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SOLVABILITY OF ITERATIVE SYSTEMS OF THREE-POINT BOUNDARY VALUE PROBLEMS

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Abstract. We establish a criterion for the existence of at least one positive solution for the iterative system of three-point boundary value problems by determining the eigenvalues λ_i, $1 \le i \le n$, using Guo–Krasnosel'skii fixed point theorem.

Keywords: Iterative system, boundary value problem, eigenvalue, positive solution, cone.

AMS Subject Classification: 34B18, 34A34

1. Introduction

In this paper, we are concerned with determining the eigenvalues λ_i, $1 \le i \le n$, for which there exist positive solutions of the iterative system of second order differential equations,

$$\left.\begin{aligned} y_i^{''}(t) + \lambda_i p_i(t) f_i(y_{i+1}(t)) =& 0, \quad 1 \le i \le n, \quad t \in [t_1, t_3], \\ y_{n+1}(t) =& y_1(t), \quad t \in [t_1, t_3], \end{aligned}\right\} \tag{1}$$

satisfying the different three-point boundary conditions,

$$\alpha_i y_i(t_1) - \beta_i y_i^{'}(t_1) = 0 \text{ and } \gamma_i y_i(t_3) + \delta_i y_i^{'}(t_3) = y_i^{'}(t_2), \ 1 \le i \le n, \tag{2}$$

where $0 \le t_1 < t_2 < t_3$, $\alpha_i > 0, \beta_i > 0, \gamma_i > 0$ and $\delta_i > 1$ are real numbers, for $1 \le i \le n$. We assume the following conditions hold throughout the paper:

(A1) $f_i : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, for $1 \le i \le n$,

(A2) $p_i : [t_1, t_3] \to \mathbb{R}^+$ is continuous and p_i does not vanish identically on any closed subinterval of $[t_1, t_3]$, for $1 \le i \le n$,

(A3) $\alpha_i > 0, \beta_i > 0, \gamma_i > 0, \delta_i > 1$ and $\gamma_i > \frac{\alpha_i \delta_i}{\alpha_i(t_2 - t_1) + \beta_i}$, for $1 \le i \le n$,

(A4) each of

$$f_{i0} = \lim_{x \to 0^+} \frac{f_i(x)}{x} \text{ and } f_{i\infty} = \lim_{x \to \infty} \frac{f_i(x)}{x},$$

for $1 \le i \le n$, exists as positive real number.

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Recently, the existence of positive solutions for the system of differential equations with multi-point boundary conditions have been studied by many authors due to their striking applications to almost all area of science, engineering and technology. In 2007, Henderson and Ntouyas [6] established the existence of positive solutions for the system of n^{th} order differential equations,

$$u^{(n)}(t) + \lambda a(t) f(v(t)) = 0,\ 0 < t < 1,$$
$$v^{(n)}(t) + \lambda b(t) g(u(t)) = 0,\ 0 < t < 1,$$

satisfying the three-point nonlocal boundary conditions,

$$u(0) = 0, u'(0) = 0, \cdots, u^{(n-2)}(0) = 0,\ u(1) = \alpha u(\eta),$$
$$v(0) = 0, v'(0) = 0, \cdots, v^{(n-2)}(0) = 0,\ v(1) = \alpha v(\eta),$$

where $0 < \eta < 1$ and $0 < \alpha\eta^{n-1} < 1$.

In 2008, Henderson, Ntouyas and Purnaras [7] deals the existence of positive solutions for the system of second order differential equations,

$$u''(t) + \lambda a(t) f(v(t)) = 0,\ 0 < t < 1,$$
$$v''(t) + \lambda b(t) g(u(t)) = 0,\ 0 < t < 1,$$

satisfying the three-point boundary conditions,

$$u(0) = \beta u(\eta),\ u(1) = \alpha u(\eta),$$
$$v(0) = \beta v(\eta),\ v(1) = \alpha v(\eta),$$

where $0 < \eta < 1$, $0 < \alpha < \frac{1}{\eta}$, $0 < \beta < \frac{1-\alpha\eta}{1-\eta}$, and also in the same year, same authors, (see [8] in references) studied the existence of positive solutions for the system of second order differential equations,

$$u''(t) + \lambda a(t) f(v(t)) = 0,\ 0 < t < 1,$$
$$v''(t) + \mu b(t) g(u(t)) = 0,\ 0 < t < 1,$$

satisfying the four-point boundary conditions,

$$u(0) = \alpha u(\xi),\ u(1) = \beta u(\eta),$$
$$v(0) = \alpha v(\xi),\ v(1) = \beta v(\eta),$$

where $0 < \xi < \eta < 1$, $0 \leq \alpha, \beta < 1$.

Till now in the literature, the authors established results for the existence of positive solutions for the system of two differential equations satisfying the same boundary conditions. We wish to extend these results to system of n differential equations satisfying the different boundary conditions.

The rest of the paper is organized as follows. In Section 2, we construct the Green's function for the homogeneous problem corresponding to (1)-(2) and estimate bounds for the Green's function. In Section 3, we determine the eigenvalues for which there exist positive solutions of the boundary value problem (1)-(2) by using Guo–Krasnosel'skii fixed point theorem for operators on a cone in a Banach space. Finally as an application, we give an example to illustrate our result.

2. Green's Function and Bounds

In this section, we construct the Green's function for the homogeneous problem corresponding to (1)-(2) and estimate bounds for the Green's function.

Let $G_i(t,s)$ be the Green's function for the homogeneous boundary value problem,

$$-y_i^{''}(t)=0, \quad t\in[t_1,t_3], \tag{3}$$

$$\alpha_i y_i(t_1)-\beta_i y_i^{'}(t_1)=0 \text{ and } \gamma_i y_i(t_3)+\delta_i y_i^{'}(t_3)=y_i^{'}(t_2), \tag{4}$$

for $1\le i\le n$.

Lemma 2.1. *Let $d_i=\alpha_i[\gamma_i(t_3-t_1)+\delta_i-1]+\beta_i\gamma_i\neq 0$, $1\le i\le n$. Then, for $1\le i\le n$, the Green's function $G_i(t,s)$ for the homogeneous boundary value problem* (3)-(4) *is given by*

$$G_i(t,s)=\begin{cases} \underset{t\in[t_1,t_2]}{G_i(t,s)}=\begin{cases} G_{i1}(t,s), & t_1\le s\le t\le t_2<t_3,\\ G_{i2}(t,s), & t_1\le t\le s\le t_2<t_3,\\ G_{i3}(t,s), & t_1<t\le t_2\le s\le t_3,\end{cases}\\ \\ \underset{t\in[t_2,t_3]}{G_i(t,s)}=\begin{cases} G_{i4}(t,s), & t_1<t_2\le s\le t\le t_3,\\ G_{i5}(t,s), & t_1<t_2\le t\le s\le t_3,\\ G_{i6}(t,s), & t_1\le s\le t_2\le t<t_3,\end{cases}\end{cases} \tag{5}$$

where

$$\begin{aligned} G_{i1}(t,s)=&\frac{1}{d_i}[\alpha_i(s-t_1)+\beta_i][\gamma_i(t_3-t)+\delta_i-1],\\ G_{i2}(t,s)=&\frac{1}{d_i}[\alpha_i(t-t_1)+\beta_i][\gamma_i(t_3-s)+\delta_i-1],\\ G_{i3}(t,s)=&\frac{1}{d_i}[\alpha_i(t-t_1)+\beta_i][\gamma_i(t_3-s)+\delta_i],\\ G_{i4}(t,s)=&\frac{1}{d_i}[(\alpha_i(s-t_1)+\beta_i)(\gamma_i(t_3-t)+\delta_i)+\alpha_i(t-s)],\\ G_{i5}(t,s)=&\frac{1}{d_i}[\alpha_i(t-t_1)+\beta_i][\gamma_i(t_3-s)+\delta_i],\\ G_{i6}(t,s)=&\frac{1}{d_i}[\alpha_i(s-t_1)+\beta_i][\gamma_i(t_3-t)+\delta_i-1]. \end{aligned}$$

Lemma 2.2. *Assume that the condition* $(A3)$ *is satisfied. Then, for $1\le i\le n$, the Green's function $G_i(t,s)$ of* (3)-(4) *is positive, for all $(t,s)\in(t_1,t_3)\times(t_1,t_3)$.*

Proof. By simple algebraic calculations, we can easily establish the positivity of the Green's function. □

Lemma 2.3. *Assume that the condition* $(A3)$ *is satisfied. Then, for $1\le i\le n$, the Green's function $G_i(t,s)$ in* (5) *satisfies the following inequality,*

$$k_iG_i(s,s)\le G_i(t,s)\le G_i(s,s), \ for\ all\ (t,s)\in[t_1,t_3]\times[t_1,t_3], \tag{6}$$

where

$$k_i=min\left\{\frac{\delta_i-1}{\gamma_i(t_3-t_1)+\delta_i-1},\frac{\beta_i}{\alpha_i(t_3-t_1)+\beta_i}\right\}<1.$$

Proof. For $1\le i\le n$, the Green's function $G_i(t,s)$ is given in (5). In each case, we prove the inequality as in (6).

Case 1. For $t_1\le s\le t\le t_2<t_3$,

$$\frac{G_i(t,s)}{G_i(s,s)}=\frac{G_{i1}(t,s)}{G_{i1}(s,s)}=\frac{\gamma_i(t_3-t)+\delta_i-1}{\gamma_i(t_3-s)+\delta_i-1}\le 1$$

and also

$$\frac{G_i(t,s)}{G_i(s,s)} = \frac{G_{i1}(t,s)}{G_{i1}(s,s)} = \frac{\gamma_i(t_3-t)+\delta_i-1}{\gamma_i(t_3-s)+\delta_i-1} \geq \frac{\delta_i-1}{\gamma_i(t_3-t_1)+\delta_i-1}.$$

Case 2. For $t_1 \leq t \leq s \leq t_2 < t_3$,

$$\frac{G_i(t,s)}{G_i(s,s)} = \frac{G_{i2}(t,s)}{G_{i2}(s,s)} = \frac{\alpha_i(t-t_1)+\beta_i}{\alpha_i(s-t_1)+\beta_i} \leq 1$$

and also

$$\frac{G_i(t,s)}{G_i(s,s)} = \frac{G_{i2}(t,s)}{G_{i2}(s,s)} = \frac{\alpha_i(t-t_1)+\beta_i}{\alpha_i(s-t_1)+\beta_i} \geq \frac{\beta_i}{\alpha_i(t_3-t_1)+\beta_i}.$$

Case 3. For $t_1 < t \leq t_2 \leq s \leq t_3$,

$$\frac{G_i(t,s)}{G_i(s,s)} = \frac{G_{i3}(t,s)}{G_{i3}(s,s)} = \frac{\alpha_i(t-t_1)+\beta_i}{\alpha_i(s-t_1)+\beta_i} \leq 1$$

and also

$$\frac{G_i(t,s)}{G_i(s,s)} = \frac{G_{i3}(t,s)}{G_{i3}(s,s)} = \frac{\alpha_i(t-t_1)+\beta_i}{\alpha_i(s-t_1)+\beta_i} \geq \frac{\beta_i}{\alpha_i(t_3-t_1)+\beta_i}.$$

Case 4. For $t_1 < t_2 \leq s \leq t \leq t_3$,

$$\begin{aligned}\frac{G_i(t,s)}{G_i(s,s)} = \frac{G_{i4}(t,s)}{G_{i4}(s,s)} &= \frac{[\alpha_i(s-t_1)+\beta_i][\gamma_i(t_3-t)+\delta_i]+\alpha_i(t-s)}{[\alpha_i(s-t_1)+\beta_i][\gamma_i(t_3-s)+\delta_i]} \\ &\leq \frac{\gamma_i(t_3-s)+\delta_i}{\gamma_i(t_3-s)+\delta_i} = 1\end{aligned}$$

and also

$$\begin{aligned}\frac{G_i(t,s)}{G_i(s,s)} = \frac{G_{i4}(t,s)}{G_{i4}(s,s)} &= \frac{[\alpha_i(s-t_1)+\beta_i][\gamma_i(t_3-t)+\delta_i]+\alpha_i(t-s)}{[\alpha_i(s-t_1)+\beta_i][\gamma_i(t_3-s)+\delta_i]} \\ &\geq \frac{\delta_i-1}{\gamma_i(t_3-t_1)+\delta_i-1}.\end{aligned}$$

Similarly, we can easily establish the inequality, when the Green's function $G_i(t,s) = G_{i5}(t,s)$ and $G_i(t,s) = G_{i6}(t,s)$ as in case 3 and case 1 respectively. Hence the inequality (6). $\square$

Lemma 2.4. *Assume that the condition (A3) is satisfied. Then, for $1 \leq i \leq n$, the Green's function $G_i(t,s)$ in (5) satisfies the following inequality,*

$$G_i(t,s) \geq kG_i(s,s), \ \ for\ all\ \ (t,s) \in [t_1,t_3] \times [t_1,t_3],$$

where $k = min\{k_1, k_2, \cdots, k_n\}$.

We note that an n-tuple $(y_1(t), y_2(t), \cdots, y_n(t))$ is a solution of the boundary value problem (1)-(2) if and only if

$$\begin{aligned} y_1(t) =& \lambda_1 \int_{t_1}^{t_3} G_1(t,s_1)p_1(s_1)f_1\Big(\lambda_2 \int_{t_1}^{t_3} G_2(s_1,s_2)p_2(s_2)\cdots \\ & f_{n-1}\Big(\lambda_n \int_{t_1}^{t_3} G_n(s_{n-1},s_n)p_n(s_n)f_n(y_1(s_n))ds_n\Big)\cdots ds_2\Big)\, ds_1, \ t \in [t_1,t_3], \end{aligned}$$

$$y_i(t) = \lambda_i \int_{t_1}^{t_3} G_i(t,s)p_i(s)f_i(y_{i+1}(s))ds, \ 2 \leq i \leq n, \ t \in [t_1,t_3],$$

where

$$y_{n+1}(t) = y_1(t), \ t \in [t_1, t_3].$$

To determine the eigenvalues for which the boundary value problem (1)-(2) has at least one positive solution, we will employ the following Guo–Krasnosel'skii fixed point theorem.

Theorem 2.1. *Let X be a Banach Space, $\kappa \subseteq X$ be a cone and suppose that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $T : \kappa \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \kappa$ is completely continuous operator such that either*

(i) $\|Tu\| \leq \|u\|, \ u \in \kappa \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|, \ u \in \kappa \cap \partial\Omega_2$, *or*
(ii) $\|Tu\| \geq \|u\|, \ u \in \kappa \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|, \ u \in \kappa \cap \partial\Omega_2$ *holds.*

Then T has a fixed point in $\kappa \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Positive Solutions in a Cone

In this section, we establish criteria to determine the eigenvalues for which the boundary value problem (1)-(2) has at least one positive solution in a cone.

For our construction, let $B = \{x \mid x \in C[t_1, t_3]\}$ be a Banach space with the norm

$$\|x\| = \sup_{t \in [t_1, t_3]} |x(t)|.$$

Define a cone $P \subset B$ by

$$P = \Big\{x \in B \mid x(t) \geq 0 \text{ on } [t_1, t_3] \text{ and } \min_{t \in [t_1, t_3]} x(t) \geq k\|x\|\Big\},$$

where k is given in Lemma 2.4.

Now, we define an integral operator $T : P \to B$, for $y_1 \in P$, by

$$\begin{aligned} Ty_1(t) =&\lambda_1 \int_{t_1}^{t_3} G_1(t, s_1)p_1(s_1)f_1\Big(\lambda_2 \int_{t_1}^{t_3} G_2(s_1, s_2)p_2(s_2) \cdots \\ & f_{n-1}\Big(\lambda_n \int_{t_1}^{t_3} G_n(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))ds_n\Big) \cdots ds_2\Big)ds_1. \end{aligned} \tag{7}$$

Notice from $(A1)$, $(A2)$ and Lemma 2.2 that, for $y_1 \in P$, $Ty_1(t) \geq 0$ on $[t_1, t_3]$. Also, for $y_1 \in P$, we have from Lemma 2.3 that

$$\begin{aligned} Ty_1(t) \leq \lambda_1 &\int_{t_1}^{t_3} G_1(s_1, s_1)p_1(s_1)f_1\Big(\lambda_2 \int_{t_1}^{t_3} G_2(s_1, s_2)p_2(s_2) \cdots \\ & f_{n-1}\Big(\lambda_n \int_{t_1}^{t_3} G_n(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))ds_n\Big) \cdots ds_2\Big)ds_1 \end{aligned}$$

so that

$$\begin{aligned} \|Ty_1\| \leq \lambda_1 &\int_{t_1}^{t_3} G_1(s_1, s_1)p_1(s_1)f_1\Big(\lambda_2 \int_{t_1}^{t_3} G_2(s_1, s_2)p_2(s_2) \cdots \\ & f_{n-1}\Big(\lambda_n \int_{t_1}^{t_3} G_n(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))ds_n\Big) \cdots ds_2\Big)ds_1. \end{aligned} \tag{8}$$

Next, if $y_1 \in P$, we have from Lemma 2.4 and (8) that

$$\begin{aligned}\min_{t\in[t_1,t_3]} Ty_1(t) &= \min_{t\in[t_1,t_3]} \Big\{\lambda_1 \int_{t_1}^{t_3} G_1(t,s_1)p_1(s_1)f_1\Big(\lambda_2 \int_{t_1}^{t_3} G_2(s_1,s_2)p_2(s_2)\cdots \\ &\quad f_{n-1}\Big(\lambda_n \int_{t_1}^{t_3} G_n(s_{n-1},s_n)p_n(s_n)f_n(y_1(s_n))ds_n\Big)\cdots ds_2\Big)ds_1\Big\} \\ &\geq \lambda_1 k \int_{t_1}^{t_3} G_1(s_1,s_1)p_1(s_1)f_1\Big(\lambda_2 \int_{t_1}^{t_3} G_2(s_1,s_2)p_2(s_2)\cdots \\ &\quad f_{n-1}\Big(\lambda_n \int_{t_1}^{t_3} G_n(s_{n-1},s_n)p_n(s_n)f_n(y_1(s_n))ds_n\Big)\cdots ds_2\Big)ds_1 \\ &\geq k\|Ty_1\|.\end{aligned}$$

Hence, $Ty_1 \in P$ and so $T : P \to P$. Further, the operator T is completely continuous operator by an application of the Ascoli-Arzela Theorem.

Now, we seek suitable fixed points of T belonging to the cone P. For our first result, define positive numbers M_1 and M_2 by

$$M_1 = \max_{1\leq i\leq n}\left\{\left[k^2 \int_{t_1}^{t_3} G_i(s,s)p_i(s)ds f_{i\infty}\right]^{-1}\right\}$$

and

$$M_2 = \min_{1\leq i\leq n}\left\{\left[\int_{t_1}^{t_3} G_i(s,s)p_i(s)ds f_{i0}\right]^{-1}\right\}.$$

Theorem 3.1. *Assume that the conditions (A1)-(A4) are satisfied. Then, for each $\lambda_1, \lambda_2, \cdots, \lambda_n$ satisfying*

$$M_1 < \lambda_j < M_2,\ 1 \leq j \leq n, \tag{9}$$

there exists an n-tuple $(y_1, y_2, \cdots, y_n)$ satisfying (1)-(2) such that $y_j(t) > 0$, $1 \leq j \leq n$, on (t_1, t_3).

Proof. Let λ_j, $1 \leq j \leq n$, be given as in (9). Now, let $\epsilon > 0$ be chosen such that

$$\max_{1\leq i\leq n}\left\{\left[k^2 \int_{t_1}^{t_3} G_i(s,s)p_i(s)ds(f_{i\infty}-\epsilon)\right]^{-1}\right\} \leq \min_{1\leq j\leq n}\lambda_j$$

and

$$\max_{1\leq j\leq n}\lambda_j \leq \min_{1\leq i\leq n}\left\{\left[\int_{t_1}^{t_3} G_i(s,s)p_i(s)ds(f_{i0}+\epsilon)\right]^{-1}\right\}.$$

We seek fixed points of the completely continuous operator $T : P \to P$ defined by (7). Now, from the definitions of f_{i0}, $1 \leq i \leq n$, there exists an $H_1 > 0$ such that, for each $1 \leq i \leq n$,

$$f_i(x) \leq (f_{i0}+\epsilon)x,\ 0 < x \leq H_1.$$

Let $y_1 \in P$ with $\|y_1\| = H_1$. We first have from Lemma 2.3 and the choice of ϵ, for $t_1 \le s_{n-1} \le t_3$,

$$\begin{aligned}\lambda_n \int_{t_1}^{t_3} G_n(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))ds_n &\le \lambda_n \int_{t_1}^{t_3} G_n(s_n, s_n)p_n(s_n)f_n(y_1(s_n))ds_n \\ &\le \lambda_n \int_{t_1}^{t_3} G_n(s_n, s_n)p_n(s_n)(f_{n0} + \epsilon)y_1(s_n)ds_n \\ &\le \lambda_n \int_{t_1}^{t_3} G_n(s_n, s_n)p_n(s_n)ds_n(f_{n0} + \epsilon)\|y_1\| \\ &\le \|y_1\| = H_1.\end{aligned}$$

It follows in a similar manner from Lemma 2.3 and the choice of ϵ that, for $t_1 \le s_{n-2} \le t_3$,

$$\begin{aligned}&\lambda_{n-1} \int_{t_1}^{t_3} G_{n-1}(s_{n-2}, s_{n-1})p_{n-1}(s_{n-1}) \\ &\qquad f_{n-1}\Big(\lambda_n \int_{t_1}^{t_3} G_n(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))ds_n\Big)ds_{n-1} \\ &\qquad \le \lambda_{n-1} \int_{t_1}^{t_3} G_{n-1}(s_{n-1}, s_{n-1})p_{n-1}(s_{n-1})ds_{n-1}(f_{n-1,0} + \epsilon)H_1 \\ &\qquad \le H_1.\end{aligned}$$

Continuing with this bootstrapping argument, we have, for $t_1 \le t \le t_3$,

$$\lambda_1 \int_{t_1}^{t_3} G_1(t, s_1)p_1(s_1)f_1\Big(\lambda_2 \int_{t_1}^{t_3} G_2(s_1, s_2)p_2(s_2) \cdots f_n(y_1(s_n))ds_n \cdots ds_2\Big)ds_1 \le H_1,$$

so that, for $t_1 \le t \le t_3$,

$$Ty_1(t) \le H_1.$$

Hence, $\|Ty_1\| \le H_1 = \|y_1\|$. If we set

$$\Omega_1 = \{x \in B \mid \|x\| < H_1\},$$

then

$$\|Ty_1\| \le \|y_1\|, \text{ for } y_1 \in P \cap \partial\Omega_1. \tag{10}$$

Next, from the definitions of $f_{i\infty}$, $1 \le i \le n$, there exists $\overline{H}_2 > 0$ such that, for each $1 \le i \le n$,

$$f_i(x) \ge (f_{i\infty} - \epsilon)x, \ x \ge \overline{H}_2.$$

Let

$$H_2 = \max\Big\{2H_1, \frac{\overline{H}_2}{k}\Big\}.$$

Choose $y_1 \in P$ and $\|y_1\| = H_2$. Then,

$$\min_{t \in [t_1, t_3]} y_1(t) \ge k\|y_1\| \ge \overline{H}_2.$$

From Lemma 2.4 and choice of ϵ, for $t_1 \le s_{n-1} \le t_3$, we have that

$$\begin{aligned}\lambda_n \int_{t_1}^{t_3} G_n(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))ds_n &\ge k\lambda_n \int_{t_1}^{t_3} G_n(s_n, s_n)p_n(s_n)(f_{n\infty} - \epsilon)y_1(s_n)ds_n \\ &\ge k^2\lambda_n \int_{t_1}^{t_3} G_n(s_n, s_n)p_n(s_n)ds_n(f_{n\infty} - \epsilon)\|y_1\| \\ &\ge \|y_1\| = H_2.\end{aligned}$$

It follows in a similar manner from Lemma 2.4 and choice of ϵ, for $t_1 \le s_{n-2} \le t_3$,

$$\begin{aligned}
&\lambda_{n-1} \int_{t_1}^{t_3} G_{n-1}(s_{n-2}, s_{n-1}) p_{n-1}(s_{n-1}) \\
&\quad f_{n-1}\Big(\lambda_n \int_{t_1}^{t_3} G_n(s_{n-1}, s_n) p_n(s_n) f_n(y_1(s_n)) ds_n\Big) ds_{n-1} \\
&\quad \ge k\lambda_{n-1} \int_{t_1}^{t_3} G_{n-1}(s_{n-1}, s_{n-1}) p_{n-1}(s_{n-1}) ds_{n-1} (f_{n-1,\infty} - \epsilon) H_2 \\
&\quad \ge k^2\lambda_{n-1} \int_{t_1}^{t_3} G_{n-1}(s_{n-1}, s_{n-1}) p_{n-1}(s_{n-1}) ds_{n-1} (f_{n-1,\infty} - \epsilon) H_2 \\
&\quad \ge H_2.
\end{aligned}$$

Again, using a bootstrapping argument, we have

$$\lambda_1 \int_{t_1}^{t_3} G_1(t, s_1) p_1(s_1) f_1\Big(\lambda_2 \int_{t_1}^{t_3} G_2(s_1, s_2) p_2(s_2) \cdots f_n(y_1(s_n)) ds_n \cdots ds_2\Big) ds_1 \ge H_2,$$

so that

$$Ty_1(t) \ge H_2 = \|y_1\|.$$

Hence, $\|Ty_1\| \ge \|y_1\|$. So, if we set

$$\Omega_2 = \{x \in B \mid \|x\| < H_2\},$$

then

$$\|Ty_1\| \ge \|y_1\|, \text{ for } y_1 \in P \cap \partial\Omega_2. \tag{11}$$

Applying Theorem 2.1 to (10) and (11), we obtain that T has a fixed point $y_1 \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$. As such, setting $y_{n+1} = y_1$, we obtain a positive solution $(y_1, y_2, \cdots, y_n)$ of (1)-(2) given iteratively by

$$y_j(t) = \lambda_j \int_{t_1}^{t_3} G_j(t, s) p_j(s) f_j(y_{j+1}(s)) ds, \ j = n, n-1, \cdots, 1.$$

The proof is completed. $\square$

Prior to our next result, we define the positive numbers M_3 and M_4 by

$$M_3 = \max_{1 \le i \le n} \left\{ \left[k^2 \int_{t_1}^{t_3} G_i(s, s) p_i(s) ds f_{i0} \right]^{-1} \right\}$$

and

$$M_4 = \min_{1 \le i \le n} \left\{ \left[\int_{t_1}^{t_3} G_i(s, s) p_i(s) ds f_{i\infty} \right]^{-1} \right\}.$$

Theorem 3.2. *Assume that the conditions (A1)-(A4) are satisfied. Then, for each $\lambda_1, \lambda_2, \cdots, \lambda_n$ satisfying*

$$M_3 < \lambda_j < M_4, \ 1 \le j \le n, \tag{12}$$

there exists an n-tuple $(y_1, y_2, \cdots, y_n)$ satisfying (1)-(2) such that $y_j(t) > 0$, $1 \le j \le n$, on (t_1, t_3).

Proof. Let λ_j, $1 \le j \le n$ be given as in (12). Now, let $\epsilon > 0$ be chosen such that

$$\max_{1 \le i \le n} \left\{ \left[k^2 \int_{t_1}^{t_3} G_i(s, s) p_i(s) ds (f_{i0} - \epsilon) \right]^{-1} \right\} \le \min_{1 \le j \le n} \lambda_j$$

and

$$\max_{1\le j\le n}\lambda_j \le \min_{1\le i\le n}\left\{\left[\int_{t_1}^{t_3} G_i(s,s)p_i(s)ds(f_{i\infty}+\epsilon)\right]^{-1}\right\}.$$

Let T be the cone preserving, completely continuous operator that was defined by (7). From the definitions of f_{i0}, $1\le i\le n$, there exists $\overline{H}_3>0$ such that, for each $1\le i\le n$,

$$f_i(x)\ge (f_{i0}-\epsilon)x,\ 0<x\le \overline{H}_3.$$

Also, from the definitions of f_{i0}, it follows that $f_{i0}(0)=0,\ 1\le i\le n$, and so there exist $0<K_n<K_{n-1}<\cdots<K_2<\overline{H}_3$ such that

$$\lambda_i f_i(t)\le \frac{K_{i-1}}{\int_{t_1}^{t_3} G_i(s,s)p_i(s)ds},\ t\in[0,K_i],\ 3\le i\le n,$$

and

$$\lambda_2 f_2(t)\le \frac{\overline{H}_3}{\int_{t_1}^{t_3} G_2(s,s)p_2(s)ds},\ t\in[0,K_2].$$

Choose $y_1\in P$ with $\|y_1\|=K_n$. Then, we have

$$\begin{aligned}\lambda_n\int_{t_1}^{t_3} G_n(s_{n-1},s_n)p_n(s_n)f_n(y_1(s_n))ds_n &\le \lambda_n\int_{t_1}^{t_3} G_n(s_n,s_n)p_n(s_n)f_n(y_1(s_n))ds_n\\ &\le \frac{\int_{t_1}^{t_3} G_n(s_n,s_n)p_n(s_n)K_{n-1}ds_n}{\int_{t_1}^{t_3} G_n(s_n,s_n)p_n(s_n)ds_n}\\ &= K_{n-1}.\end{aligned}$$

Continuing with this bootstrapping argument, it follows that

$$\lambda_2\int_{t_1}^{t_3} G_2(s_1,s_2)p_2(s_2)f_2\Big(\lambda_3\int_{t_1}^{t_3} G_3(s_2,s_3)p_3(s_3)\cdots f_n(y_1(s_n))ds_n\cdots ds_3\Big)ds_2\le \overline{H}_3.$$

Then,

$$\begin{aligned}Ty_1(t) &= \lambda_1\int_{t_1}^{t_3} G_1(t,s_1)p_1(s_1)f_1\Big(\lambda_2\int_{t_1}^{t_3} G_2(s_1,s_2)p_2(s_2)\cdots f_n(y_1(s_n))ds_n\cdots ds_2\Big)ds_1\\ &\ge k^2\lambda_1\int_{t_1}^{t_3} G_1(s_1,s_1)p_1(s_1)(f_{1,0}-\epsilon)\|y_1\|ds_1\\ &\ge \|y_1\|.\end{aligned}$$

So, $\|Ty_1\|\ge\|y_1\|$. If we put

$$\Omega_3=\{x\in B \mid \|x\|<K_n\},$$

then

$$\|Ty_1\|\ge\|y_1\|,\ \text{for } y_1\in P\cap\partial\Omega_3. \tag{13}$$

Since each $f_{i\infty}$ is assumed to be a positive real number, it follows that f_i, $1\le i\le n$, is unbounded at ∞.

For each $1\le i\le n$, set

$$f_i^*(x)=\sup_{0\le s\le x} f_i(s).$$

Then, it is straightforward that, for each $1\le i\le n$, f_i^* is a nondecreasing real-valued function, $f_i\le f_i^*$ and

$$\lim_{x\to\infty}\frac{f_i^*(x)}{x}=f_{i\infty}.$$

Next, by definition of $f_{i\infty}$, $1 \le i \le n$, there exists $\overline{H}_4 > 0$ such that, for each $1 \le i \le n$,

$$f_i^*(x) \le (f_{i\infty} + \epsilon)x, \ \ x \ge \overline{H}_4.$$

It follows that there exists $H_4 > \max\{2\overline{H}_3, \overline{H}_4\}$ such that, for each $1 \le i \le n$,

$$f_i^*(x) \le f_i^*(H_4), \ \ 0 < x \le H_4.$$

Choose $y_1 \in P$ with $\|y_1\| = H_4$. Then, using the usual bootstrapping argument, we have

$$\begin{aligned} Ty_1(t) &= \lambda_1 \int_{t_1}^{t_3} G_1(t, s_1)p_1(s_1)f_1(\lambda_2 \cdots)ds_1 \\ &\le \lambda_1 \int_{t_1}^{t_3} G_1(t, s_1)p_1(s_1)f_1^*(\lambda_2 \cdots)ds_1 \\ &\le \lambda_1 \int_{t_1}^{t_3} G_1(s_1, s_1)p_1(s_1)f_1^*(H_4)ds_1 \\ &\le \lambda_1 \int_{t_1}^{t_3} G_1(s_1, s_1)p_1(s_1)ds_1(f_{1\infty} + \epsilon)H_4 \\ &\le H_4 = \|y_1\|. \end{aligned}$$

Hence, $\|Ty_1\| \le \|y_1\|$. So, if we let

$$\Omega_4 = \{x \in B \mid \|x\| < H_4\},$$

then

$$\|Ty_1\| \le \|y_1\|, \ \text{for } y_1 \in P \cap \partial\Omega_4. \tag{14}$$

Applying Theorem 2.1 to (13) and (14), we obtain that T has a fixed point $y_1 \in P \cap (\overline{\Omega}_4 \backslash \Omega_3)$, which in turn with $y_{n+1} = y_1$, yields an n-tuple $(y_1, y_2, \cdots, y_n)$ satisfying (1)-(2) for the chosen values of λ_i, $1 \le i \le n$. The proof is completed. □

Example 3.1. *Let us consider an example to illustrate the above result. Take* $n = 3, t_1 = 0, t_2 = \frac{1}{2}, t_3 = 1, \alpha_1 = 2, \beta_1 = 1, \gamma_1 = 4, \delta_1 = 3, \alpha_2 = 1, \beta_2 = 2, \gamma_2 = 3, \delta_2 = 4, \alpha_3 = 3, \beta_3 = 1, \gamma_3 = 4, \delta_3 = 2, p_1(t) = p_2(t) = p_3(t) = 1$ *and*

$$f_1(y_2) = y_2(800 - 795.5e^{-y_2})(900 - 899.5e^{-2y_2}),$$

$$f_2(y_3) = y_3(1400 - 1398.5e^{-3y_3})(600 - 596.5e^{-y_3}),$$

$$f_3(y_1) = y_1(2000 - 1997.5e^{-2y_1})(260 - 258.5e^{-4y_1}).$$

The Green's functions $G_i(t, s)$, *for* $i = 1, 2, 3,$ *in Lemma 2.1 is*

$$G_i(t,s) = \begin{cases} \underset{t\in[0,\frac{1}{2}]}{G_i(t,s)} = \begin{cases} G_{i1}(t,s), & 0 \le s \le t \le \frac{1}{2} < 1, \\ G_{i2}(t,s), & 0 \le t \le s \le \frac{1}{2} < 1, \\ G_{i3}(t,s), & 0 < t \le \frac{1}{2} \le s \le 1, \end{cases} \\ \\ \underset{t\in[\frac{1}{2},1]}{G_i(t,s)} = \begin{cases} G_{i4}(t,s), & 0 < \frac{1}{2} \le s \le t \le 1, \\ G_{i5}(t,s), & 0 < \frac{1}{2} \le t \le s \le 1, \\ G_{i6}(t,s), & 0 \le s \le \frac{1}{2} \le t < 1, \end{cases} \end{cases}$$

where

$$\begin{aligned}
G_{11}(t,s) =& \frac{1}{16}[2s+1][6-4t], G_{12}(t,s) = \frac{1}{16}[2t+1][6-4s],\\
G_{13}(t,s) =& \frac{1}{16}[2t+1][7-4s], G_{14}(t,s) = \frac{1}{16}[(2s+1)(7-4t)+2(t-s)],\\
G_{15}(t,s) =& \frac{1}{16}[2t+1][7-4s], G_{16}(t,s) = \frac{1}{16}[2s+1][6-4t],\\
G_{21}(t,s) =& \frac{1}{12}[s+2][6-3t], G_{22}(t,s) = \frac{1}{12}[t+2][6-3s],\\
G_{23}(t,s) =& \frac{1}{12}[t+2][7-3s], G_{24}(t,s) = \frac{1}{12}[(s+2)(7-3t)+t-s],\\
G_{25}(t,s) =& \frac{1}{12}[t+2][7-3s], G_{26}(t,s) = \frac{1}{12}[s+2][6-3t],\\
G_{31}(t,s) =& \frac{1}{19}[3s+1][5-4t], G_{32}(t,s) = \frac{1}{19}[3t+1][5-4s],\\
G_{33}(t,s) =& \frac{1}{19}[3t+1][6-4s], G_{34}(t,s) = \frac{1}{19}[(3s+1)(6-4t)+3(t-s)],\\
G_{35}(t,s) =& \frac{1}{19}[3t+1][6-4s], G_{36}(t,s) = \frac{1}{19}[3s+1][5-4t].
\end{aligned}$$

From Lemma 2.4, we get $k = \frac{1}{5}$. *We found that*

$$f_{10} = 2.25, f_{20} = 5.25, f_{30} = 3.75, f_{1\infty} = 720000, f_{2\infty} = 840000, f_{3\infty} = 520000,$$

$$M_1 = \max\{0.00001129943503, 0.00002886002886, 0.00011242603\},$$

and

$$M_2 = \min\{0.8284789644, 0.184704184, 0.6235897436\}.$$

Employing Theorem 3.1, we get an eigenvalue interval $0.00011242603 < \lambda_i < 0.184704184$, $i = 1, 2, 3$, *for which the boundary value problem* (1)-(2) *has a positive solution.*

4. Conclusion

We derived sufficient conditions for the existence of positive solutions for the iterative system of second order differential equations satisfying the general three-point boundary conditions. We determine the eigenvalue intervals of the parameters for which the three-point boundary value problems possess a positive solution.

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