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NOTES ON CERTAIN HARMONIC STARLIKE MAPPINGS

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ABSTRACT. Complex-valued harmonic functions that are univalent and sense-preserving in the unit disk $\mathbb{D}$ can be written in the form $f = h + \bar{g}$, where h and g are analytic in $\mathbb{D}$. We give some inequalities for normalized harmonic functions that are starlike.

Keywords: Harmonic mappings, starlike function, Carathéodory function.

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1. INTRODUCTION

A continuous function $f = u + iv$ is said to be a complex-valued harmonic function in a complex domain $\mathcal{C}$ if both u and v are real harmonic in $\mathcal{C}$. In any simply connected domain $\mathcal{D} \subset \mathcal{C}$, we can write $f = h + \bar{g}$, where h and g are analytic in $\mathbb{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ ([1]).

Denote by $\mathcal{S_H}$ the class of $f = h + \bar{g}$ that are harmonic univalent and sense-preserving in the unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ for which $h(0) = f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in \mathcal{S_H}$,

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \;\; g(z) = \sum_{n=1}^{\infty} b_n z^n, \;\; z \in \mathbb{D}.$$

It follows from the sense-preserving property if $f \in \mathcal{S_H}$, then $|b_1| < 1$.

$f \in \mathcal{S_H}$ reduces to the class of normalized analytic univalent functions if the co-analytic part of its members is zero. In 1984 Clunie and Sheil-Small [1] investigated the class $\mathcal{S_H}$ as well as its geometric subclasses and obtained some coefficient bounds. Many studies have been done on this class and its subclasses, and continued taking place. For more references see Duren [2].

A sense-preserving harmonic mapping $f \in \mathcal{S_H}$ is said to be in the class $\mathcal{S}^*_{\mathcal{H}}$ if the range $f(\mathbb{D})$ is starlike with respect to the origin. A function $f \in \mathcal{S}^*_{\mathcal{H}}$ is called a harmonic starlike mapping in $\mathbb{D}$. A function $f = h + \overline{g}$ with such a property must satisfy the condition

$$\mathfrak{Re}\left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right) > 0$$

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for all $z \in \mathbb{D}$ [4].

Lemma 1.1. [2] *If $f = h + \overline{g} \in \mathcal{S}^*_{\mathcal{H}}$, then there exist angles α and β such that*

$$\mathfrak{Re}\left\{ \left(e^{i\alpha}\frac{h(z)}{z} - e^{-i\alpha}\frac{g(z)}{z} \right) \left(e^{i\beta} - e^{-i\beta}z^2 \right) \right\} > 0 \tag{1}$$

for all $z \in \mathbb{D}$.

In view of the above lemma, we consider the class $\mathcal{S}^{\circledast}_{\mathcal{H}}(\alpha,\beta,\gamma)$ of $f = h + \bar{g}$ which satisfy

$$\mathfrak{Re}\left\{ \left(e^{i\alpha}\frac{h(z)}{z} - e^{-i\alpha}\frac{g(z)}{z} \right) \left(e^{i\beta} - e^{-i\beta}z^2 \right) \right\} > \gamma, \ (z \in \mathbb{D})$$

for some real numbers α, β, γ $\left(0 \leq \gamma < \mathfrak{Re}\left\{ e^{i(\alpha+\beta)}\left(1 - e^{-i2\alpha}b_1\right)\right\}\right)$

2. Main Results

For the class $f(z) \in \mathcal{S}^{\circledast}_{\mathcal{H}}(\alpha,\beta,\gamma)$, we have

Theorem 2.1. *If $f(z) \in \mathcal{S}^{\circledast}_{\mathcal{H}}(\alpha,\beta,\gamma)$, then*

$$\frac{r}{1+r^2}\left\{ \frac{(1-r)(\mathfrak{Re}p(0)-\gamma)}{1+r} - \sqrt{\gamma^2 + (\mathfrak{Im}p(0))^2} \right\}$$
$$\leq |h(z) - e^{-i2\alpha}g(z)| \leq \frac{r}{1-r^2}\left\{ \frac{(1+r)(\mathfrak{Re}p(0)-\gamma)}{1-r} + \sqrt{\gamma^2 + (\mathfrak{Im}p(0))^2} \right\}$$

for $|z| = r < 1$, where $p(0) = e^{i(\alpha+\beta)}\left(1 - e^{-i2\alpha}b_1\right)$, $\mathfrak{Re}p(0) = \cos(\alpha+\beta) - |b_1|\cos(\arg(b_1) + \beta - \alpha)$ and $\mathfrak{Im}p(0) = \sin(\alpha+\beta) - |b_1|\sin(\arg(b_1) + \beta - \alpha)$.

Proof. Let

$$p(z) = \left(e^{i\alpha}\frac{h(z)}{z} - e^{-i\alpha}\frac{g(z)}{z} \right) \left(e^{i\beta} - e^{-i\beta}z^2 \right) \ (z \in \mathbb{D})$$

for $f(z) \in \mathcal{S}^{\circledast}_{\mathcal{H}}(\alpha,\beta,\gamma)$. Then $p(z)$ is analytic in $\mathbb{D}$ and

$$p(0) = e^{i(\alpha+\beta)}\left(1 - e^{-i2\alpha}b_1\right).$$

Further, let

$$\phi(z) = \frac{p(z) - \gamma - i\mathfrak{Im}p(0)}{\mathfrak{Re}p(0) - \gamma} \ (z \in \mathbb{D}).$$

Then $\phi(z)$ is analytic in $\mathbb{D}$, $\phi(0) = 1$ and $\mathfrak{Re}\phi(z) > 0$ for all z in $\mathbb{D}$. Since, $\phi(z)$ is the Carathéodory function, we can write that

$$\phi(z) \prec \frac{1+z}{1-z} \ (z \in \mathbb{D}),$$

that is, that

$$\phi(z) = \frac{1+w(z)}{1-w(z)},$$

where $w(z)$ is analytic in $\mathbb{D}$, $w(0) = 0$, and $|w(z)| < 1$ for all $z \in \mathbb{D}$. Thus, the Schwarz lemma gives us that

$$|w(z)| = \left| \frac{\phi(z)-1}{\phi(z)+1} \right| \leq |z|$$

for $|z| = r < 1$. It follows that

$$\left| \phi(z) - \frac{1+r^2}{1-r^2} \right| \leq \frac{2r}{1-r^2},$$

that is, that

$$\frac{1-r}{1+r} \leq |\phi(z)| \leq \frac{1+r}{1-r}.$$

Replacing $\phi(z)$ by $p(z)$, we have that

$$\frac{1-r}{1+r} \leq \left|\frac{p(z)-\gamma-i\mathfrak{Im}p(0)}{\mathfrak{Re}p(0)-\gamma}\right| \leq \frac{1+r}{1-r}.$$

This shows us that

$$\frac{(1-r)(\mathfrak{Re}p(0)-\gamma)}{1+r} - \sqrt{\gamma^2+(\mathfrak{Im}p(0))^2}$$
$$\leq |p(z)| \leq \frac{(1+r)(\mathfrak{Re}p(0)-\gamma)}{1-r} + \sqrt{\gamma^2+(\mathfrak{Im}p(0))^2}.$$

Noting that

$$p(z) = \frac{e^{i(\alpha+\beta)}}{z}\left(h(z)-e^{-i2\alpha}g(z)\right)\left(1-e^{-i2\beta}z^2\right)$$

and

$$1-r^2 \leq \left|1-e^{-i2\beta}z^2\right| \leq 1+r^2,$$

we have that

$$\frac{r}{1+r^2}\left\{\frac{(1-r)(\mathfrak{Re}p(0)-\gamma)}{1+r} - \sqrt{\gamma^2+(\mathfrak{Im}p(0))^2}\right\}$$
$$\leq |h(z)-e^{-i2\alpha}g(z)| \leq \frac{r}{1-r^2}\left\{\frac{(1+r)(\mathfrak{Re}p(0)-\gamma)}{1-r} + \sqrt{\gamma^2+(\mathfrak{Im}p(0))^2}\right\}.$$

Furthermore, we have that

$$\begin{aligned} p(0) &= e^{i(\alpha+\beta)}\left(1-e^{-i2\alpha}b_1\right) \\ &= e^{i(\alpha+\beta)}\left(1-|b_1|e^{i(\arg(b_1)-2\alpha)}\right) \\ &= e^{i(\alpha+\beta)} - |b_1|e^{i(\arg(b_1)+\beta-\alpha)} \\ &= \cos(\alpha+\beta) - |b_1|\cos(\arg(b_1)+\beta-\alpha) \\ &\qquad + i(\sin(\alpha+\beta) - |b_1|\sin(\arg(b_1)+\beta-\alpha)), \end{aligned}$$

that is, that

$$\mathfrak{Re}p(0) = \cos(\alpha+\beta) - |b_1|\cos(\arg(b_1)+\beta-\alpha)$$

and

$$\mathfrak{Im}p(0) = \sin(\alpha+\beta) - |b_1|\sin(\arg(b_1)+\beta-\alpha).$$

This completes the proof of the theorem. □

Theorem 2.2. *If $f = h + \bar{g} \in \mathcal{S}^{\circledast}_{\mathcal{H}}(\alpha,\beta,\gamma)$, then*

$$\left|e^{i\alpha}(a_n - e^{-i2\beta}a_{n-2}) - e^{-i\alpha}(b_n - e^{-i2\beta}b_{n-2})\right|$$
$$\leq 2\left\{\cos(\alpha+\beta) - |b_1|\cos(\arg(b_1)+\beta-\alpha) - \gamma\right\},$$

where $a_0 = b_0 = 0$ and $a_1 = 1$.

Proof. In view of the fact that

$$\phi(z) = \frac{p(z)-\gamma-i\mathfrak{Im}p(0)}{\mathfrak{Re}p(0)-\gamma} \;(z \in \mathbb{D})$$

is the Carathéodory function for $f(z) \in \mathcal{S}^{\circledast}_{\mathcal{H}}(\alpha, \beta, \gamma)$, where

$$p(z) = \left(e^{i\alpha}\frac{h(z)}{z} - e^{-i\alpha}\frac{g(z)}{z}\right)\left(e^{i\beta} - e^{-i\beta}z^2\right),$$

$$\mathfrak{Re}\,p(0) = \cos(\alpha+\beta) - |b_1|\cos(\arg(b_1)+\beta-\alpha)$$

and

$$\mathfrak{Im}\,p(0) = \sin(\alpha+\beta) - |b_1|\sin(\arg(b_1)+\beta-\alpha),$$

if we write

$$\phi(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots,$$

then we have that

$$|c_n| \leq 2$$

for $n = 1, 2, 3, \cdots$ ([3]). It follows that

$$\left(e^{i\alpha}\frac{h(z)}{z} - e^{-i\alpha}\frac{g(z)}{z}\right)\left(e^{i\beta} - e^{-i\beta}z^2\right) = (\mathfrak{Re}\,p(0) - \gamma)\phi(z) + \gamma + i\mathfrak{Im}\,p(0),$$

that is, that

$$\left\{e^{i\alpha}(1 + a_2 z + a_3 z^2 + a_4 z^3 + \cdots) - e^{-i\alpha}(b_1 + b_2 z + b_3 z^2 + b_4 z^3 + \cdots)\right\}\left(e^{i\beta} - e^{-i\beta}z^2\right)$$
$$= (\mathfrak{Re}\,p(0) - \gamma)(1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots) + \gamma + i\mathfrak{Im}\,p(0).$$

Comparing coefficients of z^{n-1} in both side of the last equality, we obtain that

$$e^{i(\alpha+\beta)}a_n - e^{-i(\beta-\alpha)}a_{n-2} - e^{i(\beta-\alpha)}b_n + e^{-i(\alpha+\beta)}b_{n-2} = (\mathfrak{Re}\,p(0) - \gamma)c_{n-1}.$$

Thus, we have that

$$\left|e^{i\alpha}(a_n - e^{-i2\beta}a_{n-2}) - e^{-i\alpha}(b_n - e^{-i2\beta}b_{n-2})\right|$$
$$\leq 2\left\{\cos(\alpha+\beta) - |b_1|\cos(\arg(b_1)+\beta-\alpha) - \gamma\right\},$$

This completes the proof of the theorem. □

Theorem 2.3. *Let*

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = b_1 z + \sum_{n=2}^{\infty} b_n z^n$$

be analytic in $\mathbb{D}$. *If* $h(z)$ *and* $g(z)$ *satisfy*

$$\sum_{n=2}^{\infty}\left|a_n + e^{-i2\alpha}b_n\right| \leq \frac{1}{4}\left\{\left|1 - \gamma + e^{i(\beta-\alpha)}b_1 + e^{i(\alpha+\beta)}\right| - \left|1 + \gamma - e^{i(\beta-\alpha)} - e^{i(\alpha+\beta)}\right|\right\}$$

for some real numbers α, β, γ *with* $0 \leq \gamma < \mathfrak{Re}\left\{e^{i(\alpha+\beta)}\left(1 + e^{-i2\alpha}b_1\right)\right\}$ *then* $f = h + \bar{g} \in \mathcal{S}^{\circledast}_{\mathcal{H}}(\alpha, \beta, \gamma)$ *for all* $z \in \mathbb{D}$.

Proof. Let us define the function $p(z)$ by

$$p(z) = e^{i(\alpha+\beta)}\left(\frac{h(z)}{z} + e^{-2i\alpha}\frac{g(z)}{z}\right)\left(1 + e^{-i2\beta}z^2\right) \quad (z \in \mathbb{D}).$$

Then, if $p(z)$ satisfies

$$\mathfrak{Re}\,p(z) > \gamma \ (z \in \mathbb{D}),$$

which is equivalent to

$$\left|\frac{1-(p(z)-\gamma)}{1+(p(z)-\gamma)}\right| < 1 \ (z \in \mathbb{D}),$$

then we say that $f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}^{\circledast}(\alpha,\beta,\gamma)$. It follows that

$$\begin{aligned}
&|1+(p(z)-\gamma)| - |1-(p(z)-\gamma)| \\
&= \left|1-\gamma+e^{i(\alpha+\beta)}\left(1+\sum_{n=2}^{\infty}a_n z^{n-1}+e^{-i2\alpha}b_1+e^{-i2\alpha}\sum_{n=2}^{\infty}b_n z^{n-1}\right)\left(1+e^{-i2\beta}z^2\right)\right| \\
&\quad - \left|1+\gamma-e^{i(\alpha+\beta)}\left(1+\sum_{n=2}^{\infty}a_n z^{n-1}+e^{-i2\alpha}b_1+e^{-i2\alpha}\sum_{n=2}^{\infty}b_n z^{n-1}\right)\left(1+e^{-i2\beta}z^2\right)\right| \\
&= \left|1-\gamma+e^{i(\alpha+\beta)}\left(\left(1+e^{-i2\alpha}b_1\right)\left(1+e^{-i2\beta}z^2\right)+\sum_{n=2}^{\infty}(a_n+e^{-i2\alpha}b_n)z^{n-1}\right.\right. \\
&\quad \left.\left. +e^{-i2\beta}\sum_{n=2}^{\infty}\left(a_n+e^{-i2\alpha}b_n\right)z^{n+1}\right)\right| \\
&\quad - \left|1+\gamma-e^{i(\alpha+\beta)}\left(\left(1+e^{-i2\alpha}b_1\right)\left(1+e^{-i2\beta}z^2\right)+\sum_{n=2}^{\infty}(a_n+e^{-i2\alpha}b_n)z^{n-1}\right.\right. \\
&\quad \left.\left. +e^{-i2\beta}\sum_{n=2}^{\infty}\left(a_n+e^{-i2\alpha}b_n\right)z^{n+1}\right)\right| \\
&= \left|1-\gamma+e^{i(\alpha+\beta)}\left(1+e^{-i2\alpha}b_1\right)+e^{i(\alpha+\beta)}\left(1+e^{-i2\alpha}b_1\right)e^{-i2\beta}z^2+e^{i(\alpha+\beta)}\left(a_2+e^{-i2\alpha}b_2\right)z\right. \\
&\quad \left. +e^{i(\alpha+\beta)}\left(a_3+e^{-i2\alpha}b_3\right)z^2+e^{i(\alpha+\beta)}\sum_{n=2}^{\infty}\left(\left(a_{n+2}+e^{-i2\alpha}b_{n+2}\right)+e^{-i2\beta}\left(a_n+e^{-i2\alpha}b_n\right)\right)z^{n+1}\right| \\
&\quad - \left|1+\gamma-e^{i(\alpha+\beta)}\left(1+e^{-i2\alpha}b_1\right)-e^{i(\alpha+\beta)}\left(1+e^{-i2\alpha}b_1\right)e^{-i2\beta}z^2-e^{i(\alpha+\beta)}\left(a_2+e^{-i2\alpha}b_2\right)z\right. \\
&\quad \left. -e^{i(\alpha+\beta)}\left(a_3+e^{-i2\alpha}b_3\right)z^2-e^{i(\alpha+\beta)}\sum_{n=2}^{\infty}\left(\left(a_{n+2}+e^{-i2\alpha}b_{n+2}\right)+e^{-i2\beta}\left(a_n+e^{-i2\alpha}b_n\right)\right)z^{n+1}\right| \\
&> \left|1-\gamma+e^{i(\beta-\alpha)}b_1+e^{i(\alpha+\beta)}\right| - \left|a_2+e^{-i2\alpha}b_2\right| - \left|a_3+e^{-i2\alpha}b_3\right| \\
&\quad - \sum_{n=2}^{\infty}\left(\left|a_n+e^{-i2\alpha}b_n\right|+\left|a_{n+2}+e^{-i2\alpha}b_{n+2}\right|\right) - |1+\gamma-e^{i(\beta-\alpha)}b_1-e^{i(\alpha+\beta)}| \\
&\quad - |a_2+e^{-i2\alpha}b_2| - |a_3+e^{-i2\alpha}b_3| - \sum_{n=2}^{\infty}\left(\left|a_n+e^{-i2\alpha}b_n\right|+\left|a_{n+2}+e^{-i2\alpha}b_{n+2}\right|\right) \\
&= \left|1-\gamma+e^{i(\beta-\alpha)}b_1+e^{i(\alpha+\beta)}\right| - \left|1+\gamma-e^{i(\beta-\alpha)}-e^{i(\alpha+\beta)}\right| - 4\sum_{n=2}^{\infty}|a_n+e^{-i2\alpha}b_n|.
\end{aligned}$$

Therefore, if $h(z)$ and $g(z)$ satisfy

$$\sum_{n=2}^{\infty}\left|a_n+e^{-i2\alpha}b_n\right| \le \left|1-\gamma+e^{i(\beta-\alpha)}b_1+e^{i(\alpha+\beta)}\right| - \left|1+\gamma-e^{i(\beta-\alpha)}-e^{i(\alpha+\beta)}\right|,$$

then we see that

$$\left|\frac{1-(p(z)-\gamma)}{1+(p(z)-\gamma)}\right| < 1 \ (z \in \mathbb{D}).$$

This completes the proof of the theorem. □

Theorem 2.4. *If* $f = h + \bar{g} \in \mathcal{S}^{\circledast}_{\mathcal{H}}(\alpha, \beta, \gamma)$, *then*

$$|a_{2m} - e^{-i2\alpha} b_{2m}| \leq 2m(\mathfrak{Re}\,p(0) - \gamma) \ (m = 1, 2, 3, \cdots)$$

and

$$|a_{2m+1} - e^{-i2\alpha} b_{2m+1}| \leq 2m(\mathfrak{Re}\,p(0) - \gamma) + |1 - e^{-i2\alpha} b_1| \ (m = 1, 2, 3, \cdots)$$

where

$$\mathfrak{Re}\,p(0) = \cos(\alpha + \beta) - |b_1| \cos(\arg(b_1) + \beta - \alpha).$$

Proof. Defining

$$p(z) = \left(e^{i\alpha}\frac{h(z)}{z} - e^{-i\alpha}\frac{g(z)}{z}\right)\left(e^{i\beta} - e^{-i\beta} z^2\right) \ (z \in \mathbb{D})$$

and

$$\phi(z) = \frac{p(z) - \gamma - i\,\mathfrak{Im}\,p(0)}{\mathfrak{Re}\,p(0) - \gamma} \ (z \in \mathbb{D}).$$

for $f(z) \in \mathcal{S}^{\circledast}_{\mathcal{H}}(\alpha, \beta, \gamma)$, we know that $\phi(z)$ is the Carathéodory function in $\mathbb{D}$. Therefore, if we write

$$\phi(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots ,$$

then

$$|c_n| \leq 2$$

for $n = 1, 2, 3, \cdots$ ([3]). It follows that

$$\begin{aligned} p(z) - \gamma - i\,\mathfrak{Im}\,p(0) &= e^{i(\alpha+\beta)}(1 - e^{-i2\alpha} b_1) - i\,\mathfrak{Im}\,p(0) - \gamma \\ &+ e^{i(\alpha+\beta)}(a_2 - e^{-i2\alpha} b_2) z \\ &+ \left(e^{i(\alpha+\beta)}(a_3 - e^{-i2\alpha} b_3) - e^{i(\alpha-\beta)}(1 - e^{-i2\alpha} b_1)\right) z^2 \\ &+ \left(e^{i(\alpha+\beta)}(a_4 - e^{-i2\alpha} b_4) - e^{i(\alpha-\beta)}(a_2 - e^{-i2\alpha} b_2)\right) z^3 \\ &+ \left(e^{i(\alpha+\beta)}(a_5 - e^{-i2\alpha} b_5) - e^{i(\alpha-\beta)}(a_3 - e^{-i2\alpha} b_3)\right) z^4 \\ &+ \left(e^{i(\alpha+\beta)}(a_6 - e^{-i2\alpha} b_6) - e^{i(\alpha-\beta)}(a_4 - e^{-i2\alpha} b_4)\right) z^5 + \cdots \end{aligned}$$

and

$$(\mathfrak{Re}\,p(0) - \gamma)\phi(z) = \mathfrak{Re}\,p(0) - \gamma + (\mathfrak{Re}\,p(0) - \gamma) c_1 z + (\mathfrak{Re}\,p(0) - \gamma) c_2 z^2 + \cdots .$$

Comparing the coefficient of z^n, we have that

$$e^{i(\alpha+\beta)}(a_{n+1} - e^{-i2\alpha} b_{n+1}) - e^{i(\alpha-\beta)}(a_{n-1} - e^{-i2\alpha} b_{n-1}) = (\mathfrak{Re}\,p(0) - \gamma) c_n,$$

for $n = 1, 2, 3, \cdots$ where $a_0 = b_0 = 0$ and $a_1 = 1$.

If we take $n = 1$, then we have that

$$|a_2 - e^{-i2\alpha} b_2| \leq 2(\mathfrak{Re}\,p(0) - \gamma).$$

We also have that for $n = 2, 3, 4$

$$|a_3 - e^{-i2\alpha} b_3| \leq 2(\mathfrak{Re}\,p(0) - \gamma) + |1 - e^{-i2\alpha} b_1|,$$

$$\begin{aligned}|a_4 - e^{-i2\alpha} b_4| &\leq 2(\mathfrak{Re}p(0) - \gamma) + |a_2 - e^{-i2\alpha} b_2| \\ &\leq 4(\mathfrak{Re}p(0) - \gamma),\end{aligned}$$

and

$$\begin{aligned}|a_5 - e^{-i2\alpha} b_5| &\leq 2(\mathfrak{Re}p(0) - \gamma) + |a_3 - e^{-i2\alpha} b_3| \\ &\leq 4(\mathfrak{Re}p(0) - \gamma) + |1 - e^{-i2\alpha} b_1|,\end{aligned}$$

respectively. Thus applying the mathematical induction, we obtain that

$$|a_{2m} - e^{-i2\alpha} b_{2m}| \leq 2m(\mathfrak{Re}p(0) - \gamma) \ (m = 1, 2, 3, \cdots)$$

and

$$|a_{2m+1} - e^{-i2\alpha} b_{2m+1}| \leq 2m(\mathfrak{Re}p(0) - \gamma) + |1 - e^{-i2\alpha} b_1| \ (m = 1, 2, 3, \cdots).$$

This completes the proof of the theorem. □

Since

$$|1 - e^{-i2\alpha} b_1| \leq 1 + |b_1| < 2$$

and

$$|a_n - e^{-i2\alpha} b_n| \geq ||a_n| - |b_n|| \ (n = 2, 3, 4, \cdots),$$

we have:

Corollary 2.1. *If* $f = h + \bar{g} \in \mathcal{S}^{\circledast}_{\mathcal{H}}(\alpha, \beta, \gamma)$*, then*

$$||a_{2m}| - |b_{2m}|| \leq 2m(\mathfrak{Re}p(0) - \gamma) \ (m = 1, 2, 3, \cdots)$$

and

$$\begin{aligned}||a_{2m+1}| - |b_{2m+1}|| &\leq 2m(\mathfrak{Re}p(0) - \gamma) + 1 + |b_1| \\ &< 2m(\mathfrak{Re}p(0) - \gamma) + 2 \ (m = 1, 2, 3, \cdots).\end{aligned}$$

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