TWMS J. App. Eng. Math. V.4, No.1, 2014, pp. 39-44.

ESTIMATING COEFFICIENTS FOR SUBCLASSES OF MEROMORPHIC BI-UNIVALENT FUNCTIONS ASSOCIATED WITH LINEAR OPERATOR

FATEH S. AZIZ[1], ABDUL RAHMAN S. JUMA[2] §

Abstract. In this paper we define a differential linear operator, applying it on the subclasses $H_{\Sigma^*_{\mathfrak{B}}}(\alpha, n, \lambda)$ of meromorphic starlike bi-univalent functions of order α, and $H_{\tilde{\Sigma}^*_{\mathfrak{B}}}(\alpha, n, \lambda)$ of meromorphic strongly starlike bi-univalent functions of order α, also we find estimates on the coefficients $|b_o|$ and $|b_1|$ for functions in these subclasses.

Keywords: Analytic, univalent and Bi-univalent functions, Starlike and strongly starlike functions, Linear operator,Meromorphic functions,Coefficient estimates.

AMS Subject Classification: 30C45, 30C50.

1. Introduction

Let A be the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and let S denote the subclass of functions in A which are univalent in U. The well-known Koebe one-quarter theorem asserts that the function $f \in S$ has an inverse defined on disc $U_\rho = \{z \in \mathbb{C} : |z| < \rho\}, (\rho \geq \frac{1}{4})$. Thus, the inverse of $f \in S$ is a univalent analytic function on the disc U_ρ. The function $f \in A$ is called bi-univalent in U if f^{-1} is also univalent in the whole disc U. The class μ of bi-univalent analytic functions was introduced in 1967 by Lewin [11] and he showed that, for every function $f \in \mu$ of the form (1), the second coefficient of f satisfy the inequality $|a_2| < 1.51$. Subsequently, Brannan and Clunie [3] improved Lewin's result by showing $|a_2| \leq \sqrt{2}$. Later, Netanyahu [12] proved that $max_{f\in\mu}|a_2| = \frac{4}{3}$. Also, several authors such as Brannan and Taha [4], Taha [18] investigated subclasses of bi-univalent analytic functions and found estimates on the initial coefficients for functions in these subclasses. Recently Ali et al. [2], Frasin and Aouf [7], Srivastava et al.[16], Juma and Aziz [1] also introduced new subclasses of bi-univalent functions and found estimates on the coefficients a_2 and a_3 for functions in these classes.

[1] Department of Mathematics, Salahaddin University, Erbil, Region of Kurdistan, Iraq,
e-mail: fatehsaber@gmail
[2] Department of Mathematics, Alanbar University, Ramadi, Iraq,
e-mail:dr_juma@hotmail.com
§ Submitted for GFTA'13, held in Işık University on October 12, 2013.

Suzeini et al.[17] considered and studied the concept of bi-univalency for classes of meromorphic functions defined on $\Delta = \{z : z \in \mathbb{C} \text{ and } 1 < |z| < \infty\}$. For this purpose they denote by Σ the class of all meromorphic univalent functions g of the form

$$g(z) = z + \sum_{k=0}^{\infty} \frac{b_k}{z^k}, \tag{2}$$

defined on the domain Δ. Since $g \in \Sigma$ is univalent, it has an inverse g^{-1} that satisfy

$$g^{-1}(g(z)) = z \quad (z \in \Delta),$$

and

$$g(g^{-1}(w)) = w \quad (M < |w| < \infty, M > 0).$$

Furthermore, the inverse function g^{-1} has a series expansion of the form

$$g^{-1}(w) = w + \sum_{k=0}^{\infty} \frac{B_k}{w^k}, \tag{3}$$

where $M < |w| < \infty$. Analogous to the bi-univalent analytic functions, a function $g \in \Sigma$ is said to be meromorphic bi-univalent if $g^{-1} \in \Sigma$. The class of all meromorphic bi-univalent functions is denoted by $\Sigma_{\mathfrak{B}}$.

Estimates on the coefficients of meromorphic univalent functions were investigated in the literature; for example, Schiffer [13] obtained the estimate $|b_2| \leq \frac{2}{3}$ for meromorphic univalent functions $g \in \Sigma$ with $b_o = 0$. In 1971, Duren [6] gave an elementary proof of the inequality $|b_n| \leq \frac{2}{n+1}$ on the coefficient of meromorphic univalent functions $g \in \Sigma$ with $b_k = 0$ for $1 \leq k < \frac{n}{2}$. For the coefficients of the inverse of meromorphic univalent functions, Springer [15] proved that

$$|B_3| \leq 1 \quad \textit{and} \quad |B_3 + \frac{1}{2}B_1^2| \leq \frac{1}{2},$$

and conjectured that

$$|B_{2n-1}| \leq \frac{(2n-2)!}{n!(n-1)!} \quad (n = 1, 2, 3, ...).$$

In 1977, Kubota [10] has proved that the Springer conjecture is true fore $n = 3, 4, 5$ and subsequently Schober [14] obtained sharp bounds for the coefficients $B_{2n-1}, 1 \leq n \leq 7$, of the inverse of meromorphic univalent functions in Δ. Recently, Kapoor and Mishra [9] found the coefficient estimates for a class consisting of inverses of meromorphic starlike univalent functions of order α in Δ.

For functions $g(z) \in \Sigma$, in the form (2) we define the following linear operator

$$F_\lambda^0 g(z) = g(z) \quad (0 \leq \lambda < \frac{1}{k+1}) \quad \textit{and} \quad F_0^n g(z) = g(z) \quad (n = 0, 1.2, ...),$$

$$F_\lambda^1 g(z) = F_\lambda g(z) = (1-\lambda)g(z) + \lambda z g'(z) = z + \sum_{k=0}^{\infty}[1-(k+1)\lambda]\frac{b_k}{z^k} \quad (0 \leq \lambda < \frac{1}{k+1}),$$

and

$$F_\lambda^2 g(z) = F_\lambda[F_\lambda g(z)] = z + \sum_{k=0}^{\infty}[1-(k+1)\lambda]^2\frac{b_k}{z^k} \quad (0 \leq \lambda < \frac{1}{k+1}),$$

hence, it can be easily seen that

$$F_\lambda^n g(z) = z + \sum_{k=0}^{\infty}[1-(k+1)\lambda]^n\frac{b_k}{z^k} \quad (0 \leq \lambda < \frac{1}{k+1}, n \in \mathbb{N}_o = \{0, 1, 2, ...\}). \tag{4}$$

In the present investigation, certain subclasses of meromorphic bi-univalent functions are introduced and estimates for the coefficients b_o and b_1 of functions in these subclasses are obtained. These coefficients results are obtained by associating the given functions with the functions having positive real part. An analytic function p of the form $p(z) = 1 + c_1 z + c_2 z^2 + ...$ is called a function with positive real part in U if $Re(p(z)) > 0$ for all $z \in U$. The class of all functions with positive real part is denoted by $\mathbf{P}$.

The following lemma for functions with positive real part will be useful in the sequel.

Lemma 1.1 ([8],Theorem 3, p.80). *The coefficients c_n of a function $p \in \mathbf{P}$ satisfy the sharp inequality $|c_n| \leq 2 \quad (n \geq 1)$*

2. Coefficient estimates

In this section, certain subclasses like the subclass $H_{\Sigma^*_{\mathfrak{B}}}(\alpha, n, \lambda)$ of the meromorphic bi-univalent functions associated with the linear operator $F^n_\lambda g(z)$ are introduced and estimates on the coefficients b_o and b_1 for functions in these subclasses are obtained.

The class of all meromorphic starlike bi-univalent functions of order α is denoted by $\Sigma^*_{\mathfrak{B}}(\alpha)$.

Definition 2.1. *A function $g(z)$ given by (2) is said to be in the subclass $H_{\Sigma^*_{\mathfrak{B}}}(\alpha, n, \lambda)$ if the following conditions are satisfied:*

$$Re\left(\frac{z(F^n_\lambda g(z))'}{F^n_\lambda g(z)}\right) > \alpha \quad \left(0 \leq \alpha < 1, 0 \leq \lambda < \frac{1}{k+1}, n = 0, 1, 2, ..., z \in \Delta\right), \tag{5}$$

and

$$Re\left(\frac{w(F^n_\lambda h(w))'}{F^n_\lambda h(w)}\right) > \alpha \quad \left(0 \leq \alpha < 1, 0 \leq \lambda < \frac{1}{k+1}, n = 0, 1, 2, ..., w \in \Delta\right), \tag{6}$$

where the function $h(w)$ is the inverse of $g(z)$ given by (3).

Theorem 2.1. *Let the function $g(z)$ given by (2) be in the subclass $H_{\Sigma^*_{\mathfrak{B}}}(\alpha, n, \lambda)$. Then*

$$|b_o| \leq 2\frac{(1-\alpha)}{(1-\lambda)^{\frac{n}{2}}} \quad \textit{and} \quad |b_1| \leq \frac{(1-\lambda)^n(1-\alpha)\sqrt{1+4(1-\lambda)^{2n}(1-\alpha)^2}}{(1-\lambda)^n(1-2\lambda)^n}.$$

Proof. Let $g(z)$ be the meromorphic starlike bi-univalent function of order α given by (2). Then

$$\begin{aligned} \frac{z(F^n_\lambda g(z))'}{F^n_\lambda g(z)} &= 1 - \frac{(1-\lambda)^n b_o}{z} + \frac{(1-\lambda)^{2n} b_o^2 - 2(1-2\lambda)^n b_1}{z^2} \\ &- \frac{(1-\lambda)^{3n} b_o^3 - 3(1-\lambda)^n(1-2\lambda)^n b_o b_1 + 3(1-2\lambda)^n b_2}{z^3} + ... \quad (z \in \Delta). \end{aligned} \tag{7}$$

Since $h(w) = g^{-1}(w)$ is the inverse of $g(z)$ whose series expansion is given in (3), and, since

$$w = g(h(w)) = g(g^{-1}(w)).$$

So, some calculations gives

$$B_o = -b_o, \quad B_1 = -b_1, \quad B_2 = -b_2 - b_o b_1 \quad \textit{and} \quad B_3 = -(b_3 + 2b_o b_2 + b_o^2 b_1 + b_1^2). \tag{8}$$

Using equations of (8) in (3), shows that the series expansion of the function $g^{-1}(w)$ becomes

$$\begin{aligned} h(w) &= g^{-1}(w) \\ &= w - b_o - b_1\frac{1}{w} - (b_2 + b_o b_1)\frac{1}{w^2} - (b_3 + 2b_o b_2 + b_o^2 b_1 + b_1^2)\frac{1}{w^3} + \dots . \end{aligned} \tag{9}$$

Using (9)we have

$$\begin{aligned} \frac{w(F_\lambda^n h(w))'}{F_\lambda^n h(w)} &= 1 + \frac{(1-\lambda)^n b_o}{w} + \frac{(1-\lambda)^{2n} b_o^2 + 2(1-2\lambda)^n b_1}{w^2} \\ &+ \frac{(1-\lambda)^{3n} b_o^3 + 3(1-\lambda)^n(1-2\lambda)^n b_o b_1 + 3(1-3\lambda)^n b_2 + 3(1-3\lambda)^n b_o b_1}{w^3} \\ &+ \dots \quad (w \in \Delta). \end{aligned} \tag{10}$$

Since $g(z)$ is a bi-univalent meromorphic starlike function of order α, there exist two functions p,q with positive real parts in Δ of the forms

$$p(z) = 1+\frac{c_1}{z}+\frac{c_2}{z^2}+\frac{c_3}{z^3}+\dots \quad (z \in \Delta) \quad and \quad q(w) = 1+\frac{d_1}{w}+\frac{d_2}{w^2}+\frac{d_3}{w^3}+\dots \quad (w \in \Delta), \tag{11}$$

such that

$$\frac{z(F_\lambda^n g(z))'}{F_\lambda^n g(z)} = \alpha + (1-\alpha)p(z) \quad and \quad \frac{w(F_\lambda^n h(w))'}{F_\lambda^n h(w)} = \alpha + (1-\alpha)q(w). \tag{12}$$

From (11),(12),(7) and (10) we obtain

$$\begin{aligned} &(1-\alpha)c_1 = -(1-\lambda)^n b_o \quad , \quad (1-\alpha)c_2 = (1-\lambda)^{2n} b_o^2 - 2(1-2\lambda)^n b_1, \\ &(1-\alpha)d_1 = (1-\lambda)^n b_o \quad and \quad (1-\alpha)d_2 = (1-\lambda)^{2n} b_o^2 + 2(1-2\lambda)^n b_1. \end{aligned} \tag{13}$$

Since $Re(p(z)) > 0$ in Δ, the function $p(\frac{1}{z}) \in \mathbf{P}$ and hence the coefficients c_n and similarly the coefficients d_n of the function q satisfy the inequality in lemma 1.1 and this immediately with equations in (13) yields the following estimates:

$$|b_o| \le 2\frac{(1-\alpha)}{(1-\lambda)^{\frac{n}{2}}} \quad and \quad |b_1| \le \frac{(1-\lambda)^n(1-\alpha)\sqrt{1+4(1-\lambda)^{2n}(1-\alpha)^2}}{(1-\lambda)^n(1-2\lambda)^n}. \tag{14}$$

This completes the proof of Theorem 2.1. □

If we put $n = 0$ or $\lambda = 0$, in Theorem 2.1 then we get the following corollary due to [17].

Corollary 2.1. *Let the function $g(z)$ given by (2) be in the subclass $H_{\Sigma^*_{\mathfrak{B}}}(\alpha)$. Then*

$$|b_o| \le 2(1-\alpha), \quad and \quad |b_1| \le (1-\alpha)\sqrt{1+4(1-\alpha)^2}. \tag{15}$$

The class of all meromorphic strongly starlike bi-univalent functions of order α is denoted by $\tilde{\Sigma}^*_{\mathfrak{B}}(\alpha)$.)

Definition 2.2. *A function $g(z)$ given by (2) is said to belong to the subclass $H_{\tilde{\Sigma}^*_{\mathfrak{B}}}(\alpha, n, \lambda)$ of bi-univalent strongly starlike meromorphic functions of order $\alpha, 0 < \alpha \le 1$ if the following conditions are satisfied:*

$$|arg(\frac{z(F_\lambda^n g(z))'}{F_\lambda^n g(z)})| < \frac{\alpha\pi}{2} \quad (0 < \alpha \le 1, 0 \le \lambda < \frac{1}{k+1}, n = 0, 1, 2, ..., z \in \Delta), \tag{16}$$

and

$$|arg(\frac{w(F_\lambda^n h(w))'}{F_\lambda^n h(w)})| < \frac{\alpha\pi}{2} \quad (0 < \alpha \le 1, 0 \le \lambda < \frac{1}{k+1}, n = 0, 1, 2, ..., w \in \Delta), \tag{17}$$

where the function $h(w)$ is the inverse of $g(z)$ given by (3).

Theorem 2.2. *Let the function $g(z)$ given by (2) be in the subclass $H_{\tilde{\Sigma}^*_{\mathfrak{B}}}(\alpha, n, \lambda)$. Then*

$$|b_o| \le \frac{2\alpha}{|1-\lambda|^n} \quad and \quad |b_1| \le \sqrt{5}\frac{\alpha^2}{(1-2\lambda)^n}.$$

Proof. Consider the function $g \in H_{\tilde{\Sigma}^*_{\mathfrak{B}}}(\alpha, n, \lambda)$. Then, by Definition 2 of the subclass $H_{\tilde{\Sigma}^*_{\mathfrak{B}}}(\alpha, n, \lambda)$

$$\frac{z(F_\lambda^n g(z))'}{F_\lambda^n g(z)} = (p(z))^\alpha, \tag{18}$$

and

$$\frac{w(F_\lambda^n h(w))'}{F_\lambda^n h(w)} = (q(w))^\alpha, \tag{19}$$

where $\frac{z(F_\lambda^n g(z))'}{F_\lambda^n g(z)}$ is given by (7) and $p(z)$ is given in (11) so,

$$\begin{aligned} &1 - \frac{(1-\lambda)^n b_o}{z} + \frac{(1-\lambda)^{2n} b_o^2 - 2(1-2\lambda)^n b_1}{z^2} \\ &- \frac{(1-\lambda)^{3n} b_o^3 - 3(1-\lambda)^n(1-2\lambda)^n b_o b_1 + 3(1-2\lambda)^n b_2}{z^3} + ... \\ &= 1 + \frac{\alpha c_1}{z} + \frac{\frac{1}{2}\alpha(\alpha-1)c_1^2 + \alpha c_2}{z^2} \\ &+ \frac{\frac{1}{6}\alpha(\alpha-1)(\alpha-2)c_1^3 + \alpha(\alpha-1)c_1 c_2 + \alpha c_3}{z^3} + \dots . \end{aligned} \tag{20}$$

Equating the coefficients in both sides of equation (20) we get

$$\alpha c_1 = -(1-\lambda)^n b_o \quad and \quad \frac{1}{2}\alpha(\alpha-1)c_1^2 + \alpha c_2 = (1-\lambda)^{2n} b_o^2 - 2(1-2\lambda)^n b_1. \tag{21}$$

Applying $q(w)$ from (11) and $\frac{w(F_\lambda^n h(w))'}{F_\lambda^n h(w)}$ from (12) in (19) we get

$$\begin{aligned} 1+&\frac{(1-\lambda)^n b_o}{w} + \frac{(1-\lambda)^{2n} b_o^2 + 2(1-2\lambda)^n b_1}{w^2} \\ +&\frac{(1-\lambda)^{3n} b_o^3 + 3(1-\lambda)^n(1-2\lambda)^n b_o b_1 + 3(1-3\lambda)^n b_2 + 3(1-3\lambda)^n b_o b_1}{w^3} + ... \\ =& 1 + \frac{\alpha d_1}{w} + \frac{\frac{1}{2}\alpha(\alpha-1)d_1^2 + \alpha d_2}{w^2} \\ +&\frac{\frac{1}{6}\alpha(\alpha-1)(\alpha-2)d_1^3 + \alpha(\alpha-1)d_1 c_2 + \alpha d_3}{w^3} + \dots . \end{aligned} \tag{22}$$

Equating the coefficients in both sides of equation (22) we get

$$\alpha d_1 = (1-\lambda)^n b_o \quad and \quad \frac{1}{2}\alpha(\alpha-1)d_1^2 + \alpha d_2 = (1-\lambda)^{2n} b_o^2 + 2(1-2\lambda)^n b_1. \tag{23}$$

From (21), (23) and applying Lemma 1.1, follows that

$$|b_o| \le \frac{2\alpha}{|1-\lambda|^n} \quad and \quad |b_1| \le \sqrt{5}\frac{\alpha^2}{(1-2\lambda)^n}.$$

This completes the proof of Theorem 2.2. □

If we put $n = 0$ or $\lambda = 0$, in Theorem 2.2 then we get the following corollary due to [17].

Corollary 2.2. *Let the function $g(z)$ given by (2) be in the subclass $H_{\tilde{\Sigma}^*_{\mathfrak{B}}}(\alpha)$. Then*

$$|b_o| \leq 2\alpha, \quad and \quad |b_1| \leq \sqrt{5}\alpha^2. \tag{24}$$

References

[1] Juma A.S. and Aziz F.S., (2012), Applying Ruscheweyh derivative on two sub-classes of bi-univalent functions, Inter. J. of Basic & Appl. Sci., V.,12 no.,06 , pp. 68-74.

[2] Ali R.M., Lee S.K., Ravichandran V., Supramaniam S., Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions, preprint.

[3] Brannan D.A. and Clunie J.G., (1980), Aspects of contemporary complex analysis, (Proceedings of the NATO Advanced Study Instituteheld at the university of Durham, Durham,July 12, 1979), Academic Press, London and New York.

[4] Brannan D.A. and Taha T.S., (1986), On some classesof bi-univalent functions, Studia Univ. Babes-Bolyai Math. 31, no. 2, pp. 70-77.

[5] Duren P.L., (1983), Univalent functions, in: Grunddlehren der mathematischen Wissenschaften, Band 259, Springer -Verlag, New York, Berlin, Hidelberg and Tokyo, .

[6] Duren P.L., (1971), Coefficients of meromorphic schlicht functions, Proc. Amer. Math. Soc., 28, pp. 169-172.

[7] Frasin B.A., Aouf M.K., (2011) New subclasses of bi-univalent functions, Appl. Math. Lett., 24 (9) pp.1569-1573.

[8] Goodman A.W., (1983) Univalent functions, Vol. 1, Polygonal Puplishing House, Washington, New Jersey.

[9] Kapoor G.P. and Mishra A.K., (2007), Coefficient estimates for inverses of starlike functions of positive order, J. Math. Anal. Appl., 329 , no. 2, pp. 922-934.

[10] Kubota Y., (1976-77), Coefficientsof meromorphic univalent functions, Kōdai Math. Sem. Rep., 28, no. 2-3, pp. 253-261.

[11] Lewin M., (1967), On a coefficient problem for bi-univalent functions, Proc. Amer.Math. Soc. 18, pp. 63-68.

[12] Netanyahu E., (1969), The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, Arch. Rational Mech. Anal. 32 , pp. 100-112.

[13] Schiffer M., (1938), Sur un problème d'extrémum de la représentation conforme, Bull. Soc. Math. France, 66 ,pp. 48-55.

[14] Schober G., (1977), Coefficients of inverses of meromorphic univalent functions, Proc. Amer. Math. Soc., 67 ,no. 1, pp. 111-116.

[15] Springer G., (1951), The coefficient problem for schlicht mappings of the exterior of the unit circle, Trans. Amer. Math. Soc., 70 , pp. 421-450.

[16] Srivastava H.M., Mishra A.K., Gochhayat P., (2010), Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett., 23, no. 10, pp. 1188-1192.

[17] Suzeini Abd Halim, Samaneh G. Hamidi and Ravichandran V., (2011), Coefficient estimates for meromorphic bi-univalent functions, arXiv:1108.4089v1 [math.CV].

[18] Taha T.S., (1981) Topics in univalent function theory, Ph.D. Thesis, University of London, .