-\varphi_1(t)}\left[\partial_{x_1}^2 u_n\right]^2\right\} dx_1 \\ &= -2\int_{\varphi_1(t)}^{\varphi_2(t)} \frac{\varphi_2(t)-x_1}{\varphi_2(t)-\varphi_1(t)}\partial_{x_1}^2 u_n.\partial_{x_1}^3 u_n dx_1 \\ &\quad + \int_{\varphi_1(t)}^{\varphi_2(t)} \frac{1}{\varphi_2(t)-\varphi_1(t)}\left[\partial_{x_1}^2 u_n\right]^2 dx_1. \end{aligned}$$

Then, we have

$$\begin{aligned} I_1 &= -\int_0^{b_{N-1}} ... \int_0^{b_1} \int_{\frac{1}{n}}^T \varphi_1'(t)\left[\partial_{x_1}^2 u_n(t, \varphi_1(t), x_2, ..., x_N)\right]^2\ \omega(t)\, dtdx_2...dx_N \\ &= -\int_{Q_n} \frac{\varphi_1'(t)}{\varphi_2(t)-\varphi_1(t)}\left[\partial_{x_1}^2 u_n(t, x_1, ..., x_N)\right]^2 \omega(t)\, dtdx_1...dx_N \\ &\quad +2\int_{Q_n} \frac{\varphi_2(t)-x_1}{\varphi_2(t)-\varphi_1(t)}\varphi_1'(t)\left(\partial_{x_1}^2 u_n\right)\left(\partial_{x_1}^3 u_n\right)\ \omega(t)\, dtdx_1...dx_N. \end{aligned}$$

Thanks to Lemma 3.1, we can write

$$\int_{\varphi_1(t)}^{\varphi_2(t)} \left[\partial_{x_1}^2 u_n\right]^2 dx_1 \leq K_2\left[\varphi_2(t) - \varphi_1(t)\right]^4 \int_{\varphi_1(t)}^{\varphi_2(t)} \left[\partial_{x_1}^4 u_n\right]^2 dx_1.$$

Therefore

$$\int_{\varphi_1(t)}^{\varphi_2(t)} \left[\partial_{x_1}^2 u_n\right]^2 \frac{|\varphi_1'|}{\varphi_2(t)-\varphi_1(t)} \omega(t)\, dx_1 \quad \leq \quad K_2 |\varphi_1'| \left[\varphi_2(t)-\varphi_1(t)\right]^3 \int_{\varphi_1(t)}^{\varphi_2(t)} \left[\partial_{x_1}^4 u_n\right]^2 \omega(t)\, dx_1,$$

consequently

$$\begin{aligned} |I_1| \quad \leq \quad & K_2 \int_{Q_n} |\varphi_1'| \left[\varphi_2(t)-\varphi_1(t)\right]^3 \left(\partial_{x_1}^4 u_n\right)^2 \omega(t)\, dt dx_1 ... dx_N \\ & +2 \int_{Q_n} |\varphi_1'| \left|\partial_{x_1}^2 u_n\right| \left|\partial_{x_1}^3 u_n\right| \ \omega(t)\, dt dx_1 ... dx_N, \end{aligned}$$

since $\left|\frac{\varphi_2(t)-x_1}{\varphi_2(t)-\varphi_1(t)}\right| \leq 1$. Using the inequality

$$2\left|\varphi_1' \partial_{x_1}^2 u_n\right| \left|\partial_{x_1}^3 u_n\right| \quad \leq \quad \epsilon \left(\partial_{x_1}^3 u_n\right)^2 + \tfrac{1}{\epsilon} \left(\varphi_1'\right)^2 \left(\partial_{x_1}^2 u_n\right)^2$$

for all $\epsilon > 0$, we obtain

$$\begin{aligned} |I_1| \quad \leq \quad & K_2 \int_{Q_n} |\varphi_1'| \left[\varphi_2(t)-\varphi_1(t)\right]^3 \left(\partial_{x_1}^4 u_n\right)^2 \omega(t)\, dt dx_1 ... dx_N \\ & + \int_{Q_n} \epsilon \left(\partial_{x_1}^3 u_n\right)^2 \omega(t)\, dt dx_1 ... dx_N + \tfrac{1}{\epsilon} \int_{Q_n} \left(\varphi_1'\right)^2 \left(\partial_{x_1}^2 u_n\right)^2 \omega(t)\, dt dx_1 ... dx_N. \end{aligned}$$

Lemma 3.2 yields

$$\begin{aligned} & \tfrac{1}{\epsilon} \int_{Q_n} \left(\varphi_1'\right)^2 \left(\partial_{x_1}^2 u_n\right)^2 \omega(t)\, dt dx_1 ... dx_N \\ & \leq K_2 \tfrac{1}{\epsilon} \int_{Q_n} \left(\varphi_1'\right)^2 \left[\varphi_2(t)-\varphi_1(t)\right]^4 \left(\partial_{x_1}^4 u_n\right)^2 \omega(t)\, dt dx_1 ... dx_N. \end{aligned}$$

Thus,

$$\begin{aligned} |I_1| \quad \leq \quad & K_2 \int_{Q_n} \left[|\varphi_1'| \left[\varphi_2(t)-\varphi_1(t)\right]^3 + \tfrac{1}{\epsilon} \left(\varphi_1'\right)^2 \left[\varphi_2(t)-\varphi_1(t)\right]^4\right] \left(\partial_{x_1}^4 u_n\right)^2 \omega(t)\, dt ... dx_N \\ & + \int_{Q_n} \epsilon \left(\partial_{x_1}^3 u_n\right)^2 \omega(t)\, dt dx_1 ... dx_N \\ \leq \quad & (K_2+1)\, \epsilon \int_{Q_n} \left(\partial_{x_1}^4 u_n\right)^2 \omega(t)\, dt dx_1 ... dx_N, \end{aligned}$$

since $\left|\varphi_1'(\varphi_2(t)-\varphi_1(t))^2 \left[(\varphi_2(t)-\varphi_1(t)) - \varphi_1'(\varphi_2(t)-\varphi_1(t))^2\right]\right| \leq \epsilon$ thanks to the condition (6). Finally, taking $K_3 = (K_2+1)$, we obtain

$$|I_1| \quad \leq \quad K_3 \epsilon \left\|\partial_{x_1}^4 u_n\right\|_{L_\omega^2(Q_n)}.$$

The inequality

$$|I_2| \quad \leq \quad K_3 \epsilon \left\|\partial_{x_1}^4 u_n\right\|_{L_\omega^2(Q_n)},$$

can be proved by a similar argument.

2) Estimation of $2\langle \partial_t u_n, \partial_{x_k}^4 u_n \rangle, k = 2, ..., N :$ We have

$$\partial_t u_n . \partial_{x_k}^4 u_n \quad = \quad \partial_{x_k} \left(\partial_t u_n . \partial_{x_k}^3 u_n\right) - \partial_{x_k} \left(\partial_{x_k} \partial_t u_n . \partial_{x_k}^2 u_n\right) + \tfrac{1}{2} \partial_t \left(\partial_{x_k}^2 u_n\right)^2.$$

Then

$$\begin{aligned} 2\langle \partial_t u_n, \partial_{x_k}^4 u_n \rangle \quad = \quad & 2 \int_{Q_n} \partial_t u_n . \partial_{x_k}^4 u_n . \omega(t) \ \ dt dx_1 ... dx_N \\ = \quad & \int_{\partial Q_n} \left[\left(\partial_{x_k}^2 u_n\right)^2 \nu_t + 2 \left(\partial_t u_n . \partial_{x_k}^3 u_n - \partial_{x_k} \partial_t u_n . \partial_{x_k}^2 u_n\right) \nu_{x_k}\right] . \omega(t)\, d\sigma \\ & - \int_{Q_n} \left(\partial_{x_k}^2 u_n\right)^2 . \omega'(t) \ \ dt dx_1 ... dx_N. \end{aligned}$$

Using the Cauchy-Dirichlet boundary conditions, we see that the above boundary integral is nonnegative. Consequently in virtue of (4), we have

$$2\langle \partial_t u_n, \partial_{x_k}^4 u_n \rangle \geq 0. \qquad (8)$$

3) Estimation of $2\langle \partial_{x_j}^4 u_n, \partial_{x_k}^4 u_n \rangle, j = 1, ..., N-1, k = j+1, ..., N$: We have

$$\partial_{x_j}^4 u_n . \partial_{x_k}^4 u_n \quad = \quad \partial_{x_j} \left(\partial_{x_j}^3 u_n . \partial_{x_k}^4 u_n\right) - \partial_{x_k} \left(\partial_{x_j}^3 u_n . \partial_{x_j} \partial_{x_k}^3 u_n\right) + \partial_{x_j} \partial_{x_k}^3 u_n . \partial_{x_k} \partial_{x_j}^3 u_n.$$

Then

$$\begin{aligned}
2\langle \partial_{x_j}^4 u_n, \partial_{x_k}^4 u_n\rangle &= 2\int_{Q_n} \partial_{x_j}^4 u_n.\partial_{x_k}^4 u_n.\omega(t)\ dt\ dx_1...dx_N \\
&= 2\int_{Q_n} \partial_{x_j}\left(\partial_{x_j}^3 u_n.\partial_{x_k}^4 u_n\right).\omega(t)\ dt\ dx_1...dx_N \\
&\quad -2\int_{Q_n} \partial_{x_k}\left(\partial_{x_j}^3 u_n.\partial_{x_j}\partial_{x_k}^3 u_n\right).\omega(t)\ dt\ dx_1...dx_N \\
&\quad +2\int_{Q_n} \partial_{x_j}\partial_{x_k}^3 u_n.\partial_{x_k}\partial_{x_j}^3 u_n.\omega(t)\ dt\ dx_1...dx_N \\
&= 2\int_{Q_n} \partial_{x_j}\partial_{x_k}^3 u_n.\partial_{x_k}\partial_{x_j}^3 u_n.\omega(t)\ dt\ dx_1...dx_N \\
&\quad +2\int_{\partial Q_n}\left[\partial_{x_j}^3 u_n.\partial_{x_k}^4 u_n\nu_{x_j} - \partial_{x_j}^3 u_n.\partial_{x_j}\partial_{x_k}^3 u_n\nu_{x_k}\right]\omega(t)\, d\sigma.
\end{aligned}$$

We shall rewrite the boundary integral making use of the boundary conditions. On the parts of the boundary of Q_n where $t = \frac{1}{n}$, $x_k = 0, k = 2, ..., N$ and $x_k = b_{k-1}, k = 2, ..., N$, we have $\partial_{x_j} u_n = 0$ and consequently $\partial_{x_j}^3 u_n = 0$. The corresponding boundary integral vanishes. On the part of the boundary where $t = T$, we have $\nu_{x_k} = 0$. Accordingly the corresponding boundary integral vanishes. By using again Green formula and the Cauchy-Dirichlet boundary conditions, we obtain

$$2\int_{Q_n} \partial_{x_j}\partial_{x_k}^3 u_n.\partial_{x_k}\partial_{x_j}^3 u_n.\omega(t)\ dt\ dx_1...dx_N = 2\left\|\partial_{x_j}^2\partial_{x_k}^2 u_n\right\|_{L_\omega^2(Q_n)}^2.$$

Finally,

$$2\langle \partial_{x_j}^4 u_n, \partial_{x_k}^4 u_n\rangle = 2\left\|\partial_{x_j}^2\partial_{x_k}^2 u_n\right\|_{L_\omega^2(Q_n)}^2, j = 1, ..., N-1, k = j+1, ..., N. \tag{9}$$

□

Proof of Proposition (3.1): We have

$$\begin{aligned}
\|f_n\|_{L_\omega^2(Q_n)}^2 &= \langle \partial_t u_n + \textstyle\sum_{k=1}^N \partial_{x_k}^4 u_n, \partial_t u_n + \sum_{k=1}^N \partial_{x_k}^4 u_n\rangle \\
&= \|\partial_t u_n\|_{L_\omega^2(Q_n)}^2 + \textstyle\sum_{k=1}^N \left\|\partial_{x_k}^4 u_n\right\|_{L_\omega^2(Q_n)}^2 \\
&\quad +2\textstyle\sum_{k=1}^N \langle \partial_t u_n, \partial_{x_k}^4 u_n\rangle + 2\sum_{j=1}^{N-1}\sum_{k=j+1}^N \langle \partial_{x_j}^4 u_n, \partial_{x_k}^4 u_n\rangle.
\end{aligned}$$

Summing up the estimates (7), (8) and (9) of the inner products and making use of Lemma 3.4, we then obtain

$$\begin{aligned}
\|f_n\|_{L_\omega^2(Q_n)}^2 &\geq \|\partial_t u_n\|_{L_\omega^2(Q_n)}^2 + \textstyle\sum_{k=1}^N \left\|\partial_{x_k}^4 u_n\right\|_{L_\omega^2(Q_n)}^2 \\
&\quad - |I_1| - |I_2| \\
&\quad +2\textstyle\sum_{j=1}^{N-1}\sum_{k=j+1}^N \left\|\partial_{x_j}^2\partial_{x_k}^2 u_n\right\|_{L_\omega^2(Q_n)}^2 \\
&\geq \|\partial_t u_n\|_{L_\omega^2(\Omega_n)}^2 + (1-2K_3\epsilon)\left\|\partial_{x_1}^4 u_n\right\|_{L_\omega^2(Q_n)}^2 \\
&\quad + \textstyle\sum_{k=2}^N \left\|\partial_{x_k}^4 u_n\right\|_{L_\omega^2(Q_n)}^2 + 2\sum_{j=1}^{N-1}\sum_{k=j+1}^N \left\|\partial_{x_j}^2\partial_{x_k}^2 u_n\right\|_{L_\omega^2(Q_n)}^2.
\end{aligned}$$

Then, it is sufficient to choose ϵ such that $(1-2K_3\epsilon) > 0$ to get a constant $K_0 > 0$ independent of n such that

$$\|f_n\|_{L_\omega^2(Q_n)} \geq K_0 \|u_n\|_{H_\omega^{1,4}(Q_n)},$$

and since

$$\|f_n\|_{L_\omega^2(Q_n)} \leq \|f\|_{L_\omega^2(Q)},$$

there exists a constant $M > 0$, independent of n satisfying

$$\|u_n\|_{H_\omega^{1,4}(Q_n)} \leq M\|f_n\|_{L_\omega^2(Q_n)} \leq M\|f\|_{L_\omega^2(Q)}.$$

This completes the proof of Proposition (3.1).

4. Main results

We are now able to prove the main results of the paper.

4.1. **Local in time result.**

Theorem 4.1. *Assume that φ_1 and φ_2 fulfil condition (2) and the weight function ω verifies assumptions (3) and (4). Then for T small enough, the fourth order parabolic operator*

$$L = \partial_t + \sum_{k=1}^{N} \partial_{x_k}^4$$

is an isomorphism from $H_{0,\omega}^{1,4}(Q)$ into $L_\omega^2(Q)$.

Proof. **1) Injectivity of the operator L:** Let us consider $u \in H_{0,\omega}^{1,4}(Q)$ a solution of the problem (1) with a null right-hand side term. So,

$$\partial_t u + \sum_{k=1}^{N} \partial_{x_k}^4 u = 0 \text{ in } Q.$$

In addition u fulfils the boundary condtions

$$u|_{t=0} = u|_{\Sigma_i} = \partial_{x_1} u|_{\Sigma_i} = u|_{\Sigma_0 \cup \Sigma_b} = \partial_{x_2} u|_{\Sigma_0 \cup \Sigma_b} = ... = \partial_{x_N} u|_{\Sigma_0 \cup \Sigma_b} = 0,\ i = 1{,}2.$$

Using Green formula, we have

$$\begin{aligned}
&\textstyle\int_Q \left(\partial_t u + \sum_{k=1}^{N} \partial_{x_k}^4 u\right) u.\omega(t)\ dt\ dx_1...dx_N \\
&= \textstyle\int_{\partial Q} \left[\frac{1}{2}|u|^2 \nu_t + \sum_{k=1}^{N} \left(\partial_{x_k}^3 u.u - \partial_{x_k}^2 u.\partial_{x_k} u\right) \nu_{x_k}\right] \omega(t)\, d\sigma \\
&+ \textstyle\int_Q \left(\sum_{k=1}^{N} \left|\partial_{x_k}^2 u\right|^2\right) \omega(t)\, dt\ dx_1...dx_N - \int_Q \frac{1}{2}|u|^2 \omega'(t)\, dt\ dx_1...dx_N
\end{aligned}$$

where ν_t, ν_{x_1},...,ν_{x_N} are the components of the unit outward normal vector at ∂Q. Taking into account the boundary conditions, all the boundary integrals vanish except $\int_{\partial Q} |u|^2 \omega(t)\, \nu_t\ d\sigma$. We have

$$\int_{\partial Q} |u|^2 \omega(t)\, \nu_t d\sigma = \int_0^{b_{N-1}} ... \int_0^{b_1} \int_{\varphi_1(T)}^{\varphi_2(T)} |u|^2 \omega(T)\, dx_1 dx_2 ... dx_N.$$

Then

$$\begin{aligned}
&\textstyle\int_Q \left(\partial_t u + \sum_{k=1}^{N} \partial_{x_k}^4 u\right) .u\ \omega(t)\, dt\ dx_1...dx_N \\
&= \textstyle\int_0^{b_{N-1}} ... \int_0^{b_1} \int_{\varphi_1(T)}^{\varphi_2(T)} \frac{1}{2}|u|^2 \omega(T)\, dx_1 dx_2...dx_N - \int_Q \frac{1}{2}|u|^2 \omega'(t)\, dt\ dx_1...dx_N \\
&+ \textstyle\int_Q \left(\sum_{k=1}^{N} \left|\partial_{x_k}^2 u\right|^2\right) \omega(t)\, dt\ dx_1...dx_N.
\end{aligned}$$

Consequently

$$\int_Q \left(\partial_t u + \sum_{k=1}^{N} \partial_{x_k}^4 u\right) .u\ \omega(t)\, dt\ dx_1...dx_N = 0$$

yields

$$\int_Q \left(\sum_{k=1}^{N} \left|\partial_{x_k}^2 u\right|^2\right) \omega(t)\, dt\ dx_1...dx_N = 0,$$

because

$$\int_0^{b_{N-1}} ... \int_0^{b_1} \int_{\varphi_1(T)}^{\varphi_2(T)} \frac{1}{2}|u|^2 \omega(T)\, dx_1 dx_2...dx_N - \int_Q \frac{1}{2}|u|^2 \omega'(t)\, dt\ dx_1...dx_N \geq 0$$

thanks to the conditions (3) and (4). This implies that $\sum_{k=1}^{N}\left|\partial_{x_k}^2 u\right|^2=0$ and consequently $\partial_{x_1}^4 u=...=\partial_{x_N}^4 u=0$. Then, the hypothesis $\partial_t u+\sum_{k=1}^{N}\partial_{x_k}^4 u=0$ gives $\partial_t u=0$. Thus, u is constant. The boundary conditions imply that $u=0$ in Q. This proves the uniqueness of the solution of Problem (1).

2) Surjectivity of the operator L**:** Choose a sequence $(Q_n)_{n\in\mathbb{N}^*}$ of the domains defined above (see Section 2), such that $Q_n\subseteq Q$. Then, we have $Q_n\to Q$, as $n\to\infty$. Consider the solution $u_n\in H_\omega^{1,4}(Q_n)$ of the Cauchy-Dirichlet problem

$$\begin{cases}\partial_t u_n+\sum_{k=1}^{N}\partial_{x_k}^4 u_n=f_n \text{ in } Q_n\\ u_n|_{t=\frac{1}{n}}=u_n|_{\Sigma_{i,n}}=\partial_{x_1}u_n|_{\Sigma_{i,n}}=0,\ i=1,\ 2,\\ u_n|_{\Sigma_{0,n}\cup\Sigma_{b,n}}=\partial_{x_2}u_n|_{\Sigma_{0,n}\cup\Sigma_{b,n}}=...=\partial_{x_N}u_n|_{\Sigma_{0,n}\cup\Sigma_{b,n}}=0,\end{cases}$$

where $\Sigma_{i,n}=\left\{(t,\varphi_i(t))\in\mathbb{R}^2:\frac{1}{n}<t<T\right\}\times\prod_{k=1}^{N-1}]0,b_k[,\ i=1,\ 2,\ \Sigma_{0,n}$ is the part of the boundary of Q_n where $x_k=0,\ k=2,...,N$, and $\Sigma_{b,n}$ is the part of the boundary of Q_n where $x_k=b_{k-1},\ k=2,...,N$. Such a solution u_n exists by Theorem 2.1. Let $\widetilde{u_n}$ the 0-extension of u_n to Q. In virtue of Proposition 3.1, we know that there exists a constant C such that

$$\left\|\widetilde{u_n}\right\|_{L_\omega^2(Q)}+\left\|\widetilde{\partial_t u_n}\right\|_{L_\omega^2(Q)}+\sum_{\substack{i_1,i_2,...,i_N=0\\1\le i_1+i_2+...+i_N\le 4}}^{4}\left\|\widetilde{\partial_{x_1}^{i_1}\partial_{x_2}^{i_2}...\partial_{x_N}^{i_N}u_n}\right\|_{L_\omega^2(Q)}\ \le\ C\left\|f\right\|_{L_\omega^2(Q)}.$$

This means that $\widetilde{u_n}$, $\widetilde{\partial_t u_n}$, $\widetilde{\partial_{x_1}^{i_1}\partial_{x_2}^{i_2}...\partial_{x_N}^{i_N}u_n}$ for $1\le i_1+i_2+...+i_N\le 4$ are bounded functions in $L_\omega^2(Q)$. The following compactness result is well known: A bounded sequence in a reflexive Banach space (and in particular in a Hilbert space) is weakly convergent. So for a suitable increasing sequence of integers n_k, $k=1,2,...$, there exist functions u, v and $v_{i_1,i_2,...,i_N}$ $1\le i_1+i_2+...+i_N\le 4$ in $L_\omega^2(Q)$ such that

$$\widetilde{u_{n_k}}\rightharpoonup u,\ \widetilde{\partial_t u_{n_k}}\rightharpoonup v,\ \widetilde{\partial_{x_1}^{i_1}\partial_{x_2}^{i_2}...\partial_{x_N}^{i_N}u_{n_k}}\rightharpoonup v_{i_1,i_2,...,i_N},\ 1\le i_1+i_2+...+i_N\le 4$$

weakly in $L_\omega^2(Q)$ as $k\to\infty$. Clearly,

$$v=\partial_t u,\ v_{i_1,i_2,...,i_N}=\partial_{x_1}^{i_1}\partial_{x_2}^{i_2}...\partial_{x_N}^{i_N}u,\ 1\le i_1+i_2+...+i_N\le 4$$

in the sense of distributions in Q and so in $L_\omega^2(Q)$. So, $u\in H_\omega^{1,4}(Q)$ and

$$\partial_t u+\sum_{k=1}^{N}\partial_{x_k}^4 u=f \text{ in } Q.$$

On the other hand, the solution u satisfies the boundary conditions

$$u|_{t=0}=u|_{\Sigma_i}=\partial_{x_1}u|_{\Sigma_i}=0,\ i=1,\ 2$$

and

$$u|_{\Sigma_0\cup\Sigma_b}=\partial_{x_2}u|_{\Sigma_0\cup\Sigma_b}=...=\partial_{x_N}u|_{\Sigma_0\cup\Sigma_b}=0,$$

since

$$\forall n\in\mathbb{N}^*,\qquad u|_{Q_n}=u_n.$$

This proves the existence of solution to Problem (1). This ends the proof of Theorem 4.1. □

4.2. **Global in time result.** In the case where T is not in the neighborhood of zero, we set $Q = D_1 \cup D_2 \cup \Sigma_{T_1}$ where

$$D_1 = \{(t, x_1, ..., x_N) \in Q : 0 < t < T_1\},$$
$$D_2 = \{(t, x_1, ..., x_N) \in Q : T_1 < t < T\},$$
$$\Sigma_{T_1} = \left\{(T_1, x_1) \in \mathbb{R}^2 : \varphi_1(T_1) < x_1 < \varphi_2(T_1)\right\} \times \prod_{i=1}^{N-1}]0, b_i[$$

with T_1 small enough. In the sequel, f stands for an arbitrary fixed element of $L^2_\omega(Q)$ and $f_i = f|_{D_i}$, $i = 1, 2$.

Theorem 4.1 applied to the non-regular domain D_1, shows that there exists a unique solution $v_1 \in H^{1,4}_\omega(D_1)$ of the problem

$$\begin{cases} \partial_t v_1 + \sum_{k=1}^N \partial^4_{x_k} v_1 = f_1 \in L^2_\omega(D_1), \\ v_1|_{t=0} = 0, \\ v_1|_{\Sigma_{i,1}} = \partial_{x_1} v_1|_{\Sigma_{i,1}} = 0, \ i = 1, 2, \\ v_1|_{\Sigma_{0,1}\cup\Sigma_{b,1}} = \partial_{x_2} v_1|_{\Sigma_{0,1}\cup\Sigma_{b,1}} = ... = \partial_{x_N} v_1|_{\Sigma_{0,1}\cup\Sigma_{b,1}} = 0, \end{cases} \tag{10}$$

$\Sigma_{i,1}$ are the parts of the boundary of D_1 where $x_1 = \varphi_i(t)$, $i = 1, 2$, $\Sigma_{0,1}$ is the part of the boundary of D_1 where $x_k = 0$, $k = 2, ..., N$ and $\Sigma_{b,1}$ is the part of the boundary of D_1 where $x_k = b_{k-1}$, $k = 2, ..., N$.

Hereafter, we denote the trace $v_1|_{\Sigma_{T_1}}$ by ψ which is in the Sobolev space $H^2_\omega(\Sigma_{T_1})$ because $v_1 \in H^{1,4}_\omega(D_1)$ (see [20]). Now, consider the following problem in D_2

$$\begin{cases} \partial_t v_2 + \sum_{k=1}^N \partial^4_{x_k} v_2 = f_2 \in L^2_\omega(D_2), \\ v_2|_{\Sigma_{T_1}} = \psi, \\ v_2|_{\Sigma_{i,2}} = \partial_{x_1} v_2|_{\Sigma_{i,2}} = 0, \ i = 1, 2, \\ v_2|_{\Sigma_{0,2}\cup\Sigma_{b,2}} = \partial_{x_2} v_2|_{\Sigma_{0,2}\cup\Sigma_{b,2}} = ... = \partial_{x_N} v_2|_{\Sigma_{0,2}\cup\Sigma_{b,2}} = 0, \end{cases} \tag{11}$$

$\Sigma_{i,2}$ are the parts of the boundary of D_2 where $x_1 = \varphi_i(t)$, $i = 1, 2$, $\Sigma_{0,2}$ is the part of the boundary of D_2 where $x_k = 0$, $k = 2, ..., N$ and $\Sigma_{b,2}$ is the part of the boundary of D_2 where $x_k = b_{k-1}$, $k = 2, ..., N$.

We use the following result, which is a consequence of [20, Theorem 4.3, Vol.2] to solve Problem (11).

Proposition 4.1. *Let R be the cylinder $]0, T[\times]0, 1[\times \prod_{i=1}^{N-1}]0, b_i[$, $f \in L^2_\omega(R)$ and $u_0 \in H^2_\omega(\gamma_0)$. Then, the problem*

$$\begin{cases} \partial_t u + \sum_{k=1}^N \partial^4_{x_k} u = f \text{ in } R, \\ u|_{\gamma_0} = u_0, \\ u|_{\gamma_i} = \partial_{x_1} u|_{\gamma_i} = 0, \ i = 1, 2, \\ u|_{\partial R-(\gamma_0\cup\gamma_i)} = \partial_{x_2} u|_{\partial R-(\gamma_0\cup\gamma_i)} = ... = \partial_{x_N} u|_{\partial R-(\gamma_0\cup\gamma_i)} = 0, \ i = 1, 2, \end{cases}$$

where $\gamma_0 = \{0\} \times]0, 1[\times \prod_{i=1}^{N-1}]0, b_i[$, $\gamma_1 =]0, T[\times \{0\} \times \prod_{i=1}^{N-1}]0, b_i[$ and $\gamma_2 =]0, T[\times \{1\} \times \prod_{i=1}^{N-1}]0, b_i[$, admits a(unique) solution $u \in H^{1,4}_\omega(R)$ if and only if the following compatibility conditions are fulfilled

$$\partial^k_{x_j} u_0\Big|_{\partial\gamma_0} = 0, \ k = 0, 1; \ j = 1, ..., N.$$

The transformation

$$(t, x_1, x_2, ..., x_N) \longmapsto (t, y_1, y_2, ..., y_N) = (t, (\varphi_2(t) - \varphi_1(t))x_1 + \varphi_1(t), x_2, ..., x_N)$$

leads to the following result:

Proposition 4.2. *Problem (11) admits a (unique) solution $v_2 \in H_{\omega}^{1,4}(D_2)$ if and only if the following compatibility conditions are fulfilled*

$$\partial_{x_j}^k \psi\Big|_{\partial\Sigma_{T_1}} = 0,\ k = 0,1;\ j = 1,...,N.$$

Remark 4.1. *We can observe that the boundary conditions of Problems (10) and (11) yield*

$$v_1|_{\Sigma_{T_1}} = v_2|_{\Sigma_{T_1}}$$

and $\partial_{x_j}^k v_i\Big|_{\Sigma_{T_1}} \in H_{\omega}^{\frac{3}{4}}(\Sigma_{T_1});\ k = 0,1;\ j = 1,...,N.$. *Then the compatibility conditions*

$$\partial_{x_j}^k \psi\Big|_{\partial\Sigma_{T_1}} = 0,\ k = 0,1;\ j = 1,...,N$$

are satisfied since $v_1|_{\Sigma_{T_1}} = \psi$.

Now, consider the function u in Q defined by

$$u := \begin{cases} v_1 \text{ in } D_1 \\ v_2 \text{ in } D_2 \end{cases}$$

where v_1 and v_2 are the solutions of Problem (10) and Problem (11) respectively. Observe that $v_1|_{\Sigma_{T_1}} = v_2|_{\Sigma_{T_1}}$, see Remark 4.1, so

$$\partial_{x_j}^k v_1\Big|_{\Sigma_{T_1}} = \partial_{x_j}^k v_2\Big|_{\Sigma_{T_1}},\ k = 0,1;\ j = 1,...,N.$$

This implies that $u \in H_{\omega}^{1,4}(Q)$ and u is the (unique) solution of Problem (1) for an arbitrary T.

Our second main result is as follows.

Theorem 4.2. *Under the assumptions (2), (3) and (4) on the functions φ_1, φ_2 and ω, Problem (1) admits a (unique) solution $u \in H_{\omega}^{1,4}(Q)$.*

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Arezki Kheloufi is a teacher researcher in the Department of Technology, Bejaia University, Algeria. He obtained his HDR in the year 2014. He has attended to national and international talks and conferences. His research interests are: Parabolic equations in non regular domains, the operator?s sum method, singularities in BVP. He has publications in various journals like: Electronic journal of differential equations, applied mathematics and computation, mathematical methods in the applied sciences, Georgian mathematical journal, mediterranean journal of mathematics and differential equations and applications.