

DIFFERENTIAL SUBORDINATIONS USING RUSCHEWEYH DERIVATIVE AND SĂLĂŢEAN OPERATOR

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ABSTRACT. In the present paper we study the operator defined by using the Ruscheweyh derivative $R^m f(z)$ and the SălăŢean operator $S^m f(z)$, denoted $L_\alpha^m : \mathcal{A}_n \rightarrow \mathcal{A}_n$, $L_\alpha^m f(z) = (1-\alpha)R^m f(z) + \alpha S^m f(z)$, $z \in U$, where $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \dots, z \in U\}$ is the class of normalized analytic functions. We obtain several differential subordinations regarding the operator L_α^m .

Keywords: differential subordination, convex function, best dominant, differential operator, SălăŢean operator, Ruscheweyh derivative.

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1. INTRODUCTION

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U . Let

$$\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \dots, z \in U\}$$

and

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, z \in U\}$$

for $a \in \mathbb{C}$ and $n \in \mathbb{N}$. Denote by $K = \left\{f \in \mathcal{A}_n : \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, z \in U\right\}$, the class of normalized convex functions in U .

If f and g are analytic functions in U , we say that f is subordinate to g , written $f \prec g$, if there is a function w analytic in U , with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$. Let $\psi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ and h be a univalent function in U . If p is analytic in U and satisfies the (second-order) differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \quad z \in U, \tag{1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1). A dominant \tilde{q} that satisfies $\tilde{q} \prec q$ for all dominants q of (1) is said to be the best dominant of (1). The best dominant is unique up to a rotation of U .

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Definition 1.1. (Sălăgean [6]) For $f \in \mathcal{A}_n$, $n, m \in \mathbb{N}$, the operator S^m is defined by $S^m : \mathcal{A}_n \rightarrow \mathcal{A}_n$,

$$S^0 f(z) = f(z), \quad S^1 f(z) = z f'(z), \dots$$

$$S^{m+1} f(z) = z (S^m f(z))', \quad z \in U.$$

Remark 1.1. If $f \in \mathcal{A}_n$, $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then

$$S^m f(z) = z + \sum_{j=n+1}^{\infty} j^m a_j z^j, \quad z \in U$$

Definition 1.2. ([5]) For $f \in \mathcal{A}_n$, $n, m \in \mathbb{N}$, the operator R^m is defined by $R^m : \mathcal{A}_n \rightarrow \mathcal{A}_n$,

$$R^0 f(z) = f(z), \quad R^1 f(z) = z f'(z), \dots$$

$$(m+1) R^{m+1} f(z) = z (R^m f(z))' + m R^m f(z), \quad z \in U.$$

Remark 1.2. If $f \in \mathcal{A}_n$, $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then $R^m f(z) = z + \sum_{j=n+1}^{\infty} C_{m+j-1}^m a_j z^j$, $z \in U$.

Definition 1.3. ([1]) Let $\alpha \geq 0$, $n, m \in \mathbb{N}$. Denote by L_α^m the operator given by $L_\alpha^m : \mathcal{A}_n \rightarrow \mathcal{A}_n$,

$$L_\alpha^m f(z) = (1-\alpha) R^m f(z) + \alpha S^m f(z), \quad z \in U.$$

Remark 1.3. If $f \in \mathcal{A}_n$, $f(z) = z + \sum_{j=n+1}^{\infty} a_j z^j$, then

$$L_\alpha^m f(z) = z + \sum_{j=n+1}^{\infty} \left(\alpha j^m + (1-\alpha) C_{m+j-1}^m \right) a_j z^j, \quad z \in U.$$

This operator was studied also in [1], [2].

Lemma 1.1. (Hallenbeck and Ruscheweyh [4, Th. 3.1.6, p. 71]) Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C} \setminus \{0\}$ be a complex number with $\operatorname{Re} \gamma \geq 0$. If $p \in \mathcal{H}[a, n]$ and $p(z) + \frac{1}{\gamma} z p'(z) \prec h(z)$, $z \in U$, then $p(z) \prec g(z) \prec h(z)$, $z \in U$, where $g(z) = \frac{\gamma}{nz\gamma/n} \int_0^z h(t) t^{\gamma/n-1} dt$, $z \in U$.

Lemma 1.2. (Miller and Mocanu [4]) Let g be a convex function in U and let $h(z) = g(z) + n\alpha z g'(z)$, for $z \in U$, where $\alpha > 0$ and n is a positive integer.

If $p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \dots$, $z \in U$, is holomorphic in U and $p(z) + \alpha z p'(z) \prec h(z)$, $z \in U$, then $p(z) \prec g(z)$, $z \in U$, and this result is sharp.

2. MAIN RESULTS

Theorem 2.1. Let g be a convex function, $g(0) = 1$ and let h be the function $h(z) = g(z) + \frac{nz}{\delta} g'(z)$, $z \in U$. If $\alpha, \delta \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ and satisfies the differential subordination

$$\left(\frac{L_\alpha^m f(z)}{z} \right)^{\delta-1} (L_\alpha^m f(z))' \prec h(z), \quad z \in U, \quad (2)$$

then $\left(\frac{L_\alpha^m f(z)}{z} \right)^\delta \prec g(z)$, $z \in U$, and this result is sharp.

Theorem 2.2. Let h be an holomorphic function which satisfies the inequality

$\operatorname{Re} \left(1 + \frac{zh''(z)}{h'(z)} \right) > -\frac{1}{2}$, $z \in U$, and $h(0) = 1$. If $\alpha, \delta \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ and satisfies the differential subordination

$$\left(\frac{L_\alpha^m f(z)}{z} \right)^{\delta-1} (L_\alpha^m f(z))' \prec h(z), \quad z \in U, \quad (3)$$

then $\left(\frac{L_\alpha^m f(z)}{z}\right)^\delta \prec q(z)$, $z \in U$, where $q(z) = \frac{\delta}{nz^{\frac{\delta}{n}}} \int_0^z h(t)t^{\frac{\delta}{n}-1} dt$. The function q is convex and it is the best dominant.

Corollary 2.1. Let $h(z) = \frac{1+(2\beta-1)z}{1+z}$ be a convex function in U , where $0 \leq \beta < 1$. If $\alpha, \delta \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ and satisfies the differential subordination

$$\left(\frac{L_\alpha^m f(z)}{z}\right)^{\delta-1} (L_\alpha^m f(z))' \prec h(z), \quad z \in U, \tag{4}$$

then $\left(\frac{L_\alpha^m f(z)}{z}\right)^\delta \prec q(z)$, $z \in U$, where q is given by $q(z) = (2\beta - 1) + \frac{2(1-\beta)\delta}{nz^{\frac{\delta}{n}}} \int_0^z \frac{t^{\frac{\delta}{n}-1}}{1+t} dt$, $z \in U$. The function q is convex and it is the best dominant.

Remark 2.1. For $n = 1$, $m = 1$, $\alpha = 2$, $\delta = 1$ we obtain the same example as in [3, Example 2.2.1, p. 26].

Theorem 2.3. Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + \frac{nz}{\gamma} g'(z)$, $z \in U$, where $\gamma > 0$. If $\alpha \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ and the differential subordination

$$\frac{(\gamma + 1)z}{\gamma} \frac{L_\alpha^m f(z)}{(L_\alpha^{m+1} f(z))^2} + \frac{z^2}{\gamma} \frac{L_\alpha^m f(z)}{(L_\alpha^{m+1} f(z))^2} \left[\frac{(L_\alpha^m f(z))'}{L_\alpha^m f(z)} - 2 \frac{(L_\alpha^{m+1} f(z))'}{L_\alpha^{m+1} f(z)} \right] \prec h(z), \quad z \in U \tag{5}$$

holds, then $z \frac{L_\alpha^m f(z)}{(L_\alpha^{m+1} f(z))^2} \prec g(z)$, $z \in U$, and this result is sharp.

Theorem 2.4. Let h be an holomorphic function which satisfies the inequality

$\operatorname{Re} \left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, and $h(0) = 1$. If $\alpha \geq 0$, $\gamma \in \mathbb{C} \setminus \{0\}$ be a complex number with $\operatorname{Re} \gamma \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ and satisfies the differential subordination

$$\frac{(\gamma + 1)z}{\gamma} \frac{L_\alpha^m f(z)}{(L_\alpha^{m+1} f(z))^2} + \frac{z^2}{\gamma} \frac{L_\alpha^m f(z)}{(L_\alpha^{m+1} f(z))^2} \left[\frac{(L_\alpha^m f(z))'}{L_\alpha^m f(z)} - 2 \frac{(L_\alpha^{m+1} f(z))'}{L_\alpha^{m+1} f(z)} \right] \prec h(z), \quad z \in U, \tag{6}$$

then $z \frac{L_\alpha^m f(z)}{(L_\alpha^{m+1} f(z))^2} \prec q(z)$, $z \in U$, where $q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\gamma/n-1} dt$. The function q is convex and it is the best dominant.

Theorem 2.5. Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + \frac{nz}{\gamma} g'(z)$, $z \in U$, where $\gamma > 0$. If $\alpha \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ and the differential subordination

$$\frac{(\gamma + 2)z^2}{\gamma} \frac{(L_\alpha^m f(z))'}{L_\alpha^m f(z)} + \frac{z^3}{\gamma} \left[\frac{(L_\alpha^m f(z))''}{L_\alpha^m f(z)} - \left(\frac{(L_\alpha^m f(z))'}{L_\alpha^m f(z)} \right)^2 \right] \prec h(z), \quad z \in U \tag{7}$$

holds, then $z^2 \frac{(L_\alpha^m f(z))'}{L_\alpha^m f(z)} \prec g(z)$, $z \in U$. This result is sharp.

Theorem 2.6. Let h be an holomorphic function which satisfies the inequality

$\operatorname{Re} \left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$, $z \in U$, and $h(0) = 1$. If $\alpha \geq 0$, $\gamma \in \mathbb{C} \setminus \{0\}$ be a complex number with $\operatorname{Re} \gamma \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ and satisfies the differential subordination

$$\frac{(\gamma + 2)z^2}{\gamma} \frac{(L_\alpha^m f(z))'}{L_\alpha^m f(z)} + \frac{z^3}{\gamma} \left[\frac{(L_\alpha^m f(z))''}{L_\alpha^m f(z)} - \left(\frac{(L_\alpha^m f(z))'}{L_\alpha^m f(z)} \right)^2 \right] \prec h(z), \quad z \in U, \tag{8}$$

then $z^2 \frac{(L_\alpha^m f(z))'}{L_\alpha^m f(z)} \prec q(z)$, $z \in U$, where $q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\gamma/n-1} dt$. The function q is convex and it is the best dominant.

Theorem 2.7. Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + n z g'(z)$, $z \in U$. If $\alpha \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ and the differential subordination

$$1 - \frac{L_\alpha^m f(z) \cdot (L_\alpha^m f(z))''}{[(L_\alpha^m f(z))']^2} \prec h(z), \quad z \in U \quad (9)$$

holds, then $\frac{L_\alpha^m f(z)}{z(L_\alpha^m f(z))'} \prec g(z)$, $z \in U$. This result is sharp.

Theorem 2.8. Let h be an holomorphic function which satisfies the inequality

$\operatorname{Re} \left(1 + \frac{z h''(z)}{h'(z)} \right) > -\frac{1}{2}$, $z \in U$, and $h(0) = 1$. If $\alpha \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ and satisfies the differential subordination

$$1 - \frac{L_\alpha^m f(z) \cdot (L_\alpha^m f(z))''}{[(L_\alpha^m f(z))']^2} \prec h(z), \quad z \in U, \quad (10)$$

then $\frac{L_\alpha^m f(z)}{z(L_\alpha^m f(z))'} \prec q(z)$, $z \in U$, where $q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt$. The function q is convex and it is the best dominant.

Corollary 2.2. Let $h(z) = \frac{1+(2\beta-1)z}{1+z}$ be a convex function in U , where $0 \leq \beta < 1$. If $\alpha \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ and satisfies the differential subordination

$$1 - \frac{L_\alpha^m f(z) \cdot (L_\alpha^m f(z))''}{[(L_\alpha^m f(z))']^2} \prec h(z), \quad z \in U, \quad (11)$$

then $\frac{L_\alpha^m f(z)}{z(L_\alpha^m f(z))'} \prec q(z)$, $z \in U$, where q is given by $q(z) = (2\beta - 1) + \frac{2(1-\beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t} dt$, $z \in U$. The function q is convex and it is the best dominant.

Example 2.1. Let $h(z) = \frac{1-z}{1+z}$ a convex function in U with $h(0) = 1$ and $\operatorname{Re} \left(\frac{z h''(z)}{h'(z)} + 1 \right) > -\frac{1}{2}$.

Let $f(z) = z + z^2$, $z \in U$. For $n = 1$, $m = 1$, $\alpha = 2$, we obtain

$$L_2^1 f(z) = -R^1 f(z) + 2S^1 f(z) = -z f'(z) + 2z f'(z) = z f'(z) = z + 2z^2.$$

$$\text{Then } (L_2^1 f(z))' = 1 + 4z,$$

$$\frac{L_2^1 f(z)}{z(L_2^1 f(z))'} = \frac{z + 2z^2}{z(1 + 4z)} = \frac{1 + 2z}{1 + 4z},$$

$$1 - \frac{L_2^1 f(z) \cdot (L_2^1 f(z))''}{[(L_2^1 f(z))']^2} = 1 - \frac{(z + 2z^2) \cdot 4}{(1 + 4z)^2} = \frac{8z^2 + 4z + 1}{(1 + 4z)^2}.$$

We have

$$q(z) = \frac{1}{z} \int_0^z \frac{1-t}{1+t} dt = -1 + \frac{2 \ln(1+z)}{z}.$$

Using Theorem 2.8 we obtain $\frac{8z^2+4z+1}{(1+4z)^2} \prec \frac{1-z}{1+z}$, $z \in U$, induce $\frac{1+2z}{1+4z} \prec -1 + \frac{2 \ln(1+z)}{z}$, $z \in U$.

Theorem 2.9. Let g be a convex function such that $g(0) = 0$ and let h be the function $h(z) = g(z) + n z g'(z)$, $z \in U$. If $\alpha \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ and the differential subordination

$$[(L_\alpha^m f(z))']^2 + L_\alpha^m f(z) \cdot (L_\alpha^m f(z))'' \prec h(z), \quad z \in U \quad (12)$$

holds, then $\frac{L_\alpha^m f(z) \cdot (L_\alpha^m f(z))'}{z} \prec g(z)$, $z \in U$. This result is sharp.

Theorem 2.10. Let h be an holomorphic function which satisfies the inequality $\operatorname{Re} \left(1 + \frac{zh''(z)}{h'(z)} \right) > -\frac{1}{2}$, $z \in U$, and $h(0) = 0$. If $\alpha \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ and satisfies the differential subordination

$$\left[(L_\alpha^m f(z))' \right]^2 + L_\alpha^m f(z) \cdot (L_\alpha^m f(z))'' \prec h(z), \quad z \in U, \quad (13)$$

then $\frac{L_\alpha^m f(z) \cdot (L_\alpha^m f(z))'}{z} \prec q(z)$, $z \in U$, where $q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)t^{\frac{1}{n}-1} dt$. The function q is convex and it is the best dominant.

Corollary 2.3. Let $h(z) = \frac{1+(2\beta-1)z}{1+z}$ be a convex function in U , where $0 \leq \beta < 1$. If $\alpha \geq 0$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ and satisfies the differential subordination

$$\left[(L_\alpha^m f(z))' \right]^2 + L_\alpha^m f(z) \cdot (L_\alpha^m f(z))'' \prec h(z), \quad z \in U, \quad (14)$$

then $\frac{L_\alpha^m f(z) \cdot (L_\alpha^m f(z))'}{z} \prec q(z)$, $z \in U$, where q is given by $q(z) = (2\beta - 1) + \frac{2(1-\beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t} dt$, $z \in U$. The function q is convex and it is the best dominant.

Example 2.2. Let $h(z) = \frac{1-z}{1+z}$ a convex function in U with $h(0) = 1$ and $\operatorname{Re} \left(\frac{zh''(z)}{h'(z)} + 1 \right) > -\frac{1}{2}$. Let $f(z) = z + z^2$, $z \in U$. For $n = 1$, $m = 1$, $\alpha = 2$, we obtain

$$L_2^1 f(z) = -R^1 f(z) + 2S^1 f(z) = -zf'(z) + 2zf'(z) = zf'(z) = z + 2z^2, z \in U$$

Then

$$(L_2^1 f(z))' = 1 + 4z,$$

$$\frac{L_2^1 f(z) \cdot (L_2^1 f(z))'}{z} = \frac{(z + 2z^2)(1 + 4z)}{z} = 8z^2 + 6z + 1,$$

$$\left[(L_2^1 f(z))' \right]^2 + L_2^1 f(z) \cdot (L_2^1 f(z))'' = (1 + 4z)^2 + (z + 2z^2) \cdot 4 = 24z^2 + 12z + 1.$$

We have

$$q(z) = \frac{1}{z} \int_0^z \frac{1-t}{1+t} dt = -1 + \frac{2 \ln(1+z)}{z}.$$

Using Theorem 2.10 we obtain $24z^2 + 12z + 1 \prec -1 + \frac{2 \ln(1+z)}{z}$, $z \in U$, induce

$$8z^2 + 6z + 1 \prec -1 + \frac{2 \ln(1+z)}{z}, \quad z \in U.$$

Theorem 2.11. Let g be a convex function such that $g(0) = 0$ and let h be the function $h(z) = g(z) + \frac{nz}{1-\delta} g'(z)$, $z \in U$. If $\alpha \geq 0$, $\delta \in (0, 1)$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ and the differential subordination

$$\left(\frac{z}{L_\alpha^m f(z)} \right)^\delta \frac{L_\alpha^{m+1} f(z)}{1-\delta} \left(\frac{(L_\alpha^{m+1} f(z))'}{L_\alpha^{m+1} f(z)} - \delta \frac{(L_\alpha^m f(z))'}{L_\alpha^m f(z)} \right) \prec h(z), \quad z \in U \quad (15)$$

holds, then $\frac{L_\alpha^{m+1} f(z)}{z} \cdot \left(\frac{z}{L_\alpha^m f(z)} \right)^\delta \prec g(z)$, $z \in U$. This result is sharp.

Theorem 2.12. Let h be an holomorphic function which satisfies the inequality $\operatorname{Re} \left(1 + \frac{zh''(z)}{h'(z)} \right) > -\frac{1}{2}$, $z \in U$, and $h(0) = 1$. If $\alpha \geq 0$, $\delta \in (0, 1)$, $n, m \in \mathbb{N}$, $f \in \mathcal{A}_n$ and satisfies the differential subordination

$$\left(\frac{z}{L_\alpha^m f(z)} \right)^\delta \frac{L_\alpha^{m+1} f(z)}{1-\delta} \left(\frac{(L_\alpha^{m+1} f(z))'}{L_\alpha^{m+1} f(z)} - \delta \frac{(L_\alpha^m f(z))'}{L_\alpha^m f(z)} \right) \prec h(z), \quad z \in U, \quad (16)$$

then $\frac{L_{\alpha}^{m+1}f(z)}{z} \cdot \left(\frac{z}{L_{\alpha}^m f(z)}\right)^{\delta} \prec q(z)$, $z \in U$, where $q(z) = \frac{1-\delta}{nz^{\frac{1-\delta}{n}}} \int_0^z h(t)t^{\frac{1-\delta}{n}-1} dt$. The function q is convex and it is the best dominant.

REFERENCES

- [1] A. Alb Lupaş, *On special differential subordinations using Sălăgean and Ruscheweyh operators*, Mathematical Inequalities and Applications, Volume 12, Issue 4, 2009, 781-790.
- [2] A. Alb Lupaş, D. Breaz, *On special differential superordinations using Sălăgean and Ruscheweyh operators*, Geometric Function Theory and Applications' 2010 (Proc. of International Symposium, Sofia, 27-31 August 2010), 98-103.
- [3] D.A. Alb Lupaş, *Subordinations and Superordinations*, Lap Lambert Academic Publishing, 2011.
- [4] S.S. Miller, P.T. Mocanu, *Differential Subordinations. Theory and Applications*, Marcel Dekker Inc., New York, Basel, 2000.
- [5] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc., 49(1975), 109-115.
- [6] G. St. Sălăgean, *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, Berlin, 1013(1983), 362-372.