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# ON CERTAIN CLASSES OF UNIVALENT MEROMORPHIC FUNCTIONS ASSOCIATED WITH INTEGRAL OPERATORS

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Abstract. This paper illustrates how some inclusion relationships of certain class of univalent meromorphic functions may be defined by using the linear operator. Further, a property preserving integrals is considered for the final outcome of the study.

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## 1. Introduction

A meromorphic function is a single-valued function that is analytic in all but possibly a discrete subset of its domain, and at those singularities it must go to infinity like a polynomial (i.e., these exceptional points must be poles and not essential singularities). A simpler definition states that a meromorphic function $f(z)$ is a function of the form

$$f(z) = \frac{g(z)}{h(z)},$$

where $g(z)$ and $h(z)$ are entire functions with $h(z) \neq 0$ (see [10], p. 64). A meromorphic function therefore may only have finite-order, isolated poles and zeros and no essential singularities in its domain. A meromorphic function with an infinite number of poles is exemplified by $csc\frac{1}{z}$ on the punctured disk $U^* = \{z : 0 < |z| < 1\}$. An equivalent definition of a meromorphic function is a complex analytic map to the Riemann sphere. For example the Gamma function is meromorphic in the whole complex plane, see [9] and [10].

In this paper, the linear operator is used to define some inclusion relationships of mermorophic fucntions. Moreover, the final outcome of the study has a property preserving integrals considered.

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## 2. Preliminaries

Let $\Sigma$ denote the class of meromorphic functions $f(z)$ normalized by

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the punctured unit disk $U = \{z : 0 < |z| < 1\}$. For $0 \leq \beta$, we denote by $S^*(\beta)$ and $k(\beta)$, the subclasses of $\Sigma$ consisting of all meromorphic functions which are, respectively, starlike of order $\beta$ and convex of order $\beta$ in $U$.

For functions $f_j(z)(j = 1; 2)$ defined by

$$f_j\left(z\right) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n, \tag{2}$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^n. \tag{3}$$

Analogous to the integral operator defined by Jung et al. [3] introduced and investigated the following integral operator:

$$Q_{\alpha,\beta} : \Sigma \to \Sigma$$

defined, in terms of the familiar Gamma function, by

$$Q_{\alpha,\beta} f\left(z\right) = \frac{\Gamma\left(\beta+\alpha\right)}{\Gamma\left(\alpha\right)\Gamma\left(\beta\right)} \frac{1}{z^{\beta+1}} \int_0^z t^{\beta} \left(1 - \frac{t}{z}\right)^{\alpha-1} f\left(t\right) dt$$

$$= \frac{1}{z} + \frac{\Gamma\left(\beta+\alpha\right)}{\Gamma\left(\beta\right)} \sum_{n=1}^{\infty} \frac{\Gamma\left(n+\beta+1\right)}{\Gamma\left(n+\beta+\alpha+1\right)} z^k \quad , \left(\alpha > 0;\ \beta > 0;\ z \in U^*\right). \tag{4}$$

By setting

$$f_{\alpha,\beta}(z) = \frac{1}{z} + \frac{\Gamma\left(\beta\right)}{\Gamma\left(\beta+\alpha\right)} \sum_{n=1}^{\infty} \frac{\Gamma\left(n+\beta+\alpha+1\right)}{\Gamma\left(n+\beta+1\right)} z^k \quad , \qquad \left(\alpha > 0;\ \beta > 0;\ z \in U^*\right), \tag{5}$$

we define a new function $f_{\alpha,\beta}^{\lambda}\left(z\right)$ in terms of the Hadamard product (or convolution):

$$f_{\alpha,\beta}(z) * f_{\alpha,\beta}^{\lambda}\left(z\right) = \frac{1}{z\left(1-z\right)^{\lambda}}, \qquad \left(\alpha > 0;\ \beta > 0; \lambda > 0;\ z \in U^*\right). \tag{6}$$

Then, motivated essentially by the operator $Q_{\alpha,\beta}$, we now introduce the operator

$$Q_{\alpha,\beta}^{\lambda} : \Sigma \to \Sigma$$

which is defined as

$$Q_{\alpha,\beta}^{\lambda} := f_{\alpha,\beta}^{\lambda}\left(z\right) * f\left(z\right), \qquad \left(\alpha > 0;\ \beta > 0; \lambda > 0;\ z \in U^*, f \in \Sigma\right). \tag{7}$$

Let us put

$$q_{\lambda,\mu}\left(z\right) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{\lambda}{n+1+\lambda}\right)^{\mu} z^n, \qquad \left(\lambda > 0,\ \mu \geq 0\right). \tag{8}$$

Corresponding to the functions $Q^{\lambda}_{\alpha,\beta}$ and $q_{\lambda,\mu}(z)$, and using the Hadamard product again for $f(z) \in \Sigma$, we define a new linear operator

$$Q^{a,\lambda,\mu}_{\alpha,\beta} f(z) = \frac{1}{z} + \frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)} \sum_{n=1}^{\infty} \frac{(a)_{n+1}}{(n+1)!} \frac{\Gamma(n+\beta+1)}{\Gamma(n+\beta+\alpha+1)} \left(\frac{\lambda}{n+1+\lambda}\right)^{\mu} a_n z^n \tag{9}$$

$(z \in U^*)$, where is $(a)_n$ the Pochhammer symbol defined by

$$(a)_n := \begin{cases} 1, & n=0 \\ a(a+1)\ldots(a+n-1) & (n \in := \{1,2,\ldots\}). \end{cases}$$

Clearly, $Q^{1,\lambda,0}_{\alpha,\beta} = Q_{\alpha,\beta}$. The meromorphic functions with the integral operators were considered recently by [1],[2], [5], [6] and [7].

It is readily verified from (9) that

$$z\left(Q^{a,\lambda,\mu}_{\alpha,\beta} f\right)'(z) = aQ^{a+1,\lambda,\mu}_{\alpha,\beta} f(z) - (a+1)\, Q^{a,\lambda,\mu}_{\alpha,\beta} f(z), \tag{10}$$

$$z\left(Q^{\lambda,\mu}_{\alpha+1,\beta} f\right)'(z) = (\beta+\alpha)\, Q^{a,\lambda,\mu}_{\alpha,\beta} f(z) - (\beta+\alpha+1)\, Q^{a,\lambda,\mu}_{\alpha,\beta} f(z). \tag{11}$$

**Definition 2.1.** *We say that a function $f \in \Sigma$ is in the class $\Sigma^{a,\lambda,\mu}_{\alpha,\beta}(\gamma)$ if it satisfies the following condition:*

$$\Re\left\{z^2\left(Q^{a,\lambda,\mu}_{\alpha,\beta} f(z)\right)'\right\} > \gamma, \qquad (z \in U^*) \tag{12}$$

*where $\alpha > 0$, $\beta > 0$, $\lambda > 0, \mu \geq 0$, and $0 \leq \gamma < 1$.*
*Using (10) condition (12) can be written in the form*

$$\Re\left\{-azQ^{a+1,\lambda,\mu}_{\alpha,\beta} f(z) + (a+1)\, Q^{a,\lambda,\mu}_{\alpha,\beta} f(z)\right\} > \gamma \qquad 0 \leq \gamma < 1, z \in U^*. \tag{13}$$

## 3. Main results

We will assume in the reminder of this paper that $\Sigma^{a,\lambda,\mu}_{\alpha,\beta}(\gamma)$. We begin by recalling the following result (Jack's lemma), which we shall apply in proving our inclusion theorems below.

**Lemma 3.1.** [4] *Let the (nonconstant) function $w(z)$ be analytic in $U$, with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in U$, then $z_0 w'(z_0) = \xi w(z_0)$, where $\xi$ is a real number and $\xi \geq 1$.*

**Theorem 3.1.** *The following inclusion property holds true for the class $\Sigma^{a,\lambda,\mu}_{\alpha,\beta}(\gamma)$*

$$\Sigma^{a+1,\lambda,\mu}_{\alpha,\beta}(\gamma) \subset \Sigma^{a,\lambda,\mu}_{\alpha,\beta}(\gamma). \tag{14}$$

*Proof.* Let $f(z) \in \Sigma^{a+1,\lambda,\mu}_{\alpha,\beta}(\gamma)$ and define a regular function $w(z)$ in $U$ such that $w(0) = 0$, $w(z) \neq -1$ by

$$-azQ^{a+1,\lambda,\mu}_{\alpha,\beta} f(z) + (a+1)\, Q^{a,\lambda,\mu}_{\alpha,\beta} f(z) = \frac{1+(2\gamma-1)\,w(z)}{z\,(1+w(z))}. \tag{15}$$

Differentiating (15) with respect to $z$, we obtain

$$-z_0^2\left(Q^{a+1,\lambda,\mu}_{\alpha,\beta} f(z)\right)' = \frac{1+(2\gamma-1)\,w(z)}{1+w(z)} - \frac{2\,(1-\gamma)}{\lambda} \frac{zw'(z)}{(1+w(z_0))^2}. \tag{16}$$

We claim that $|w(z)| < 1$ for $z \in U$. Otherwise there exists a point $z_0 \in U$ such that $\max_{|z|\le|z_0|} |w(z)| = |w(z)| = 1$. Applying Jack's lemma, we have

$$z_0 w'(z_0) = \xi w(z_0), \qquad \xi \ge 1. \tag{17}$$

From (16) and (17) we have

$$-z_0^2 \left(Q_{\alpha,\beta}^{a+1,\lambda,\mu} f(z)\right)' = \frac{1 + (2\gamma - 1) w(z_0)}{1 + w(z_0)} - \frac{2(1-\gamma)}{\lambda} \frac{z w'(z_0)}{(1 + w(z_0))^2}. \tag{18}$$

Since $\Re\left\{\frac{1+(2\gamma-1)w(z_0)}{1+w(z_0)}\right\} = \gamma$, $\xi \ge 1$ and $\frac{zw'(z_0)}{(1+w(z_0))^2}$ is real and positive, we see that

$$\Re\left\{-z_0^2 \left(Q_{\alpha,\beta}^{a+1,\lambda,\mu} f(z)\right)'\right\} < \gamma,$$

which obviously contradicts $f(z) \in \Sigma_{\alpha,\beta}^{a+1,\lambda,\mu}(\gamma)$. Hence $|w(z)| < 1$ for $z \in U$, and it follows from (15) that $f(z) \in \Sigma_{\alpha,\beta}^{a,\lambda,\mu}(\gamma)$. This completes the proof of Theorem 3.1. □

**Theorem 3.2.** *Let $c$ be any real number and $c > 0$. If $f(z) \in \Sigma_{\alpha,\beta}^{a,\lambda,\mu}(\gamma)$, then*

$$J_c(z) = \frac{c}{z^c} \int_0^z t^c f(t) dt \in \Sigma_{\alpha,\beta}^{a,\lambda,\mu}(\gamma), \qquad (c > 0). \tag{19}$$

*Proof.* From (19), we have

$$z \left(Q_{\alpha,\beta}^{a,\lambda,\mu} J_c(z)\right)' = c Q_{\alpha,\beta}^{a+1,\lambda,\mu} J_c(z) - (c+1) Q_{\alpha,\beta}^{a,\lambda,\mu} J_c(z). \tag{20}$$

Define a regular function $w(z)$ in $U$ such that $w(0) = 0$, $w(z) \ne -1$ by

$$-z_0^2 \left(Q_{\alpha,\beta}^{a,\lambda,\mu} J(z)\right)' = \frac{1 + (2\gamma - 1) w(z)}{1 + w(z)}. \tag{21}$$

From (20) and (21) we have

$$c Q_{\alpha,\beta}^{a+1,\lambda,\mu} J(z) - (c+1) Q_{\alpha,\beta}^{a,\lambda,\mu} J(z) = \frac{1 + (2\gamma - 1) w(z)}{z(1 + w(z))}. \tag{22}$$

Differentiating (22) with respect to $z$, and using (21) we obtain

$$-z_0^2 \left(Q_{\alpha,\beta}^{a,\lambda,\mu} J(z)\right)' = \frac{1 + (2\gamma - 1) w(z)}{1 + w(z)} - \frac{2(1-\gamma)}{c} \frac{z w'(z_0)}{(1 + w(z_0))^2}. \tag{23}$$

The remaining part of the proof of Theorem 3.2 is similar to that of Theorem 3.1. □

**Theorem 3.3.** *If $f(z) \in \Sigma_{\alpha,\beta}^{a+1,\lambda,\mu}(\gamma)$, and satisfy the condition*

$$\Re\left\{-z^2 \left(Q_{\alpha,\beta}^{a+1,\lambda,\mu} f(z)\right)'\right\} > \gamma - \frac{(1-\gamma)}{2c} \qquad (c > 0). \tag{24}$$

*Then the function*

$$J_c(z) = \frac{c}{z^c} \int_0^z t^c f(t) dt \in \Sigma_{\alpha,\beta}^{a,\lambda,\mu}(\gamma) \qquad (c > 0).$$

*Proof.* The proof of Theorem 3.3 is similar to that of Theorem 3.2 and hence, it will not be elaborated. □

**Theorem 3.4.** *Let $f(z)$ be defined by*

$$J_c(z) = \frac{c}{z^c}\int_0^z t^c f(t)dt \in \Sigma_{\alpha,\beta}^{a,\lambda,\mu}(\gamma) \qquad (c>0)\,. \tag{25}$$

*If $J_c(z) \in \Sigma_{\alpha,\beta}^{a,\lambda,\mu}(\gamma)$, then $f(z) \in \Sigma_{\alpha,\beta}^{a,\lambda,\mu}(\gamma)$ in $|z| < \frac{c}{1+\sqrt{c^2+1}}$.*

*Proof.* Since $J_c(z) \in \Sigma_{\alpha,\beta}^{a,\lambda,\mu}(\gamma)$ we can write

$$-z\left(Q_{\alpha,\beta}^{a,\lambda,\mu}J(z)\right)' = \frac{\gamma + (1-\gamma)\,u(z)}{z}, \tag{26}$$

where $u(z) \in P$, the class of functions with positive real part in the unit disk $U$ and normalized by $u(0) = 1$. We can re-write (26) as

$$-aQ_{\alpha,\beta}^{a+1,\lambda,\mu}J(z) + (a+1)\,Q_{\alpha,\beta}^{a,\lambda,\mu}J(z) = \frac{\gamma + (1-\gamma)\,u(z)}{z} \tag{27}$$

Differentiating (27) with respect to $z$, and using (20) we obtain

$$\frac{-z^2\left(Q_{\alpha,\beta}^{a,\lambda,\mu}J(z)\right)' - \gamma}{(1-\gamma)} = u(z) + \frac{1}{c}zu'(z). \tag{28}$$

Using the well-known estimate (see[[11]]) $|zu'(z)| \le \frac{2r}{1-r^2}\ \Re\, u(z)\,, |z| = r$ (28) yields

$$\Re\left\{\frac{-z^2\left(Q_{\alpha,\beta}^{a,\lambda,\mu}J(z)\right)' - \gamma}{(1-\gamma)}\right\} \ge \left(1 - \frac{2r}{c\,(1-r^2)}\right)\Re\, u(z) \tag{29}$$

The right-hand side of (29) is positive if $r < \frac{c}{1+\sqrt{c^2+1}}$. This completes the proof of Theorem 3.4. $\square$

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