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MULTIPLE SEQUENCES IN CONE METRIC SPACES

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Abstract. In this study, we introduce the ordinary and statistical convergence of double and multiple sequences in cone metric spaces. Moreover, the relationships between these convergence types are also invastigated.

Keywords: cone metric, multiple sequence, convergence, statistical convergence.

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1. Introduction

Cone metric spaces (CMS) are one of the generalizations of metric spaces. This generalization is made by means of a partial ordering $\preceq$ on a Banach spaces $(E, \|\cdot\|, P)$ via cone P. In 2007, the definition of CMS was given by Huang and Zhang [1]. Moreover, Huang and Zhang defined the convergence and being a Cauchy sequence via interior points of the cone in ordered Banach space E [2]. In light of this study, some fixed point theorems were proved by the other authors [3, 4, 5, 6, 7]. In the paper [8], Du introduced the concept of the CMS over topological vector space (*tvs* for short) for the first time and showed that a kind of equivalence can be established between metric and cone metric spaces by using the nonlinear scalarization function. Recently, studying on CMS over *tvs* has become more popular among the authors who are interested in this subject [9, 10].

Being a Cauchy sequence and convergence of single sequences in CMS were first introduced in [1]. But these subjects haven't been studied for double sequences yet. Subsection 3.1 of this study is devoted to introduce the ordinary convergence of double sequences.

As is well known, to extend the set of convergent sequences, Fast and Schoenberg independently introduced the concept of statistical convergence [13, 14]. Since then, this concept was studied by Šalát [15], Fridy [16] and Connor [17] and many others [18, 19, 11, 12].

The concept "statistical convergence" was first introduced for double sequences by Mursaleen & Edely and, Móricz, independently [20, 21]. At the same time, Móricz investigated it for multiple sequences [21]. The concept of statistical convergence hasn't also been studied in CMS yet.

In this paper, statistical convergence of double sequences in CMS and some basic results are introduced, for the first time. Finally, we introduce the ordinary and statistical convergence for $n-$tuple sequences in CMS in the last section of the paper.

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2. Preliminaries

In this section we review some basic definitions and theorems about CMS, ordinary and statistical convergence of double sequences.

Let E be a Hausdorff *tvs* with the zero vector θ. $P \subset E$ is called (convex) cone the following conditions hold:

i. $P \neq \{\theta\}$, P is non-empty and closed,
ii. $P + P \subset P$ and $\lambda P \subset P$ for all non-negative λ,
iii. $P \cap (-P) = \{\theta\}$.

The partial ordering $\preceq$ is defined with respect to P by $x \preceq y$ if and only if $y - x \in P$, for any cone $P \subset E$. And $x \prec y$ will stand for $x \preceq y$ and $x \neq y$, $x \ll y$ will stand for $y - x \in \mathrm{int}P$, where $\mathrm{int}P$ denotes the interior of P. For $x, y \in E$ such that $x \preceq y$, the set *order-intervals* is defined as the following:

$$[x, y] = \{z \in E : x \preceq z \preceq y\}.$$

The *order-intervals* are convex sets and $A \subset E$ is called *order-convex* if $[x, y] \subset A$ whenever $x, y \in A$ and $x \preceq y$.

Ordered tvs (E, P) is called order-convex if it has a base of *order-convex* neighborhoods of θ. In this case, the cone P is called normal. In the case of a normed space this condition is equivalent to the following: There is a number M such that $x, y \in E$ and $\theta \preceq x \preceq y$ implies that $\|x\| \leq M\|y\|$. The smallest constant M is called the normal constant of P [1, 9].

Throughout the paper, we always assume $d : X \times X \to E$ is a Banach space (as a special Hausdorff *tvs*) valued cone metric, (X, d) is a CMS, $P \subset E$ is a cone with in $P \neq \emptyset$ and $\preceq$ is partial ordering with respect to P.

Definition 2.1. [1, 8, 9] *Let $X \neq \emptyset$ and the mapping $d : X \times X \to E$ satisfies*

(d_1) *$\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$,*
(d_2) *$d(x, y) = d(y, x)$ for all $x, y \in X$,*
(d_3) *$d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.*

Then d is called a cone metric on X, and (X, d) is called a CMS.

Example 2.1. *Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = \mathbb{R}$ and $d(x, y) = (\alpha|x - y|, \beta|x - y|)$, where $\alpha, \beta \geq 0$ is a constant. Then, it is easily seen that (X, d) is a CMS.*

In [22], the concept of convergence for double sequences was first introduced by Pringsheim. A double sequence $\{x_{ij}\}_{i,j\in\mathbb{N}} = \{x_{ij}\}$ is said to be convergent in Pringsheim sense if for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x_{ij} - L| < \varepsilon$ for every $i, j \geq N$. In this case, L is called the Pringsheim limit of $\{x_{ij}\}$.

Recall that a subset K of the set $\mathbb{N} \times \mathbb{N}$ is said to have double natural density

$$\delta_2(E) = \lim_{n,m\to\infty} \frac{1}{mn} \left|\{i \leq m \text{ and } j \leq n : (n, m) \in K\}\right|,$$

where vertical bars denote the cardinality of enclosed set.

Definition 2.2. [20, 21] *$\{x_{ij}\}$ is said to be statistically convergent to ξ if for each $\varepsilon > 0$, the set*

$$\{(i, j),\ i \leq m \textit{ and } j \leq n : |x_{ij} - \xi| \geq \varepsilon\}$$

has double natural density zero.

3. Main Results

3.1. Convergence (Ordinary) of Double Sequence.

Definition 3.1. *$\{x_{ij}\}$ in (X,d) is called convergent to ξ if for each $c \in E$ with $\theta \ll c$ there exists $N \in \mathbb{N}$ such that $d(x_{ij},\xi) \ll c$ for all $i,j \geq N$. ξ is called the limit of the sequence $\{x_{ij}\}_{i,j\in\mathbb{N}}$.*

Lemma 3.1. *Suppose that (X,d) is CMS, P is a normal cone with M and $\{x_{ij}\}$ be a double sequence in X. Then $\{x_{ij}\}$ converges to ξ if and only if $d(x_{ij},\xi) \to \theta$ $(i,j \to \infty)$.*

Proof. Let $\{x_{ij}\}$ converges to ξ. For every $\varepsilon > 0$, there exits $c \in E$ such that $\theta \ll c$ and $M\|c\| < \varepsilon$. Then, there is $N \in \mathbb{N}$, such that for all $i,j \geq N$, $d(x_{ij},\xi) \ll c$. When $i,j > N$, $\|d(x_{ij},\xi)\| \leq M\|c\| < \varepsilon$. This means $d(x_{ij},\xi) \to \theta$ $(i,j \to \infty)$.

Conversely, $d(x_{ij},\xi) \to \theta$ as $i,j \to \infty$. For $c \in E$ with $\theta \ll c$, there is $\delta > 0$ such that $\|t\| < \delta$ implies $c - t \in \text{int}P$. For this δ there exists $N \in \mathbb{N}$, such that $\forall i,j \geq N$, $\|d(x_{ij},\xi)\| < \delta$. Thus, $c - d(x_{ij},\xi) \in \text{int}P$ and $d(x_{ij},\xi) \ll c$. Hence $\{x_{ij}\}$ converges to ξ. □

Theorem 3.1. *If $\{x_{ij}\}$ is convergent in (X,d) and P is a normal cone with M then the limit is unique.*

Proof. For every $c \in E$ with $\theta \ll c$, there exits $N \in \mathbb{N}$ such that $i,j > N \Rightarrow d(x_{ij},\xi) \ll c$ and $d(x_{ij},\eta) \ll c$. We have

$$d(\xi,\eta) \preceq d(x_{ij},\xi) + d(x_{ij},\eta) \preceq 2c.$$

Thus, $\|d(\xi,\eta)\| \leq 2M\|c\|$. Since c is arbitrary $d(\xi,\eta) = 0$; therefore $\xi = \eta$. □

Definition 3.2. *$\{x_{ij}\}$ is called bounded in (X,d) if there exists $\theta \ll K$ such that $d(x_{ij},0) \preceq K$ for all $i,j \in \mathbb{N}$.*

Remark 3.1. *On the contrary to the case for single sequences, convergent double sequences need not to be bounded in CMS.*

Example 3.1. *Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \left(\mathbb{R}^2, \|\cdot\|, P\right)$ with $P = \left\{(x,y) \in \mathbb{R}^2 : x,y \geq 0\right\}$ be a cone metric on $\mathbb{R}^2$. Define*

$$x_{ij} = \begin{cases} (1,j) & , \text{if } i = 2, j \in \mathbb{N} \\ \left(\frac{1}{i+j}, \frac{2}{i^2+j^2}\right) & , \text{otherwise} \end{cases} . \tag{1}$$

It is easily seen that $\{x_{ij}\}$ converges to point $(0,0)$ but this sequence is not bounded in CMS.

Definition 3.3. *$\{x_{ij}\}$ is called double Cauchy sequence in (X,d) if and only if for every $c \gg 0$, there exists $N \in \mathbb{N}$ such that*

$$d(x_{ij},x_{nm}) \ll c, \quad \forall i \geq n \geq N \text{ and } \forall j \geq m \geq N.$$

Theorem 3.2. *Let $\{x_{ij}\}$ is convergent in (X,d). Then $\{x_{ij}\}$ is a Cauchy sequence.*

Proof. Let $\{x_{ij}\}$ converges to ξ as $n \to \infty$. Then, for any $c \in E$ with $\theta \ll c$, there is $N \in \mathbb{N}$ such that for all $i,j,n,m \geq N$, $d(x_{ij},\xi) \ll \frac{c}{2}$ and $d(x_{nm},\xi) \ll \frac{c}{2}$. Hence $d(x_{ij},x_{nm}) \preceq d(x_{ij},\xi) + d(x_{nm},\xi) \ll c$, then $\{x_{ij}\}$ is double Cauchy sequence. □

Lemma 3.2. *Let $\{x_{ij}\}$ be a double sequence in (X,d) and P be a normal cone with M. Then $\{x_{ij}\}$ is a Cauchy sequence if and only if $d(x_{ij},x_{nm}) \to \theta$ $(i,j,n,m \to \infty)$.*

Proof. Assume that $\{x_{ij}\}$ is Cauchy sequence. Then, for each $\varepsilon > 0$ there exists $c \in E$ with $\theta \ll c$ and $M\|c\| < \varepsilon$. So, there exists $N \in \mathbb{N}$ such that $\forall i, j, n, m \geq N$, $d(x_{ij}, x_{nm}) \ll c$. When $i, j, n, m > N$, $\|d(x_{ij}, x_{nm})\| \leq M\|c\| < \varepsilon$. This means $d(x_{ij}, \xi) \to \theta$ $(i, j \to \infty)$.

Conversely, $d(x_{ij}, x_{nm}) \to \theta$ as $i, j, n, m \to \infty$. For any $c \in E$ with $\theta \ll c$, there exists $\delta > 0$, such that $\|t\| < \delta$ implies $c - t \in \operatorname{int}P$. For this δ there exists $N \in \mathbb{N}$ such that for all $i, j, n, m \geq N$, $\|d(x_{ij}, x_{nm})\| < \delta$. Thus, $c - d(x_{ij}, x_{nm}) \in \operatorname{int}P$ and $d(x_{ij}, x_{n,m}) \ll c$. Hence $\{x_{ij}\}$ is a Cauchy sequence. □

Theorem 3.3. *Let $\{x_{ij}\}$ and $\{y_{ij}\}$ be two double sequences in (X, d), P be a normal cone with M and $x_{ij} \to \xi$, $y_{ij} \to \eta$. Then*

$$d(x_{ij}, y_{ij}) \to d(\xi, \eta) \text{ as } i, j \to \infty.$$

Proof. Foe each ε, choose $c \in E$ with $\theta \ll c$ and $\|c\| < \frac{\varepsilon}{4M+2}$. If $x_{ij} \to \xi$ and $y_{ij} \to \eta$ then there exists $N \in \mathbb{N}$ such that $\forall i, j > N$, $d(x_{ij}, \xi) \ll c$ and $d(y_{ij}, \eta) \ll c$. We have

$$\begin{aligned} d(x_{ij}, y_{ij}) &\preceq d(x_{ij}, \xi) + d(\xi, \eta) + d(y_{ij}, \eta) \preceq d(\xi, \eta) + 2c, \\ d(\xi, \eta) &\preceq d(x_{ij}, \xi) + d(x_{ij}, y_{ij}) + d(y_{ij}, \eta) \preceq d(x_{ij}, y_{ij}) + 2c. \end{aligned}$$

Thus

$$\theta \preceq d(\xi, \eta) + 2c - d(x_{ij}, y_{ij}) \preceq 4c,$$

and

$$\|d(x_{ij}, y_{ij}) - d(\xi, \eta)\| \leq \|d(\xi, \eta) + 2c - d(x_{ij}, y_{ij})\| + \|2c\| \leq (4M + 2)\|c\| < \varepsilon.$$

Therefore $d(x_{ij}, y_{ij}) \to d(\xi, \eta)$ as $i, j \to \infty$. □

3.2. **Statistical Convergence of Double Sequence.** Hereby, we introduce statistical convergence of double sequences in CMS.

Definition 3.4. *$\{x_{ij}\}$ in (X, d) is called statistically convergent to ξ, if for each $c \gg \theta$,*

$$\lim_{n,m\to\infty} \frac{1}{mn} |\{i \leq m \text{ and } j \leq n : d(x_{ij}, \xi) \gg c\}| = 0. \tag{2}$$

Definition 3.5. *$\{x_{ij}\}$ in (X, d) is called statistically bounded if there exists $C \in E$ with $C \gg \theta$ such that*

$$\lim_{n,m\to\infty} \frac{1}{mn} |\{i \leq m \text{ and } j \leq n : d(x_{ij}, 0) \gg C\}| = 0.$$

Definition 3.6. *$\{x_{ij}\}$ in (X, d) is called statistically Cauchy, if for each $\varepsilon > 0$ and $l \geq 0$ there exists $M \geq l$ and $N \geq l$ such that*

$$\lim_{n,m\to\infty} \frac{1}{mn} |\{i \leq m \text{ and } j \leq n : d(x_{ij}, x_{MN}) \gg c\}| = 0.$$

Theorem 3.4. *Let (X, d) be a CMS, P be a normal cone with M. If $\{x_{ij}\}$ is statistically convergent in (X, d) then $\{x_{ij}\}_{i,j\in\mathbb{N}}$ is statistically Cauchy.*

Proof. Let $\{x_{ij}\}$ be statistically convergent to ξ. Then for every $\varepsilon > 0$, the set

$$\delta_2^c(\{(i, j), i \leq m \text{ and } j \leq n : \|d(x_{ij}, \xi)\| \geq \varepsilon\}) = 0.$$

Choose m_ε and n_ε such that $\|d(x_{m_\varepsilon n_\varepsilon}, \xi)\| \geq \varepsilon$. Let

$$\begin{aligned} A_\varepsilon &= \{(i, j), i \leq m, j \leq n : \|d(x_{ij}, x_{m_\varepsilon n_\varepsilon})\| \geq \varepsilon\}, \\ B_\varepsilon &= \{(i, j), i \leq m, j \leq n : \|d(x_{ij}, \xi)\| \geq \varepsilon\}, \\ C_\varepsilon &= \{(i, j), i = m_\varepsilon \leq m, j = n_\varepsilon \leq n : \|d(x_{m_\varepsilon n_\varepsilon}, \xi)\| \geq \varepsilon\}. \end{aligned}$$

Then, $A_\varepsilon \subseteq B_\varepsilon \cup C_\varepsilon$ and therefore

$$\delta_2^c(A_\varepsilon) \leq \delta_2^c(B_\varepsilon) + \delta_2^c(C_\varepsilon) = 0.$$

Hence $\{x_{ij}\}$ is statistically Cauchy. $\square$

Theorem 3.5. *Let (X,d) be a CMS, P be a normal cone with M. Then $\{x_{ij}\}$ in (X,d) is statistically convergent to ξ if and only if there exists a subset $S = \{(i,j)\} \subset \mathbb{N}\times\mathbb{N}$ such that $\delta_2^c(S) = 1$ and*

$$\lim_{\substack{(i,j)\in S \\ i,j\to\infty}} x_{ij} = \xi.$$

That is, for every $c \gg \theta$ there exists $N \in \mathbb{N}$ such that $d(x_{ij},\xi) \ll c$ if $i,j \geq N$ and $(i,j) \in S$.

Proof. Let $\{x_{ij}\}_{i,j\in\mathbb{N}}$ be statistically convergent to ξ. Let

$$S_n = \left\{(i,j) \in \mathbb{N}\times\mathbb{N} : d(x_{ij},\xi) \gg c \Leftrightarrow \|d(x_{ij},\xi)\| \geq \frac{1}{n} \text{ such that } \|c\| \leq \frac{1}{n}\right\},$$

and

$$T_n = \left\{(i,j) \in \mathbb{N}\times\mathbb{N} : d(x_{ij},\xi) \ll c \Leftrightarrow \|d(x_{ij},\xi)\| < \frac{1}{n} \text{ such that } \|c\| \leq \frac{1}{n}\right\}, \quad n \in \mathbb{N}.$$

Then $\delta_2^c(S_n) = 0$,

$$T_1 \supset T_2 \supset \cdots \supset T_i \supset T_{i+1} \supset \cdots \tag{3}$$

and hence

$$\delta_2^c(T_n) = 1, \quad n = 1,2,\cdots. \tag{4}$$

To complete necessity side of the proof we have to show that for $(i,j) \in T_n$, $\{x_{ij}\}$ is convergent to ξ. Suppose that $\{x_{ij}\}$ is not convergent to ξ. Thus, there is ε such that the inequality $\|d(x_{ij},\xi)\| \geq \varepsilon$ holds for infinitely many terms. Let

$$T_\varepsilon = \{(i,j) : \|d(x_{ij},\xi)\| < \varepsilon\} \text{ and } \varepsilon > \frac{1}{n} \quad (n = 1,2,\cdots).$$

Then $\delta_2^c(T_\varepsilon) = 0$ and by (3), $T_n \subset T_\varepsilon$. Hence $\delta_2^c(T_n) = 0$ which contradicts with (4). Therefore $\{x_{ij}\}$ is convergent to ξ.

Conversely, assume that there exists a subset $S = \{(i,j)\} \subset \mathbb{N}\times\mathbb{N}$ such that $\delta_2^c(S) = 1$ and $\lim_{i,j\to\infty} x_{ij} = \xi$. Then there exists $N \in \mathbb{N}$ such that for every $\varepsilon > 0$,

$$\|d(x_{ij},\xi)\| < \varepsilon, \quad \forall i,j \geq N.$$

Now set

$$K_\varepsilon = \{(i,j) : \|d(x_{ij},\xi)\| \geq \varepsilon\} \subset \mathbb{N}\times\mathbb{N} - \{(i_{N+1},j_{N+1}),(i_{N+2},j_{N+2}),\dots\}.$$

We get

$$\delta_2^c(K_\varepsilon) \leq 1 - 1 = 0.$$

Hence $\{x_{ij}\}$ is convergent to ξ. $\square$

Theorem 3.6 (Decomposition Theorem)**.** *Let $\{x_{ij}\}$ in CMS (X,d) is statistically convergent to ξ if and only if there exists two sequence $\{u_{ij}\}$ and $\{v_{ij}\}$ such that*

$$x_{ij} = u_{ij} + v_{ij}, i,j \in \mathbb{N}, \tag{5}$$

$$\lim_{i,j\to\infty} u_{ij} = \xi, \tag{6}$$

and

$$\lim_{m,n\to\infty} \frac{1}{mn} |\{i \leq m \text{ and } j \leq n : v_{ij} \neq 0\}| = 0. \tag{7}$$

Moreover, if $\{x_{ij}\}$ is bounded then $\{u_{ij}\}$ and $\{v_{ij}\}$ are also bounded.

Proof. Necessity: Let a double sequence $\{x_{ij}\}$ is convergent to ξ in CMS (X, d). By Theorem 3.5, there exists a set $S \subseteq \mathbb{N} \times \mathbb{N}$ such that $\delta_2^c(S) = 1$ and $\lim\limits_{\substack{(i,j)\in S \\ i,j\to\infty}} x_{ij} = \xi$. Now construct the sequences u_{ij} and v_{ij} as follows:

$$u_{ij} = \begin{cases} x_{ij} & \text{if } (i,j) \in S \\ \xi & \text{if otherwise} \end{cases}$$

and

$$v_{ij} = \begin{cases} 0 & \text{if } (i,j) \in S \\ x_{ij} - \xi & \text{otherwise} \end{cases}.$$

It is easy to see that the condition (5) holds and (6) is true. In fact, for every $c \in E$ with $c \gg 0$

$$d(u_{ij}, \xi) = \begin{cases} d(x_{ij}, \xi) \ll c & \text{if } (i,j) \in S \\ d(\xi, \xi) = 0 & \text{otherwise} \end{cases}.$$

It remains to prove (3). Since $\delta_2^c(\mathbb{N} \times \mathbb{N} \setminus S) = \delta_2^c(\{(i,j) \in \mathbb{N} \times \mathbb{N} : v_{ij} \neq 0\}) = 0$, the result is obtained.

Sufficiency: It is straightforward. Indeed, from (3), statistical limit of $\{v_{ij}\}$ is 0. By (6) and considering the above facts the result follows via additivity. □

4. Generalization to Multiple Sequences

In this section, we introduce some results for $n-$tuple sequences in CMS in analogy to F. Moricz [21]. The notions and results of the previous sections can be extended to $n-$tuple sequences. Let $\mathbb{N}^n$ be the set of $n-$tuples $\mathbf{i} := (i_1, i_2, \ldots, i_n)$ with nonnegative integers for coordinates i_j, where n is a fixed positive integer. $\mathbf{i}$ and $\mathbf{m} := (m_1, m_2, \ldots, m_n)$ are distinct if and only if $i_j \neq m_j$ for at least one j. $\mathbb{N}^n$ is partially ordered by agreeing that $\mathbf{i} \leq \mathbf{m}$ if and only if $i_j \leq m_j$ for each $j = 1, \ldots, n$. We say that a n-tuple sequence $\{x_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{N}^n} = \{x_{\mathbf{i}}\}$ is statistically convergent to ξ in CMS if for each $c \gg 0$,

$$\lim_{\min_j\{m_j\}\to\infty} \frac{1}{|\mathbf{m}|} |\{\mathbf{i} \leq \mathbf{m} : d(x_{\mathbf{i}}, \xi) \gg c\}| = 0, \tag{8}$$

where

$$|\mathbf{m}| := \prod_{j=1}^{n} m_j.$$

Furthermore, we say that $\{x_{\mathbf{i}}\}$ is statistically Cauchy if for each $c \in E$, $c \gg 0$ there exists $\mathbf{k} := (k_1, k_2, \ldots, k_n) \in \mathbb{N}^n$ such that $\min\{m_j\} \geq l$ and

$$\lim_{\min\limits_j\{m_j\}\to\infty} \frac{1}{|\mathbf{m}|} |\{\mathbf{i} \leq \mathbf{m} : d(x_{\mathbf{i}}, x_{\mathbf{k}}) \gg c\}| = 0. \tag{9}$$

The natural density of a set $S \subseteq \mathbb{N}^n$ is defined as the following:

$$\delta_n^c(S) := \lim_{\min\limits_j\{m_j\}\to\infty} \frac{1}{|\mathbf{m}|} |\{\mathbf{i} \leq \mathbf{m} : \mathbf{i} \in S\}|$$

provided that this limit exists.

Subsequently, we give the results about statistical convergence for $n-$tuple sequences in CMS which are just generalizations of the results obtained above for double sequences, without proofs.

Theorem 4.1. *If $\{x_{\boldsymbol{i}}\}$ is statistically convergent in CMS (X, d), then $\{x_{\boldsymbol{i}}\}$ is statistically Cauchy.*

Theorem 4.2. *Let (X, d) be CMS and P be a normal cone with M. Then $\{x_{\boldsymbol{i}}\}$ in (X, d) is statistically convergent to ξ if and only if there exists a subset $S = \{\boldsymbol{i}\} \subset \mathbb{N} \times \mathbb{N}$ such that $\delta_n^c(S) = 1$ and*

$$\lim_{\substack{\boldsymbol{i} \in S \\ \min_j \{i_j\} \to \infty}} x_{\boldsymbol{i}} = \xi.$$

Theorem 4.3 (Decomposition Theorem)**.** *$\{x_{\boldsymbol{i}}\}$ in CMS (X, d) is statistically convergent to ξ if and only if there exists two sequence $\{u_{\boldsymbol{i}}\}_{\boldsymbol{i} \in \mathbb{N}}$ and $\{v_{\boldsymbol{i}}\}_{\boldsymbol{i} \in \mathbb{N}}$ such that*

$$x_{\boldsymbol{i}} = u_{\boldsymbol{i}} + v_{\boldsymbol{i}}, \boldsymbol{i} \in \mathbb{N}^n, \tag{10}$$

$$\lim_{\min_j \{i_j\} \to \infty} u_{\boldsymbol{i}} = \xi, \tag{11}$$

and

$$\lim_{\min_j \{m_j\} \to \infty} \frac{1}{|\boldsymbol{m}|} |\{\boldsymbol{i} \leq \boldsymbol{m} : v_{\boldsymbol{m}} \neq 0\}| = 0. \tag{12}$$

Moreover, if $\{x_{\boldsymbol{i}}\}$ is bounded then $\{u_{\boldsymbol{i}}\}$ and $\{v_{\boldsymbol{i}}\}$ are also bounded.

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