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NEW SUFFICIENT CONDITIONS FOR STARLIKE AND CONVEX FUNCTIONS

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ABSTRACT. Let $\mathcal{A}$ be the class of analytic functions $f(z)$ in the open unit disc. Applying the subordination, some sufficient conditions for starlikeness and convexity are discussed.

Keywords: Analytic function, Starlike of order α, Convex of order α, Subordination.

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1. Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z) \in \mathcal{A}$ is said to be the starlike function of order α if it satisfies

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U})$$

for some α $(0 \leq \alpha < 1)$. Also a function $f(z) \in \mathcal{A}$ is said to be the convex function of order α if it satisfies

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathbb{U})$$

for some α $(0 \leq \alpha < 1)$. These classes are denoted by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, respectively. We well-known that $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ and $\mathcal{K}(0) \equiv \mathcal{K}$, and that the relation $f(z) \in \mathcal{K}$ if and only if $zf'(z) \in \mathcal{S}^*$.

By investigating expressions

$$\frac{z^2 f'(z)}{(f(z))^2} - (1+\gamma)\frac{z}{f(z)}, \; \frac{zf''(z)}{(f'(z))^2} - \gamma\frac{1}{f'(z)}$$

and

$$\frac{f(z)f''(z)}{(f'(z))^2} - (1+\gamma)\frac{f(z)}{zf'(z)},$$

we would like to introduce some sufficient conditions for the classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$.

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2. Sufficient conditions for starlikeness and convexity

For analitic functions $f(z)$ and $g(z)$ in $\mathbb{U}$, $f(z)$ is said to be subordidnate to $g(z)$ in $\mathbb{U}$ if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that $w(0) = 0$, $|w(z)| < 1$ and $f(z) = g(w(z))$. We denote this subordination by

$$f(z) \prec g(z).$$

If $g(z)$ is univalent in $\mathbb{U}$, $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.
we make use of the case which γ is a non-negative real number of Theorem 2 of Miller, Mocanu and Reade [1] as following:

Lemma 2.1. *Let $F(z)$ and $G(z)$ be analytic functions in $\mathbb{U}$, $\gamma \geq 0$ and $G'(0) \neq 0$. Furthermore, in the case of $\gamma = 0$, $F(0) = G(0) = 0$. If*

$$\operatorname{Re}\left(1 + \frac{zG''(z)}{G'(z)}\right) > k(\gamma) = \begin{cases} -\dfrac{\gamma}{2} & (\gamma \leq 1) \\ -\dfrac{1}{2\gamma} & (\gamma \geq 1) \end{cases} \qquad (z \in \mathbb{U})$$

and

$$F(z) \prec G(z),$$

then

$$z^{-\gamma} \int_0^z t^{\gamma-1} F(t)dt \prec z^{-\gamma} \int_0^z t^{\gamma-1} G(t)dt.$$

For $F(z) = 1 - \gamma p(z) - zp'(z)$, the following lemma was studied by Singh and Tuneski [3].

Lemma 2.2. *Let $p(z)$ and $G(z)$ be analytic functions in $\mathbb{U}$, $\gamma \geq 0$ and $G'(0) \neq 0$. If*

$$\operatorname{Re}\left(1 + \frac{zG''(z)}{G'(z)}\right) > k(\gamma) \qquad (z \in \mathbb{U})$$

and

$$1 - \gamma p(z) - zp'(z) \prec G(z),$$

then

$$p(z) - Cz^{-\gamma} \prec z^{-\gamma} \int_0^z t^{\gamma-1}(1 - G(t))dt,$$

where $C = p(0)$ for $\gamma = 0$ and $C = 0$ for $\gamma > 0$.

Lemma 2.3. (Tuneski [4]) *Let us $f(z) \in \mathcal{A}$. If it satisfies*

$$\left| f'(z) - (1-\gamma)\frac{f(z)}{z} - \gamma \right| < \lambda \qquad (z \in \mathbb{U})$$

for some γ ($\gamma \geq 0$) and λ ($\lambda > 0$), then

$$\left| \frac{f(z)}{z} - 1 \right| < \frac{\lambda}{1+\gamma} \qquad (z \in \mathbb{U})$$

and

$$|f(z)| < 1 + \frac{\lambda}{1+\gamma} \qquad (z \in \mathbb{U}).$$

Using similer manner of Lemma 2.3, Our first result is following

Theorem 2.1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - (1+\gamma)\frac{z}{f(z)} + \gamma\right| < \lambda \qquad (z \in \mathbb{U}) \tag{1}$$

for some γ $(\gamma \geq 0)$ and λ $(\lambda > 0)$, then

$$\left|\frac{z}{f(z)} - 1\right| < \frac{\lambda}{1+\gamma} \qquad (z \in \mathbb{U}). \tag{2}$$

Proof. Let us define the function $G(z)$ by $G(z) = 1 - \gamma + \lambda z$, then $G'(0) = \lambda$ and

$$\mathrm{Re}\left(1 + \frac{zG''(z)}{G'(z)}\right) = 1 \qquad (z \in \mathbb{U}).$$

Furthermore, let us suppose that $p(z) = \dfrac{z}{f(z)}$, then $p(0) = 1$ and

$$1 - \gamma p(z) - zp'(z) = 1 - (1+\gamma)\frac{z}{f(z)} + \frac{z^2 f'(z)}{(f(z))^2}.$$

On the other hand, we have

$$1 - (1+\gamma)\frac{z}{f(z)} + \frac{z^2 f'(z)}{(f(z))^2} \prec 1 - \gamma + \lambda z$$

from inequality (1). Applying Lemma 2, we obtain

$$\begin{aligned}\frac{z}{f(z)} - Cz^{-\gamma} &\prec z^{-\gamma}\int_0^z t^{\gamma-1}(1 - G(t))dt \\ &= 1 - \frac{\lambda}{1+\gamma}z - C_1 z^{-\gamma},\end{aligned}$$

where $C = C_1 = 1$ for $\gamma = 0$ and $C = C_1 = 0$ for $\gamma > 0$. Thus, we arrive

$$\left|\frac{z}{f(z)} - 1\right| < \frac{\lambda}{1+\gamma} \qquad (z \in \mathbb{U}).$$

The left hand side of the inequality (1) holds true for λ if we take the function

$$f(z) = \frac{z}{1 + \dfrac{\lambda}{1+\gamma}e^{i\theta}z}$$

from inequality (2), implying that this result is sharp. □

By vertue of Theorem 2.1, we obtain the sufficient condition of starlikeness below

Theorem 2.2. *If $f(z) \in \mathcal{A}$ satisfies*

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - (1+\gamma)\frac{z}{f(z)} + \gamma\right| < \lambda \qquad (z \in \mathbb{U}) \tag{3}$$

for some γ $(\gamma \geq 0)$ and λ $\left(0 < \lambda \leq \frac{1}{2}\right)$, then $f(z) \in \mathcal{S}^*\left(\dfrac{(1+\gamma)(1-2\lambda)}{1+\gamma-\lambda}\right)$.

Proof. Supposing that a function $f(z)$ satisfies the inequality (3) and that there exists an analytic function $w(z)$ in $\mathbb{U}$ such that $w(0) = 0$ and $|w(z)| < 1$, then we see

$$\frac{zf'(z)}{f(z)} - (1+\gamma) = \frac{f(z)}{z}(\lambda w(z) - \gamma).$$

It follows that

$$\left|\frac{zf'(z)}{f(z)} - (1+\gamma)\right| = \left|\frac{f(z)}{z}\right| |\lambda w(z) - \gamma| < \frac{(1+\gamma)(\gamma+\lambda)}{1+\gamma-\lambda}.$$

This shows that

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > 1+\gamma - \frac{(1+\gamma)(\gamma+\lambda)}{1+\gamma-\lambda} = \frac{(1+\gamma)(1-2\lambda)}{1+\gamma-\lambda}.$$

We complete the proof of the theorem. □

Taking $\lambda = \frac{1}{2}$ in Theorem 2.2, we have

Corollary 2.1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - (1+\gamma)\frac{z}{f(z)} + \gamma\right| < \frac{1}{2} \qquad (z \in \mathbb{U})$$

for some γ $(\gamma \geq 0)$, then $f(z) \in \mathcal{S}^*$.

Putting $zf'(z)$ instead of $f(z)$ in Theorem 2.1, we get

Theorem 2.3. *If $f(z) \in \mathcal{A}$ satisfies*

$$\left|\frac{zf''(z)}{(f'(z))^2} - \gamma\frac{1}{f'(z)} + \gamma\right| < \lambda \qquad (z \in \mathbb{U}) \tag{4}$$

for some γ $(\gamma \geq 0)$ and λ $(\lambda > 0)$, then

$$\left|\frac{1}{f'(z)} - 1\right| < \frac{\lambda}{1+\gamma} \qquad (z \in \mathbb{U}). \tag{5}$$

Proof. Letting $p(z) = \frac{1}{f'(z)}$ in the proof of Theorem 2.1, we arrive

$$\left|\frac{1}{f'(z)} - 1\right| < \frac{\lambda}{1+\gamma} \qquad (z \in \mathbb{U}).$$

The left hand side of the inequality (4) holds true for λ if we take the function

$$f(z) = \frac{1+\gamma}{\lambda e^{i\theta}} \log\left(1 + \frac{\lambda}{1+\gamma}e^{i\theta}z\right)$$

from inequality (5), implying that this result is sharp. □

In view of Theorem 2.3, we obtain the sufficient condition of convexity below

Theorem 2.4. *If* $f(z) \in \mathcal{A}$ *satisfies*

$$\left| \frac{zf''(z)}{(f'(z))^2} - \gamma \frac{1}{f'(z)} + \gamma \right| < \lambda \qquad (z \in \mathbb{U})$$

for some γ $(\gamma \geq 0)$ *and* λ $\left(0 < \lambda \leq \frac{1}{2}\right)$, *then* $f(z) \in \mathcal{K}\left(\frac{(1+\gamma)(1-2\lambda)}{1+\gamma-\lambda}\right)$.

Proof. As the same technique in the proof of Theorem 2.2, we see

$$\left| \frac{zf''(z)}{f'(z)} - \gamma \right| < \frac{(1+\gamma)(\gamma+\lambda)}{1+\gamma+\lambda}.$$

This shows that

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{(1+\gamma)(1-2\lambda)}{1+\gamma-\lambda} \qquad (z \in \mathbb{U})$$

which proves the theorem. □

Taking $\lambda = \frac{1}{2}$ in Theorem 2.4, we have

Corollary 2.2. *If* $f(z) \in \mathcal{A}$ *satisfies*

$$\left| \frac{zf''(z)}{(f'(z))^2} - \gamma \frac{1}{f'(z)} + \gamma \right| < \frac{1}{2} \qquad (z \in \mathbb{U})$$

for some γ $(\gamma \geq 0)$, *then* $f(z) \in \mathcal{K}$.

Applying the same way as the proof of Theorem 2.1, we get

Theorem 2.5. *If* $f(z) \in \mathcal{A}$ *satisfies*

$$\left| \frac{f(z)f''(z)}{(f'(z))^2} - (1-\gamma)\frac{f(z)}{zf'(z)} + \gamma - 1 \right| < \lambda \qquad (z \in \mathbb{U}) \tag{6}$$

for some γ $(\gamma \geq 0)$ *and* λ $(\lambda > 0)$, *then*

$$\left| \frac{f(z)}{zf'(z)} - 1 \right| < \frac{\lambda}{1+\gamma} \qquad (z \in \mathbb{U}). \tag{7}$$

Proof. Letting $p(z) = \frac{f(z)}{zf'(z)}$ in the proof of Theorem 2.1, we arrive

$$\left| \frac{f(z)}{zf'(z)} - 1 \right| < \frac{\lambda}{1+\gamma} \qquad (z \in \mathbb{U}).$$

The left hand side of the inequality (6) holds true for λ if we take the function

$$f(z) = \frac{z}{1 + \frac{\lambda}{1+\gamma} e^{i\theta} z}$$

from the inequality (7), implying that this result is sharp. □

In view of Theorem 2.5, we obtain the sufficient condition of convexity below

Theorem 2.6. *If* $f(z) \in \mathcal{A}$ *satisfies*

$$f(z) \in \mathcal{A}, \quad \left| \frac{f(z)f''(z)}{(f'(z))^2} - (1-\gamma)\frac{f(z)}{zf'(z)} + \gamma - 1 \right| < \lambda \qquad (z \in \mathbb{U}), \tag{8}$$

then $f(z) \in \mathcal{K}\left(1 - \dfrac{2\gamma\lambda}{1+\gamma-\lambda}\right)$ *for some* γ $(\gamma \geq 1)$ *and* λ $\left(0 \leq \lambda \leq \dfrac{1+\gamma}{2\gamma+1}\right)$, *or* $f(z) \in \mathcal{K}\left(1 + \dfrac{2\gamma^2 - 2\gamma\lambda - 2}{1+\gamma-\lambda}\right)$ *for some* γ $\left(\dfrac{1}{2} < \gamma \leq 1\right)$ *and* λ $\left(0 < \lambda \leq \dfrac{2\gamma^2+\gamma-1}{2\gamma+1}\right)$.

Proof. As the same technique in the proof of Theorem 2.2, we see

$$\left| \frac{zf''(z)}{f'(z)} + (1-\gamma) \right| < \frac{(1+\gamma)(\gamma-1+\lambda)}{1+\gamma-\lambda} \qquad (z \in \mathbb{U})$$

when $(\gamma \geq 1)$ for the inequality (8). This shows that

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 1 - \frac{2\gamma\lambda}{1+\gamma-\lambda} \qquad (z \in \mathbb{U}).$$

Moreover, we see

$$\left| \frac{zf''(z)}{f'(z)} + (1-\gamma) \right| < 1 - \frac{(1+\gamma)(1-\gamma+\lambda)}{1+\gamma-\lambda} \qquad (z \in \mathbb{U})$$

when $\left(\dfrac{1}{2} < \gamma \leq 1\right)$ for the inequality (8). This shows that

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 1 + \frac{2\gamma^2 - 2\gamma\lambda - 2}{1+\gamma-\lambda} \qquad (z \in \mathbb{U}).$$

The proof of theorem is completed. □

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