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ON CONTROLLED POISSON PROCESSES

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Abstract. We consider a special class of two-dimensional Markov processes, finding the relationship between transition probabilities of two such classes.

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1. Introduction

In this paper, we consider Markov processes $\{\alpha_t, n_t\}$, $t \geq 0$ with homogeneous second component, where at fixed α_t, process n_t is a conditioned Poisson process. Definitions and basic properties of Markov processes with homogeneous second component have been investigated in [3] and [4]. The processes under our investigation are quite useful in the study of service systems with n unreliable components, when a non-ordinary Poisson queue stream.

By a controlled unbounded Poisson process, we understand a Markov process $\{\alpha_t, n_t\}$, $t \geq 0$ with homogeneous second component in the phase space $T \times N$, where $T = \{\alpha, \beta, ...\}$ is a finite set and $N = \{0, \pm 1, \pm 2, ...\}$.

Let

$$P^k_{\alpha\beta}(t, s) = P\{\alpha_s = \beta, n_s = k + r/\alpha_t = \alpha, n_t = r\},$$

$$(\alpha, \beta \in T; k, k \in N; s \geq t \geq 0).$$

Then let us assume that the bounds

$$\lim_{s \downarrow t} \frac{P^k_{\alpha\beta}(t, s) - \delta_{\alpha\beta}\delta_{ko}}{s - t} = q^k_{\alpha\beta}(t), \quad (\alpha, \beta \in T; k \in N; t \geq 0).$$

exist and are continuous in t. By virtue of the equation

$$\sum_N \sum_T q^k_{\alpha\beta}(t) \equiv 0, \qquad (\alpha \in T; t \geq 0),$$

the functions $q^k_{\alpha\beta}(t)$ are uniformly bounded on α, β, k in any finite run of t.

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$$P_{\alpha\beta}\left(t,\, s,\, \theta\right) = \sum_N P^k_{\alpha\beta}\left(t, s\right)\theta^k, \quad P\left(t,\, s,\, \theta\right) = \left\| P_{\alpha\beta}\left(t, s, \theta\right)\right\|,$$

$$q_{\alpha\beta}\left(t,\, \theta\right) = \sum_N q^k_{\alpha\beta}\left(t\right)\theta^k, \quad Q\left(t,\, \theta\right) = \left\| q_{\alpha\beta}\left(t, \theta\right)\right\|,$$

$$P_k\left(t,\, s\right) = \left\| P^k_{\alpha\beta}\left(t, s\right)\right\|, \qquad Q_k\left(t\right) = \left\| q^k_{\alpha\beta}\left(t\right)\right\|.$$

According to the general theory of Markov processes with homogeneous second component,

$$\frac{\partial P\left(t, s, \theta\right)}{\partial s} = P\left(t, s, \theta\right) Q\left(s, \theta\right), \quad \frac{\partial P\left(t, s, \theta\right)}{\partial t} = -Q\left(t, \theta\right) P\left(t, s, \theta\right),$$

$$\left. P\left(t,\, s,\, \theta\right)\right|_{s=t} = I = \left\|\delta_{\alpha\beta}\right\|. \tag{1}$$

A multiplicative integral, i.e. a matricient [2] seems to be a general solution to forward and backward equations (1):

$$P\left(t,\, s,\, \theta\right) = \Omega^s_t\left(Q\left(u,\, \theta\right)\right),$$

where

$$\Omega^s_t\left(Q\left(u,\, \theta\right)\right) = \lim_{n\to\infty}\prod_{k=0}^{n}\left(I + \frac{s-t}{n} Q\left(t + \frac{k}{n}\left(s-t\right), \theta\right)\right).$$

Let us assume that with probability 1, $n_{t+0} - n_{t-0} \geq -2$, $t > 0$. It means that with probability 1, process n_t has no negative jumps different from -1, therefore,

$$q^k_{\alpha\beta}\left(t\right) = 0, \qquad \left(t \geq 0;\;\; \alpha, \beta \in T; k \leq -2\right).$$

Such processes in the case of integer-valued phase are naturally called "downward" continuous processes [1].

By a controlled bounded Poisson process, we understand a Markov chain $\{\beta_t,\, m_t\}$, $t \geq 0$ in the phase space $T \times N^+$, where $N^+ = \{0,\, 1,\, 2,\, ...\}$ and with the following transition probabilities in the small interval $(t,\;t+\Delta)$:

$$P\left\{\left(\alpha,\, k\right)\overset{(t,t+\Delta)}{\to}\left(\beta,\, r\right)\right\} = \delta_{\alpha\beta}\delta_{kr} +$$

$$+\begin{cases} q^{r-k}_{\alpha\beta}\left(t\right)\Delta + o\left(\Delta\right), k \geq c,\, r \geq k-1, \\ \pi^{kr}_{\alpha\beta}\left(t\right)\Delta + o\left(\Delta\right),\, 0 \leq k \leq c-1,\, r \geq 0. \end{cases} \tag{2}$$

where c is a fixed natural number and $\pi^{kr}_{\alpha\beta}\left(t\right)$ are continuous in t function and bounded by the relation

$$\sum_{r=0}^{\infty}\sum_{\beta\in T}\pi^{kr}_{\alpha\beta}\left(t\right) \equiv 0, \quad \left(t \geq 0;\;\; \alpha \in T;\, 0 \leq k \leq c-1\right).$$

It follows from (2) that as long as $m_t \geq c$, the increment of process $\{\beta_t,\, m_t\}$ is a stochastic equivalent to the increment of process $\{\alpha_t,\, n_t\}$. If $m_t \in [0,\, c-1]$, then the evolution of process $\{\beta_t,\, m_t\}$ is described by an auxiliary Markov chain with local transition probabilities $\pi^{kr}_{\alpha\beta}\left(t\right)$.

Using the transition probabilities

$$f_{\alpha\beta}^{kr}(t,\, s) = P\left\{\beta_s = \beta,\, m_s = r/\beta_t = \alpha,\, m_t = k\right\}$$

and local characteristics of $\pi_{\alpha\beta}^{kr}(t)$, let us introduce the matrices:

$$F_{kr}(t,\, s) = \left\| f_{\alpha\beta}^{kr}(t, s) \right\|,\ F_k(t,\, s,\, \theta) = \left\| f_{\alpha\beta}^{k}(t,\, s,\, \theta) \right\|,$$

$$\Pi_{kr}(t) = \left\| \pi_{\alpha\beta}^{kr}(t) \right\|, \quad \Pi_k(t,\, \theta) = \left\| \pi_{\alpha\beta}^{k}(t,\, \theta) \right\|$$

and the generating function

$$\pi_{\alpha\beta}^{kr}(t,\theta) = \sum_{r=0}^{\infty} \pi_{\alpha\beta}^{kr}(t)\, \theta^r \quad , \quad |\theta| \le 1.$$

Our goal is to find the connection between the transition probabilities of the processes $\{\alpha_t,\, n_t\}$ and $\{\beta_t,\, m_t\}$.

2. Main results

Using (2), at $\Delta \downarrow 0$ we have

$$f_{\alpha\beta}(t, s+\Delta) = f_{\alpha\beta}^{kr}(t,s) + \Delta \sum_{j=0}^{c-1} \sum_{\gamma\in T} f_{\alpha\beta}^{kj}(t,s)\, \pi_{\gamma\beta}^{jr}(s) +$$

$$+ \sigma\left\{r \ge c-1\right\} \Delta \sum_{j=c}^{r+1} \sum_{\gamma\in T} f_{\alpha\beta}^{kj}(t,s)\, q_{\gamma\beta}^{r-j}(s) + o(\Delta),$$

where

$$\sigma\left\{r \ge c-1\right\} = \begin{cases} 1, & if \quad r \ge c-1, \\ 0, & if \quad r < c-1. \end{cases}$$

Proceeding here to the bound at $\Delta \downarrow 0$ we get a forward system of differential Kolmogorov equations for transition probabilities $f_{\alpha\beta}^{kj}(t,s)$:

$$\frac{\partial f_{\alpha\beta}^{kr}(t,s)}{\partial s} = \sum_{j=0}^{c-1} \sum_{\gamma\in T} f_{\alpha\beta}^{kj}(t,s)\, \pi_{\gamma\beta}^{jr}(s) + \sigma\left\{r \ge c-1\right\} \sum_{j=c}^{r+1} \sum_{\gamma\in T} f_{\alpha\beta}^{kj}(t,s)\, q_{\gamma\beta}^{r-j}(s),$$

$$\left(\alpha,\beta \in T; \quad k,r \in N^+; \quad s \ge t \ge 0\right).$$

or in generating functions

$$\frac{\partial f_{\alpha\beta}^{kr}(t,s,\theta)}{\partial s} = \sum_{j=0}^{c-1} \sum_{\gamma\in T} f_{\alpha\beta}^{kj}(t,s)\, \pi_{\gamma\beta}^{jr}(s,\theta) +$$

$$+ \sum_{j=c}^{\infty} \sum_{\gamma\in T} \sum_{r=j-1}^{\infty} f_{\alpha\beta}^{kj}(t,s)\, \theta^j q_{\gamma\beta}^{r-j}(s) \theta^{r-j} =$$

$$= \sum_{j=0}^{c-1} \sum_{\gamma\in T} f_{\alpha\beta}^{kj}(t,s)\, \pi_{\gamma\beta}^{jr}(s,\theta) + \sum_{j=c}^{\infty} \sum_{\gamma\in T} f_{\alpha\beta}^{kj}(t,s)\, \theta^j q_{\gamma\beta}(s),$$

i.e.

$$\frac{\partial f_{\alpha\beta}^{kr}(t,s,\theta)}{\partial s} = \sum_{\gamma\in T} f_{\alpha\beta}^{k}(t,s)\, q_{\gamma\beta}(s,\theta) +$$

$$+\sum_{j=0}^{c-1}\sum_{\gamma\in T} f_{\alpha\beta}^{kj}(t,s)\left[\pi_{\gamma\beta}^{j}(s,\theta) - \theta^{j} q_{\gamma\beta}(s)\right].$$

The last equality takes the following form in matrix notation

$$\frac{\partial F_k(t,s,\theta)}{\partial s} = F_k(t,s,\theta)\, Q(s,\theta) +$$

$$+\sum_{j=0}^{c-1} F_{kj}(t,s)\left[\Pi_j(s,\theta) - \theta^j Q(s,\theta)\right].$$

In view of (1) and the boundary condition

$$F_k(t,t,\theta) = \theta^k I,$$

the solution of this equation can be represented as follows:

$$\frac{\partial F_k(t,s,\theta)}{\partial s} = \theta^k P(t,s,\theta)\, Q(s,\theta) +$$

$$+\sum_{j=0}^{c-1}\int_t^s F_{kj}(t,u)\left[\Pi_j(u,\theta) - \theta^j Q(u,\theta)\right] P(u,s,\theta)\, du.$$

Equating the coefficients at θ^r, we will get

$$F_{kr}(t,s) = P_{r-k}(t,s) +$$

$$+\sum_{j=0}^{c-1}\int_t^s F_{kj}(t,u)\sum_l [\Pi_{jl}(u) - Q_{l-j}(u)] P_{r-l}(u,s)\, du$$

or

$$F_{kr}(t,s) = P_{r-k}(t,s) + \sum_{j=0}^{c-1}\int_t^s F_{kj}(t,u)\, L_{jr}(u,s) du, \left(k, r \in N^+;\ s \geq t \geq 0\right), \quad (3)$$

where

$$L_{jr}(t,s) = \sum_l [\Pi_{jl}(t) - Q_{l-j}(t)]\, P_{r-l}(t,s). \quad (4)$$

In (4) the sum is taken in all l that yield a coefficient at θ^r.

We have established that matrices F_{kr} and P_{r-k} are bound by relations (3) and (4).

It is clear from (3) that for each $k \in N^+$, F_{kr} are expressed through $F_{kj}, j < c$ and known matrices L_{jr}.

Let us introduce the following notation

$$L^{1^\circ}(t,\, s) = L(t,\, s)$$

$$L^{n^{\circ}}(t,\, s) = \int\limits_{t}^{s} L^{(n-1)^{\circ}}(t,\, u)\, L(u,\, s) du \qquad (n \geq 2)$$

According to (3), for $n \geq 1$

$$\overrightarrow{F_k}(t,\, s) = \overrightarrow{P_k}(t,\, s) +$$

$$+ \sum_{j=1}^{n} \int\limits_{t}^{s} \overrightarrow{P_k}(t,\, u) L^{j^{\circ}}(u, s)\, du + \int\limits_{t}^{s} \overrightarrow{F_k}(t,\, u) L^{(n+1)^{\circ}}(u, s)\, du. \tag{5}$$

Estimating the elements of matrix $L^{(n+1)^{\circ}}(t,\, s)$, we have

$$L^{(n+1)^{\circ}}(t,\, s) = \int\limits_{t \leq u_1 \leq \cdots} \cdots \int\limits_{\leq u_n \leq s} L(t,\, u_1)\, L(u_1,\, u_2) \cdots L(u_n,\, s)\, du_1 \cdots du_n$$

The standard form of the product under the integral is

$$\sum_{j_1=0}^{c-1} \cdots \sum_{j_n=0}^{c-1} L_{ij_1}(t,\, u_1)\, L_{j_1 j_2}(u_1,\, u_2) \cdots L_{j_n^k}(u_n,\, s), \tag{6}$$

$$(i,\ k = 0,\ 1,\ \cdots, c-1;\ t \leq u_1 \leq \cdots \leq u_n \leq s).$$

Let

$$L_{jk}(t, s) = \left\| l_{ik}^{\alpha\beta}(t, s) \right\|, \quad (\alpha, \beta \in T).$$

The elements of matrix L_{ik} are determined by (4).
Let us assume that

$$l(t,\, s) = \max_{0 \leq i,k < s} \max_{\alpha,\beta \in T} \max_{t \leq u \leq v \leq s} \left| l_{ik}^{\alpha\beta}(t, s) \right|.$$

Due to the continuous nature of $l_{ik}^{\alpha\beta}(t, s)$, the value $l(t,\, s) < \infty$.

If d is the number of elements of the set T, then all elements of the product under the summation sign in (6) do not exceed $d^n l^{n+1}(t, s)$, which means that all elements of the total (6) in the module do not exceed $(cd)^n l^{n+1}(t, s)$, so all elements of $L^{(n+1)^{\circ}}(t,\, s)$ do not exceed

$$\frac{(cd)^n l^{n+1}(t, s)(s-t)^n}{n!}. \tag{7}$$

The latter is nearing zero at $n \to \infty$.

Proceeding to the bound in (5) at $n \to \infty$ we get

$$\overrightarrow{F_k}(t,\, s) = \overrightarrow{P_k}(t,\, s) + \int\limits_{t}^{s} \overrightarrow{P_k}(t,\, u)\, R(u, s) du, \tag{8}$$

where

$$R(t, s) = \sum_{n=1}^{\infty} L^{n^{\circ}}(t,\, s). \tag{9}$$

is the resolvent operator of integral equation (8). It should be noted that the estimate (7) guarantees the convergence of the series in the right-hand side of (8). This convergence will be uniform in any finite run of t and s $\quad (s \geq t)$, so that the elements of the left-hand side of (8) are continuous in t and s.

Thus, we have the following result.

The elements of vector $\vec{F_k}$ are determined by equalities (8), (9) and $R(t,s)$ is the resolvent operator of equation (8).

3. A particular case

All obtained results can be extended to the homogeneous case without significant changes. Thus, in the homogeneous case, the matriciant $\Omega_t^s(Q)$ looks as follows

$$\Omega_t^s(Q) = e^{(s-t)Q} = \sum_{k=0}^{\infty} \frac{[(s-t)Q]^k}{k!}.$$

It should be noted that the knowledge of the infimum distribution of process n_t is of particular importance for practical reasons. Precisely, let us consider a particular case of process $\{\beta_t,\, m_t\}$, when $c = 1$ and $\pi_{\alpha\beta}^{or}(t) = 0 \quad r \geq 0;\ \ \alpha, \beta \in T$.

The evolution of this process is described by the process $\{\alpha_t,\, n_t\}$ until n_t gets into 0 for the first time. If it happens at the instant t_0 and $\alpha_{t_0} = \alpha$, then for $t \geq t_0$, $\beta_t \equiv \alpha$, $m_t \equiv \alpha$. In that case, according to (3) and (4), we have

$$F_{kr}\,(t,s) = P_{r-k}\,(t,s) - \int_t^s F_{k0}\,(t,u)\, L_r\,(u,s)\, du,$$

where

$$L_r\,(t,s) = \sum_l Q_l(t) P_{r-l}\,(t,s).$$

According to (1)

$$L_r\,(t,s) = -\frac{\partial P_r\,(t,s)}{\partial t}.$$

Therefore

$$F_{kr}\,(t,s) = P_{r-k}\,(t,s) + \int_t^s F_{k0}\,(t,0)\,\frac{\partial P_r\,(u,s)}{\partial u} du.$$

Thus to find $F_{kr}\,(t,s)$ one only needs to know $F_{k0}\,(t,s)$.

Assuming that $r = 0$ in the latter, we have the following integral equation for $F_{k0}\,(t,s)$:

$$F_{k0}\,(t,s) = P_{-k}\,(t,s) + \int_t^s F_{k0}\,(t,0)\,\frac{\partial P_0\,(u,s)}{\partial u} du.$$

The solution to this equation can be found through the pattern built for equation (8).

References

[1] Nelson, R., (1995), Probability, stochastic processes, and queueing theory: the mathematics of computer performance modeling, Springer-Verlag, Inc. New York, 583 p.
[2] Datta, K.B., (2006), Matrix and Linear Algebra, New Delhi: Hall of India, 636 p.
[3] Yezhov, I.I. and Skorokhod, A.V., (1969), Markov processes with homogeneous second component, I– Theory of probability and its applications, 14 (1), 3-14. (in Russian)
[4] Yezhov, I.I. and Skorokhod, A.V., (1969), Markov processes with homogeneous second component, II– Theory of probability and its applications, 14 (4), 679-692. (in Russian)

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