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RESOLUTION OF AN INVERSE PARABOLIC PROBLEM USING SINC-GALERKIN METHOD

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Abstract. In this paper, a numerical method is proposed to solve an Inverse Heat Conduction Problem (IHCP) using noisy data based on Sinc-Galerkin method. A stable numerical solution is determined for the problem. To do this, we use a sensor located at a point inside the body and measure $u(x,t)$ at a point $x = a$, where $0 < a < 1$. We also show that the rate of convergence of the method is as exponential. The numerical results show the efficiency of our approach to estimate the unknown functions of the inverse problem. The function can be computed within a couple of minutes CPU time at pentium IV-2.7 GHz PC.

Keywords: Existence and Uniqueness; Stability; The Tikhonov regularization Method; SVD Method.

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Inverse problems have been appeared in many important applications in heat transfer, thermoelasticity, control theory, population dynamics, nuclear reactor dynamics, medical sciences, biochemistry, and etc [10, 9, 8, 11, 31, 12, 13, 14, 38]. These problems belong to the class of ill-posed problems, i.e, small errors in the measured data can lead to large deviations in the estimated quantities. Moreover, their solution does not satisfy the general requirement of existence, uniqueness, and stability under small changes to the input data. Hence, many researchers have focused on the design of inverse algorithms to solve such problems [2, 4, 5, 6, 7, 19, 24, 26, 27, 28, 29, 30, 36, 37, 32]. Various methods have been developed to solve these problems. Here, we mention two instances of the methods that are commonly applied to solve the inverse problems in literature [2, 4, 5, 6, 7, 19, 24, 26, 27, 28, 29, 36, 37, 32]. The first instance is to solve the inverse problems directly. The other is to transform the inverse problems into a system of integral equations. In the first instance,various techniques have been proposed to solve inverse problems. We mention some methods in this areas such as Tikhonov regularization[39], iterative regularization[2], mollification[25], Base Function Method (BFM)[28], Semi Finite Difference Method (SFDM) [24], and the Function Specification Method (FSM) [4].

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Moreover, Beck and Murio [6] presented a new method that combines Beck's function specification method with Tikhonov's regularization technique. Murio and Paloschi [26] proposed a combined procedure based on a data filtering interpretation of the mollification method and FSM. Beck et al. [4] compared the FSM, the Tikhonov regularization, and the iterative regularization using experimental data. In the other instance, sinc methods are highly efficient numerical methods developed by Frank Stenger [34]. Excellent overviews of the methods based on Sinc functions have been provided to solve ordinary and partial differential equations and integral equations in [34, 21]. Sinc methods have increasingly been recognized as powerful tools to attack problems in applied physics and engineering[34, 3]. They have also been employed as forward solvers in the solution of inverse problems[22, 33]. Shidfar et al.[32] also applied the Sinc-collocation method to solve inverse problems. However, the above methods solved the problem in a small time limitation. Moreover, the stability, existence and uniqueness of resulted solutions of sinc-collocation methods do not imply the stability, existence and uniqueness of the inverse problem, in a general case. Therefore, we are motivated to propose an efficient approach to solve the inverse problem based on Sinc-Galerkkin method to overcome the difficulty. In this approach, we obtain the solution of problem in a more extensive time range with respect to the above mentioned instances. In fact, the solutions are obtained in the whole domain. Furthermore, at most of the above papers, the numerical results are given based on noiseless data[1, 5, 12, 13, 14, 32, 19, 36]. This difficulty is overcome in this paper and the results are computed based on the noisy data. We also show that the rate of convergence of the approach is exponential.
The plan of this paper is as follows. Section 2 contains two subsections and outlines some of the main properties of sinc functions and sinc method that are necessary for the formulation of the IHCP. Section 3 is divided to two subsections ,we formulate and solve IHCP in this section. In subsection 3.1, IHCP is discretized to obtain a matrix form as a sylvester system. In the continuation, we convert the sylvester system to a general system $AX = B$ and apply $0^{th}-, 1^{st}-, and 2^{nd}$ order Tikhonov regularization method to solve the general system. The convergence of the Sinc-Galerkin method is discussed to solve IHCP, in Section 4. Also, the method is illustrated by a numerical example and its results are compared to the $0^{th}-, 1^{st}-, and 2^{nd}$ order Tikhonov regularization method. Finally, conclusions are given in Section 6.

1. Sinc method

In this section, we will review sinc function properties and the sinc method. A comprehensive review concerning sinc function properties as well as sinc method can be found in [21, 35].

Let $\mathbb{C}$ denote the set of all complex numbers. The sinc cardinal or sinc function is defined for each $z \in \mathbb{C}$ as follows:

$$sinc(z) \equiv \begin{cases} \frac{sin(\pi z)}{\pi z}, & z \neq 0, \\ 1, & z = 0. \end{cases} \tag{1.1}$$

For $h > 0$ and any integer k, the translated sinc function with evenly spaced nodes is denoted as $S(j, h)(z)$ and defined by

$$S(j, h)(z) \equiv sinc(\frac{z - jh}{h}), \ j = 0, \pm 1, \pm 2, \dots . \tag{1.2}$$

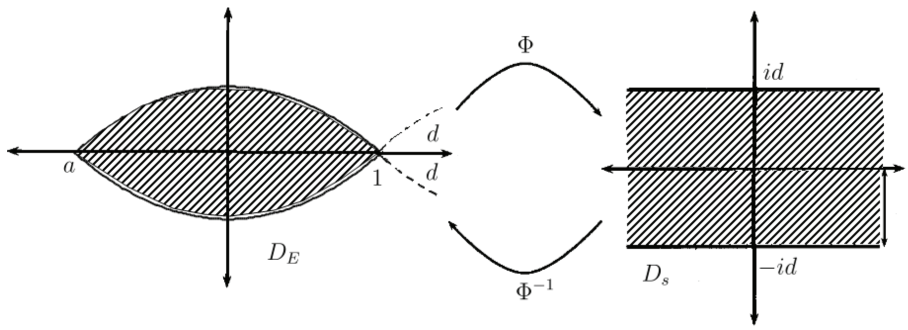

FIGURE 1. Relationship between the domains D_E and D_s

The sinc functions are cardinal for the interpolating points $z_k = kh$ in the sense that

$$S(j,h)(kh) = \delta^{(0)}_{jk} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases} \tag{1.3}$$

If f is a function defined on the real line $\mathbb{R}$ then the cardinal function of f, denoted as $C(f,h)(x)$, is as follows:

$$C(f,h)(x) \equiv \sum_{j=-\infty}^{\infty} f(jh)S(j,h)(x). \tag{1.4}$$

Whenever the series in (1.4) converges, the cardinal function interpolates f at the points $\{nh\}_{n=-\infty}^{\infty}$. The series was addressed in [42] and analyzed in details in [43].

The truncated cardinal series, denoted as $C_{M,N}(f,h)(x)$, is defined by

$$C_{M,N}(f,h)(x) \equiv \sum_{j=-M}^{N} f(jh)S(j,h)(x). \tag{1.5}$$

We will now introduce two conformal mappings to transform the eye-shaped and wedge-shaped domains to an infinite strip domain. To do this, we define the following function

$$\nu = \Phi(z) = ln(\frac{z-a}{1-z}).$$

This function Φ provides a conformal transformation of the "eye-shaped" spatial domain in the z-plane

$$D_E = \{z \in \mathbb{C} : |arg(\frac{z-a}{1-z})| < d\},$$

onto the infinite strip

$$D_s = \{w = u + iv : |v| < d \leq \frac{\pi}{2}\},$$

in the w-plane. This is shown in Fig. (1)

Define the translated sinc basis functions as follows

$$S_i(z) = S(i,h) \circ \Phi(z) \equiv sinc(\frac{\Phi(z) - ih}{h}). \tag{1.6}$$

For the temporal space, we define the function $\Upsilon(t) = ln(t)$ which is a conformal mapping from D_w the "wedge-shaped" temporal domain onto D_s, the infinite strip, where:

$$D_w = \{t = r + is : |arg(t)| < d \leq \frac{\pi}{2}\}.$$

This is shown in Figure (2). The basic function are derived from the composite translated

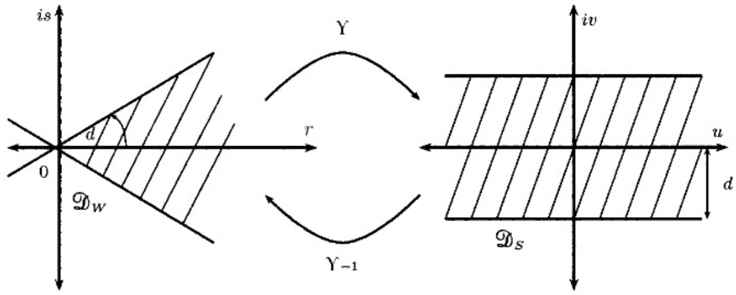

FIGURE 2. Relationship between the domains D_w and D_s

sinc functions,

$$S(j,h) \circ \Upsilon(t) \equiv sinc(\frac{\Upsilon(t) - jh}{h}). \tag{1.7}$$

for $t \in D_w$.
The function $z = \Phi^{-1}(v) = \frac{a+e^v}{1+e^v}$ is an inverse mapping of $v = \Phi(z)$.
We define the range of Φ^{-1} on the real line as

$$\Gamma = \{\psi(u) = \Phi^{-1}(u) \in D_E : -\infty < u < \infty\}.$$

Definition 1.1. [21, 34] *Let D be a domain in the $\varrho = \vartheta + i\aleph$ plane with boundary ∂D and boundary points $a \neq b$. Let $z = \phi(\varrho)$ be a one-to-one conformal map of D onto the infinite strip D_s of width $2d$ where $\phi(a) = -\infty$ and $\phi(b) = \infty$. Assume $\varrho = \psi(z)$ denotes the inverse of the mapping ϕ. Then $B(D)$ is the set of functions analytic in D that satisfy, for some constant c with $0 \leq c \leq 1$,*

$$\int_{\psi(x+L)} |F(\varrho)d\varrho| = O(|x|^c), \qquad x \to \pm\infty$$

where $L = \{iy : |y| < d\}$ and for γ a simple closed contour in D require

$$N(F, D) \equiv \lim_{\gamma \to \partial D} \int_\gamma |F(\varrho)d\varrho| < \infty.$$

1.1. **Interpolation and quadrature rules for approximation in sinc method:** For problems on a subinterval, Γ, of the real line, we employ a conformal map Φ for which $\Phi(\Gamma) = \mathbb{R}$. Suppose $d > 0$ and let Φ be a conformal map of the domain D onto D_s. Then over a subinterval $\Gamma = \Phi^{-1}(\mathbb{R})$, we apply the following method of interpolation[21]

$$f(z) \approx \sum_{k=-\infty}^{\infty} f(kh)S(k,h) \circ \Phi(z), \tag{1.8}$$

and quadrature:

$$\int_\Gamma f(z)dz \approx h \sum_{k=-\infty}^{\infty} f(z_k)/\Phi'(z_k). \tag{1.9}$$

The sinc gride points $z_k \in (a, 1)$ in D_E will be denoted by x_i because they are real. For the evenly spaced nodes $\{ih\}_{i=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$x_i = \Phi^{-1}(ih) = \frac{a + e^{ih}}{1 + e^{ih}},\ i = \pm 1, \pm 2, \ldots,$$

and in a same way

$$t_j = \Upsilon^{-1}(jh) = e^{jh},\ j = \pm 1, \pm 2, \ldots .$$

The Sinc-Galerkin method actually requires the evaluated derivatives of sinc basis functions $S(i,h) \circ \Phi(x)$ at the sinc nodes, $x = x_k$.The pth derivative of $S(i,h) \circ \Phi(x)$, with respect to Φ, evaluated at the nodal point x_k is denoted by

$$\frac{1}{h^p}\delta_{ik}^{(p)} \equiv \frac{d^p}{d\Phi^p}[S(i,h) \circ \Phi(x)] \mid_{x=x_k} . \tag{1.10}$$

Theorem 1.1. *Let Φ be a conformal one-to-one map of the simply connected domain D_E onto D_s then*

$$\delta_{ik}^{(0)} = [S(i,h) \circ \Phi(x)] \mid_{x=x_k} = \begin{cases} 1, & k = i; \\ 0, & k \neq i; \end{cases} \tag{1.11}$$

$$\delta_{ik}^{(1)} = h\frac{d}{d\Phi}[S(i,h) \circ \Phi(x)] \mid_{x=x_k} = \begin{cases} 0, & k = i; \\ \frac{(-1)^{(k-i)}}{(k-i)}, & k \neq i; \end{cases} \tag{1.12}$$

and

$$\delta_{ik}^{(2)} = h^2\frac{d^2}{d\Phi^2}[S(i,h) \circ \Phi(x)] \mid_{x=x_k} = \begin{cases} \frac{-\pi^2}{3}, & k = i; \\ \frac{-2(-1)^{(k-i)}}{(k-i)^2}, & k \neq i. \end{cases} \tag{1.13}$$

Proof. See [21]. □

The expressions in (1.10) for each i and k can be stored in a matrix $I^{(p)} = [\delta_{ik}^{(p)}]$ for $p = 0, 1, 2$:

$$I^{(0)} = [\delta_{ik}^{(0)}] = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} = I, \tag{1.14}$$

$$I^{(1)} = [\delta_{ik}^{(1)}] = \begin{pmatrix} 0 & -1 & \frac{1}{2} & \cdots & \frac{(-1)^{m-1}}{m-1} \\ 1 & & & & \vdots \\ \frac{-1}{2} & & \ddots & & \frac{1}{2} \\ \vdots & & & & -1 \\ \frac{(-1)^m}{m-1} & \cdots & \frac{-1}{2} & 1 & 0 \end{pmatrix}, \tag{1.15}$$

$$I^{(2)} = [\delta_{ik}^{(2)}] = \begin{pmatrix} \frac{-\pi^2}{3} & 2 & \frac{-2}{2^2} & \cdots & \frac{-2(-1)^{m-1}}{(m-1)^2} \\ 2 & & & & \vdots \\ \frac{-2}{2^2} & & \ddots & & \frac{-2}{2^2} \\ \vdots & & & & 2 \\ \frac{-2(-1)^{m-1}}{(m-1)^2} & \cdots & \frac{-2}{2^2} & 2 & \frac{-\pi^2}{3} \end{pmatrix}, \tag{1.16}$$

the above matrices are the $m \times m (m = M + N + 1)$ toeplitz matrices where

$$-M \leq k \leq N, \qquad -M \leq i \leq N.$$

If function g is evaluated at the sinc nodes $x = x_k$ for $-M_x \le i \le N_x$ then the $m_x \times m_x$ square diagonal matrix $D_{m_x}(g)$ is written by

$$D_{m_x}(g) = \begin{pmatrix} g(x_{-M_x}) & & & & \\ & \ddots & & & \\ & & g(x_0) & & \\ & & & \ddots & \\ & & & & g(x_{N_x}) \end{pmatrix}. \quad (1.17)$$

1.2. **Parameter selections for the Sinc-Galerkin method.** The matrices that comprise the discrete system in the Sinc-Galerkin method are full matrices. More sinc grid points lead to larger matrices and make an expensive computation. Some found cases in [17] show how to choose an appropriate sinc grid in space and time and those selections will be used here. If the exact solution satisfies the condition

$$|u(x,t)| \le Cx^{\alpha_s+\frac{1}{2}}(1-x)^{\beta_s+\frac{1}{2}}t^{\gamma_s+\frac{1}{2}}e^{-\delta t}, \quad (1.18)$$

for$(x,t) \in (a,1) \times (0,\infty)$, we should make the following selections

$$N_x = [|\frac{\alpha_s}{\beta_s}M_x + 1|],\ M_t = [|\frac{\alpha_s}{\gamma_s}M_x + 1|],\ N_t = [|\frac{1}{h}ln(\frac{\alpha_s}{\delta}M_x h) + 1|], \quad (1.19)$$

where $[|.|]$ denotes the greatest integer operation, $h \equiv h_x = h_t$ and

$$h = (\frac{\pi d}{\alpha_s M_x})^{\frac{1}{2}}. \quad (1.20)$$

For a given problem with a known real or complex solution, one can determine α, β, γ, and δ using (1.18) where

$$\alpha_s = \alpha - \frac{1}{2}\ and\ \beta_s = \beta - \frac{1}{2}.$$

Then (1.19) and (1.20) provide the computational parameters. In practice, one can sets $\alpha = \beta = \gamma = 1$ and $d = \frac{\pi}{2}$. Then from (1.19) and (1.20), $M_x = N_x = N_t$ and $h = \frac{\pi}{2\sqrt{M_x}}$, respectively. Numerical experiments suggest the choice $N_t = \frac{1}{2}M_x$ for the infinite time interval instead of that given in (1.19).

2. Inverse Problem for the Heat Equation

One example of the inverse heat conduction problem is the estimation of the heating history experienced by a shuttle or missile reentering the earth's atmosphere from space. The heat flux at the heated surface is needed [4]. To estimate the surface heat flux history, it is necessary to have a mathematical model of the heat transfer process. For example, it is assumed that the section of the skin is of a single material, homogeneous, and isotropic. It closely approximates a flat plate. Then a possible mathematical model for the temperature in the plate is a one-dimensional inverse heat conduction problem as follows [4]:

$$P^{(2)}u(x,t) \equiv u_t(x,t) - u_{xx}(x,t) = f(x,t), \qquad 0 < x < 1, \quad 0 < t < \infty, \quad (2.1a)$$

$$u(x,0) = \phi(x), \qquad 0 \le x \le 1, \quad (2.1b)$$

$$u(0,t) = p(t), \qquad 0 \le t \le \infty, \quad (2.1c)$$

$$u(1,t) = q(t), \qquad 0 \le t \le \infty, \quad (2.1d)$$

and the overspecified condition

$$u(a,t) = g(t), \qquad 0 \le t \le \infty, \tag{2.1e}$$

where $0 < a < 1$ is a fixed point, $\phi(x)$, $g(t)$ and $q(t)$ are known functions on $0 < x < 1$, $0 < t < \infty$, and the function $p(t)$ is unknown which remains to be determined from some interior temperature measurements.
We now show the application of the fully Sinc-Galerkin method to solve the inverse problem for the heat equation. The approximate solution is written as

$$u_{m_x,m_t}(x,t) = \sum_{j=-M_t-1}^{N_t} \sum_{i=-M_x-1}^{N_x+1} u_{ij}\chi_i(x)\theta_j(t), \tag{2.2}$$

where $m_x = M_x + N_x + 3$ and $m_t = M_t + N_t + 2$. The basis functions $\{s_{ij}(x,t)\}$ for $-M_x - 1 \le i \le N_x + 1$ and $-M_t - 1 \le j \le N_t$ are given as the product of basis functions for the appropriate one-dimensional problem. They are given by

$$s_{ij}(x,t) \equiv [s(i,h_x) \circ \Phi(x)][s(j,h_t) \circ \Upsilon(t)],$$

where

$$\Phi(x) = ln(\frac{x-a}{1-x}), \;\; \Upsilon(t) = ln(t). \tag{2.3}$$

Two linear functions are added to the sinc basis in the spatial dimension

$$\chi_i(x) = \begin{cases} \frac{1-x}{1-a}, & i = -M_x - 1, \\ s(i,h) \circ \Phi(x), & -M_x \le i \le N_x, \\ \frac{x-a}{1-a}, & i = N_x + 1, \end{cases}$$

and one rational function is appended to the temporal base

$$\theta_j(t) = \begin{cases} \frac{t+1}{t^2+1}, & j = -M_t - 1, \\ s(j,h) \circ \Upsilon(t), & -M_t \le j \le N_t. \end{cases}$$

Interpolating the boundary and initial conditions in (2.2) dictates that

$$u_{m_x,m_t}(a,t) = \sum_{j=-M_t-1}^{N_t} u_{-M_x-1,j}\theta_j(t) = g(t),$$

$$u_{m_x,m_t}(1,t) = \sum_{j=-M_t-1}^{N_t} u_{N_x+1,j}\theta_j(t) = q(t),$$

$$u_{m_x,m_t}(x,0) = \sum_{i=-M_x-1}^{N_x+1} u_{i,-M_t-1}\chi_i(x) = \phi(x).$$

The sinc approximation to (2.2) is defined by

$$u_{m_x,m_t}(x,t) = \sum_{j=-M_t}^{N_t} \sum_{i=-M_x}^{N_x} u_{ij}s_{ij}(x,t) + g^*(t)\chi_{-M_x-1}(x) +$$
$$q^*(t)\chi_{N_x+1}(x) + \phi(x)\theta_{-M_t-1}(t),$$

where

$$g^*(t) = g(t) - \phi(a)\theta_{-M_t-1}(t),$$
$$q^*(t) = q(t) - \phi(1)\theta_{-M_t-1}(t),$$

and the intervals of i and j confined to $-M_x \leq i \leq N_x$ and $-M_t \leq j \leq N_t$, respectively. Therefore, we have $m_x = M_x + N_x + 1$ and $m_t = M_t + N_t + 1$.

Define the inner product by

$$< \eta, \zeta >\equiv \int_0^\infty \int_a^1 \eta(x,t)\zeta(x,t)\nu(x)\omega(t)dxdt,$$

where the product $\nu(x)\omega(t)$ plays the role of a weight function. Assume that the product is given by

$$\nu(x)\omega(t) = \sqrt{\frac{\Upsilon'}{\Phi'}},$$

where

$$\omega(t) = \sqrt{\Upsilon'}, \qquad \nu(x) = \sqrt{\frac{1}{\Phi'}}.$$

Since $g^*(t)$ and $q^*(t)$ are known functions, the orthogonalization of the residual

$$< P^{(2)}u_{m_x,m_t} - f, s_{kl} >= 0,$$

for $-M_x \leq k \leq N_x$, $-M_t \leq l \leq N_t$ may be written

$$< P^{(2)}u_h - f^*, s_{kl} >= 0, \tag{2.4}$$

where the homogeneous part of the approximate solution is given by

$$u_h(x,t) = \sum_{j=-M_t}^{N_t} \sum_{i=-M_x}^{N_x} u_{ij}s_{ij}(x,t).$$

f^* is also given by

$$f^*(x,t) = f(x,t) - P^{(2)}[g^*(t)\chi_{-M_x-1}(x) + q^*(t)\chi_{N_x+1}(x) + \phi(x)\theta_{-M_t-1}(t)]. \tag{2.5}$$

2.1. **Discrete System Assembly.** Now we want to discrete the system of (2.4):

$$< u_t, s_{kl} > - < u_{xx}, s_{kl} > - < f^*, s_{kl} >= 0.$$

The inner product of sinc basis elements is given by

$$< u_t, s_{kl} >= \int_0^\infty \int_a^1 u_t s_{kl}\nu(x)\omega(t)dxdt,$$

This expression contains derivative of u with respect to t. We can remove derivative from the dependent variable u by integrating by parts, once doing this in t. We obtain the following term,

$$B_{T_1} - \int_0^\infty \int_a^1 u_h(x,t)[s(k,h_x) \circ \Phi(x)]\nu(x)([s(l,h_t) \circ \Upsilon(t)]\omega(t))'dxdt,$$

where the boundary term

$$B_{T_1} = \int_a^1 [s(k,h_x) \circ \Phi(x)]\nu(x)([s(l,h_t) \circ \Upsilon(t)]\omega(t)u_h(x,t)) \mid_{t=0}^{\infty} dx = 0.$$

If we do the similar calculations for $< u_{xx}, s_{kl} >$, then we have

$$< u_{xx}, s_{kl} >= \int_0^\infty \int_a^1 u_{xx}s_{kl}\nu(x)\omega(t)dxdt.$$

This expression contains the derivatives of the dependant variable u, twice in x. We can similarly remove u_{xx} by integrating by parts, as follows:

$$B_{T_2} - \int_0^\infty \int_a^1 u_h(x,t)[s(l,h_t)\circ\Upsilon(t)]\omega(t)([s(k,h_x)\circ\Phi(x)]\nu(x))''dxdt,$$

where the boundary term

$$\begin{aligned} B_{T_2} &= \int_0^\infty [s(l,h_t)\circ\Upsilon(t)]\omega(t)([s(k,h_x)\circ\Phi(x)]\nu(x))'u_h(x,t))\mid_{x=a}^1 dt \\ &- \int_0^\infty [s(l,h_t)\circ\Upsilon(t)]\omega(t)([s(k,h_x)\circ\Phi(x)]\nu(x)u_x(x,t))\mid_{x=a}^1 dt \\ &= 0. \end{aligned}$$

Remove the derivatives from the dependent variable u by integrating by parts, twice in x and once in t, to arrive at the identity

$$\int_0^\infty \int_a^1 u_h(x,t)(-\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2})(s_k(x)s_l(t)\nu(x)\omega(t))dxdt =$$

$$\int_0^\infty \int_a^1 f^* s_k(x)s_l(t)\nu(x)\omega(t)dxdt.$$

We apply the quadrature rule [21] to the iterated integrals and delete the error terms.We also replace $u_h(x,t)$ by u_{ij} and dividing by $h_x h_t$. Hence, we obtain the following discrete sinc system:

$$\frac{\omega(t_l)}{\Upsilon'(t_l)} \sum_{i=-M_x}^{N_x} [-\frac{1}{h_x^2}\delta_{ki}^{(2)}\Phi'(x_i)\nu(x_i) - \frac{1}{h_x}\delta_{ki}^{(1)}(\frac{\Phi''(x_i)\nu(x_i)}{\Phi'(x_i)} + 2\nu'(x_i)) - \delta_{ki}^{(0)}\frac{\nu''(x_i)}{\Phi'(x_i)}]u_{il} \quad (2.6)$$

$$+\frac{\nu(x_k)}{\Phi'(x_k)} \sum_{j=-M_t}^{N_t} [-\frac{1}{h_t}\delta_{lj}^{(1)}\omega(t_j) + \delta_{lj}^{(0)}\frac{\omega'(t_j)}{\Upsilon'(t_j)}]u_{kj} \quad (2.7)$$

$$= \frac{f^*(x_k,t_l)\nu(x_k)\omega(t_l)}{\Phi'(x_k)\Upsilon'(t_l)}. \quad (2.8)$$

This system is identical to the system generated by orthogonalizing the residual via

$$< p^{(2)}u_h - f^*, s_{kl} >= 0.$$

We apply the notation of Section 2 and obtain the following matrix form:

$$\begin{aligned} &[\frac{-1}{h_x^2}I_{m_x}^{(2)}D(\Phi'\nu) - \frac{1}{h_x}I_{m_x}^{(1)}D(\frac{\Phi''\nu}{\Phi'} + 2\nu') - I_{m_x}^{(0)}D(\frac{\nu''}{\Phi'})]U^{(2)}D(\frac{\omega}{\Upsilon'}) \\ &+ D(\frac{\nu}{\Phi'})U^{(2)}[\frac{-1}{h_t}I_{m_t}^{(1)}D(\omega) + I_{m_t}^{(0)}D(\frac{\omega'}{\Upsilon'})] \\ &= D(\frac{\nu}{\Phi'})F^{(2)}D(\frac{\omega}{\Upsilon'}), \end{aligned}$$

premultiplying by $D(\Phi')$ and postmultiplying by $D(\Upsilon')$ yields the equivalent system

$$\begin{aligned} &D(\Phi')[\frac{-1}{h_x^2}I_{m_x}^{(2)}D(\Phi'\nu) - \frac{1}{h_x}I_{m_x}^{(1)}D(\frac{\Phi''\nu}{\Phi'} + 2\nu') - I_{m_x}^{(0)}D(\frac{\nu''}{\Phi'})]U^{(2)}D(\omega) \\ &+ D(\nu)U^{(2)}[\frac{-1}{h_t}I_{m_t}^{(1)}D(\omega) + I_{m_t}^{(0)}D(\frac{\omega'}{\Upsilon'})]^T D(\Upsilon') \\ &= D(\nu)F^{(2)}D(\omega). \end{aligned}$$

It is helpful to single out the portion of the coefficient matrix in this system that corresponds to the second derivative. This is defined by

$$A(v) \equiv \frac{-1}{h_x^2} I_{m_x}^{(2)} - \frac{1}{h_x} I_{m_x}^{(1)} D(\frac{\Phi''}{(\Phi')^2} + \frac{2\nu'}{\Phi'\nu}) - D(\frac{\nu''}{(\Phi')^2\nu}), \tag{2.9}$$

$$B(\sqrt{\Upsilon'}) = \frac{-1}{h_t} I_{m_t}^{(1)} - D(\frac{\omega'}{\omega\Upsilon'}) = \frac{-1}{h_t} I_{m_t}^{(1)} + D(\frac{1}{2}). \tag{2.10}$$

The second equality follows $\frac{\omega'}{\omega\Upsilon'} = \frac{\Upsilon''}{2(\Upsilon')^2} \equiv \frac{-1}{2}$, where

$$\Upsilon = ln(t).$$

The representation of the system is simplified based on recalling the definition of the matrix $A(\nu)$ and $B(\sqrt{\Upsilon'})$ in (2.9) and (2.10), respectively. So we have

$$D(\Phi'\nu)D(\Phi')A(v)U^{(2)}D(\omega) + D(\omega)D(\nu)U^{(2)}B(\sqrt{\Upsilon'})D(\Upsilon')$$
$$= D(\nu)F^{(2)}D(\omega). \tag{2.11}$$

Dividing the system of (2.11) by $(\sqrt{\Upsilon'})$ yields

$$D(\Phi')A(v)D(\Phi')D(\nu)U^{(2)}D(\frac{\omega}{\sqrt{\Upsilon'}}) + D(\nu)U^{(2)}D(\frac{\omega}{\sqrt{\Upsilon'}})D((\Upsilon')^{\frac{1}{2}})B^T(\sqrt{\Upsilon'})D((\Upsilon')^{\frac{1}{2}})$$
$$= D(\nu)F^{(2)}D(\frac{\omega}{\sqrt{\Upsilon'}}). \tag{2.12}$$

We can now write (2.12) in the simplified form as follows:

$$A_x\nu^{(2)} + \nu^{(2)}B_t^T = G^{(2)}, \tag{2.13}$$

where

$$A_x = D(\Phi')A(v)D(\Phi') = D(\Phi')[\frac{-1}{h}I^{(2)} + D(\frac{-1}{(\Phi')^{\frac{3}{2}}}(\frac{1}{\sqrt{\Phi'}})'')]D(\Phi'),$$
$$B_t = D((\Upsilon')^{\frac{1}{2}})[-\frac{1}{h}I^{(1)} + D(\frac{1}{2})]D((\Upsilon')^{\frac{1}{2}}),$$
$$\nu^{(2)} = D(\nu)U^{(2)} = D((\Phi')^{\frac{-1}{2}})U^{(2)},$$
$$G^{(2)} = D(\nu)F^{(2)} = D((\Phi')^{\frac{-1}{2}})F^{(2)}.$$

In the latter, the matrix $F^{(2)}$ is now the matrix of point evaluations of f^* in (2.5). In this notation, the system (2.13) is the sylvester equation. In this section, we discretize the system and find its matrix form, in the next section. We try to solve the sylvester equation and transform the sylvester equation to the system in the following form:

$$A\Theta = B,$$

where A is a matrix and B is a vector.

2.2. **Transform of the sylvester equation to** $A\Theta = B$**.** The formulation (2.13) may be written as the large sparse discrete system using the Kronecker sum notation and the concatenation (co) of matrices [21]. Hence,we have:

$$B^{(2)}co(\nu^{(2)}) = co(G^{(2)}), \tag{2.14}$$

where

$$B^{(2)} \equiv I_{m_t} \otimes A_x + B_t \otimes I_{m_x},$$

and

$$co(\nu^{(2)}) = co(D((\Phi')^{\frac{-1}{2}})U^{(2)}) = (I_{m_x} \otimes D((\Phi')^{\frac{-1}{2}}))co(U^{(2)}),$$

$$co(G^{(2)}) = co(D((\Phi')^{\frac{-1}{2}})F^{(2)}) = (I_{m_x} \otimes D((\Phi')^{\frac{-1}{2}}))co(F^{(2)}).$$

The Schur decomposition or the full diagonalization procedure is applicable to (2.13). One of difficulties of different methods of solution is the necessity of dealing with the complex eigenvalues and eigenvectors of B_t. To circumvent this issue, note that the block structure of $B^{(2)}$ is graphically illustrated in below matrix:

$$\left(\begin{array}{ccc|ccc|ccc|ccc}
x & x & x & x & 0 & 0 & x & 0 & 0 & x & 0 & 0 \\
x & x & x & 0 & x & 0 & 0 & x & 0 & 0 & x & 0 \\
x & x & x & 0 & 0 & x & 0 & 0 & x & 0 & 0 & x \\
\hline
x & 0 & 0 & x & x & x & x & 0 & 0 & x & 0 & 0 \\
0 & x & 0 & x & x & x & 0 & x & 0 & 0 & x & 0 \\
0 & 0 & x & x & x & x & 0 & 0 & x & 0 & 0 & x \\
\hline
x & 0 & 0 & x & 0 & 0 & x & x & x & x & 0 & 0 \\
0 & x & 0 & 0 & x & 0 & x & x & x & 0 & x & 0 \\
0 & 0 & x & 0 & 0 & x & x & x & x & 0 & 0 & x \\
\hline
x & 0 & 0 & x & 0 & 0 & x & 0 & 0 & x & x & x \\
0 & x & 0 & 0 & x & 0 & 0 & x & 0 & x & x & x \\
0 & 0 & x & 0 & 0 & x & 0 & 0 & x & x & x & x
\end{array}\right),$$

A scheme designed to take advantage of this structure and to avoid the possible complex arithmetic is a "partial diagonalization" procedure [21]. Let P denote the orthogonal diagonalizer of A_x. Then (via a premultiplication by $I_{m_t} \otimes P^T$) the system (2.14) is equivalent to

$$B_p^{(2)} co(W^{(2)}) = co(H^{(2)}), \tag{2.15}$$

where

$$B_p^{(2)} = I_{m_t} \otimes \Lambda_x + B_t \otimes I_{m_t},$$

and Λ_x is the diagonal matrix of eigenvalues of A_x . The concatenated matrix of coefficients and forcing vectors are, respectively, as follows,

$$co(W^{(2)}) = (I_{m_t} \otimes p^T)co(\nu^{(2)}),$$

and

$$co(H^{(2)}) = (I_{m_t} \otimes p^T)co(G^{(2)}).$$

The $m_x m_t \times m_x m_t$ matrix $B_p^{(2)}$ can be pictorially displayed as:

$$\begin{pmatrix}
\Lambda_x + b_{-M_t,M_t} I_{m_x} & \dots & b_{-M_t,N_t} I_{m_x} \\
b_{-M_t+1,M_t} I_{m_x} & \dots & \vdots \\
\vdots & \ddots & \vdots \\
b_{N_t,M_t} I_{m_x} & \dots & \Lambda_x + b_{N_t,N_t} I_{m_x}
\end{pmatrix},$$

where the m2 numbers (b_{ij}) , $M_t < i,\ j < N_t$, are the entries in the matrix B_t and each block of Bp is a diagonal matrix. A method of solution of (2.15) is a block Gauss-Jordan routine. At one level of programming, the blocks of $B_p^{(2)}$ are treated as if they were constants so that divisions are simply inverses of diagonal matrices and the system (2.15) is naturally suited for vector computation [21].

Thus the linear system corresponding to the sinc coefficient u_{ij} can be expressed as

$$A\Theta = B. \tag{2.16}$$

The Matrix A is ill-conditioned. On the other hand, as $g(t)$ is affected by measurement errors, the estimate of Θ by (2.16) will be unstable so that the Tikhonov regularization method must be used to control this measurement errors. The Tikhonov regularized solution ([18], [20], [40] and [39]) to the system of linear algebraic equation (2.16) is given by

$$F_\alpha(\Theta) = \|A\Theta - B\|_2^2 + \alpha\|R^{(s)}\Theta\|_2^2.$$

In the case of the zero -, first-, and second-order Tikhonov regularization method the matrix $R^{(s)}$, for $s = 0, 1, 2$, is given as follows [23]:

$$R^{(0)} = I_{M_1 \times M_1} \in \mathbb{R}^{M_1 \times M_1},$$

$$R^{(1)} = \begin{pmatrix} -1 & 1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 1 & 0 \\ 0 & 0 & \dots & 0 & -1 & 1 \end{pmatrix} \in \mathbb{R}^{(M_1-1)\times M_1},$$

$$R^{(2)} = \begin{pmatrix} 1 & -2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -2 & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 & -2 & 1 \end{pmatrix} \in \mathbb{R}^{(M_1-2)\times M_1},$$

where $M_1 = (\gamma + 1) \times (\iota + 1)$.

Therefore, we obtain the Tikhonov regularized solution of the regularized equation as follows,

$$\Theta_\alpha = \left[A^T A + \alpha (R^{(s)})^T R^{(s)}\right]^{-1} A^T B.$$

In our computation, we use the GCV scheme to determine a suitable value of α ([15], [16] and [41]).

3. Convergence analysis

In this section, we discuss the convergence of the Sinc-Galerkin method for IHCP.

Theorem 3.1. *Consider the maps $\Phi : D_E \longrightarrow D_s$ and $\Upsilon : D_w \longrightarrow D_s$ given in (2.3). For the weight function*

$$\nu(x)\omega(t) = \sqrt{\frac{\Upsilon'}{\Phi'}},$$

assume that $\frac{f}{\sqrt{\Phi'}} \in B(D_E)$ and that $uF \in B(D_E)$, where

$$F = \sqrt{\Phi'}\left(\frac{1}{\Phi'}\right)'', (\Phi')^{\frac{3}{2}}, \frac{\Phi''}{\sqrt{\Phi'}}, \left(\frac{1}{\sqrt{\Phi'}}\right)''.$$

Also assume that $f\sqrt{\Upsilon'} \in B(D_w)$ and $uF \in B(D_w)$, where

$$F = \sqrt{\Upsilon'}, (\Upsilon')^{\frac{3}{2}}, \frac{\Upsilon''}{\sqrt{\Upsilon'}}.$$

Furthermore assume

$$|u(x,t)| \le C(x-a)^{\alpha_s + \frac{1}{2}}(1-x)^{\beta_s + \frac{1}{2}} t^{\gamma + \frac{1}{2}} e^{-\delta t},$$

$$(x,t) \in (a,1) \times (0,\infty).$$

Making the following selections

$$N_x = [|\frac{\alpha_s}{\beta_s}M_x + 1|],\ h_s = (\frac{\pi d}{\alpha_s M_x})^{\frac{1}{2}}$$

and

$$M_t = [|\frac{\alpha_s}{\gamma}M_x + 1|],\ N_t = [|\frac{1}{h}ln(\frac{\alpha_s}{\delta}M_x h) + 1|],$$

we have

$$\|u - u_{m_x,m_t}\|_\infty \leq KM_x^2 exp(-(\pi d\alpha_s M_x)^{\frac{1}{2}}).$$

Proof. With regard to the discrete system (2.7) and $|\delta_{lj}^{(1)}| \leq 1$, we can write

$$\begin{aligned}
&| \int_0^\infty \int_a^1 u_t S_{kl}\nu(x)\omega(t)dxdt + h_x h_t \frac{\nu(x_k)}{\Phi'(x_k)} \sum_{j=-M_t}^{N_t} [\frac{1}{h_t}\delta_{lj}^{(1)}\omega(t_j) + \frac{\omega'(t_j)}{\Upsilon'(t_j)}\delta_{lj}^{(0)}]u_{kj} | \\
&\leq h_x h_t \mid \frac{\nu(x_k)}{\Phi'(x_k)} \mid [\sum_{j=-M_t}^{N_t} \frac{1}{h_t} \mid \delta_{lj}^{(1)} \mid\mid \omega(t_j)u_{kj} \mid + \mid \frac{\omega'(t_l)}{\Upsilon'(t_l)} \mid\mid u_{kl} \mid] \\
&\leq h_x h_t \mid \frac{\nu(x_k)}{\Phi'(x_k)} \mid [\sum_{j=-M_t}^{N_t} \frac{1}{h_t} \mid \omega(t_j)u_{kj} \mid + \mid \frac{\omega'(t_l)}{\Upsilon'(t_l)} \mid\mid u_{kl} \mid] \\
&\leq h_x \mid \frac{\nu(x_k)}{\Phi'(x_k)} \mid \sum_{j=M_t+1}^{\infty} \mid \omega(t_{-j})u_{k,-j} \mid \\
&+ \sum_{j=N_t+1}^{\infty} \mid \omega(t_j)u_{k,j} \mid + h_x h_t \mid \frac{\omega'(t_l)}{\Upsilon'(t_l)} \mid\mid u_{kl} \mid\mid \frac{\nu(x_k)}{\Phi'(x_k)} \mid \\
&\leq h_x \mid \frac{(1-a)^{\frac{3}{2}}e^{\frac{3}{2}kh}}{(1+e^{kh})^3} \mid (\sum_{j=M_t+1}^{\infty} \mid e^{\frac{1}{2}jh} \mid\mid \frac{(1-a)^{\alpha_s+\beta_s+1}e^{kh(\alpha_s+\frac{1}{2})}}{(1+e^{kh})^{\alpha_s+\beta_s+1}} e^{-jh(\gamma+\frac{1}{2})} \mid \\
&+ \sum_{j=N_t}^{\infty} \mid e^{-\frac{1}{2}jh} \mid\mid \frac{(1-a)^{\alpha_s+\beta_s+1}e^{kh(\alpha_s+\frac{1}{2})}}{(1+e^{kh})^{\alpha_s+\beta_s+1}} e^{-\delta e^{jh}} \mid) \\
&+ h_x h_t \mid -\frac{1}{2}e^{-\frac{1}{2}lh} \mid\mid \frac{(1-a)^{\alpha_s+\beta_s+1}e^{kh(\alpha_s+\frac{1}{2})}}{(1+e^{kh})^{\alpha_s+\beta_s+1}} e^{-\delta e^{lh}} e^{lh(\delta+\frac{1}{2})} \mid\mid \frac{(1-a)^{\frac{3}{2}}e^{\frac{3}{2}kh}}{(1+e^{kh})^3} \mid \\
&\leq h_x \mid \frac{(1-a)^{\frac{5}{2}+\alpha_s+\beta_s}e^{kh(\alpha_s+2)}}{(1+e^{kh})^{\alpha_s+\beta_s+4}} \mid \\
&(\sum_{j=M_t+1}^{\infty} \mid e^{-\gamma jh} \mid + \sum_{j=N_t+1}^{\infty} \mid e^{jh}e^{-\delta e^{jh}} \mid + h_t \mid \frac{-1}{2}e^{\gamma lh}e^{-\delta e^{lh}} \mid). \qquad (3.1)
\end{aligned}$$

The exponential convergence of the approximating sum is maintained with those M_t and N_t and verified by the following processes.

$$\begin{aligned}
\sum_{N_t+1}^{\infty} e^{jh}e^{-\delta e^{jh}} &= lim_{b\to\infty} \int_{N_t+1}^{b} e^{jh}e^{-\delta e^{jh}} dj \\
&= \frac{-1}{\delta h} lim_{b\to\infty}(e^{-\delta e^{bh}} - e^{-\delta e^{N_t h+h}}) \\
&= \frac{-1}{\delta h}(-e^{-\delta e^{N_t h}e^h})
\end{aligned}$$

$$= \frac{1}{\delta h} e^{-\delta e^{N_t h e^h}}, \tag{3.2}$$

and

$$\sum_{j=M_t+1}^{\infty} e^{-\gamma j h} = \frac{1}{h_t \gamma} e^{-M_t h_t \gamma}, \tag{3.3}$$

We can find following bound by replacing (3.2) and (3.3) in (3.1) as follows:

$$K((\frac{1}{\gamma} + \frac{1}{\delta}) + \frac{1}{2}(\frac{\pi d}{\alpha_s})M_x^{-1})e^{-(\pi d \alpha_s M_x)^{\frac{1}{2}}}$$

$$\leq K_1 L_1 e^{-(\pi d \alpha_s M_x)^{\frac{1}{2}}},$$

where K_1 is a positive constant. We apply a similar approach and obtain bounds for the two other parts of our discrete system. Replacing all bounds in the discrete system, we find the maximum bound as follows:

$$\| u - u_{m_x,m_t} \| \leq K M_x^2 exp(-(\pi d \alpha_s M_x)^{\frac{1}{2}}),$$

and the proof is completed. □

4. Numerical Result

In this section, we illustrate the Sinc-Galerkin method to solve IHCP with the unknown boundary condition in the inverse problem (2.1). Since inverse problems are ill-posed, it is necessary to investigate the stability of the proposed method with a test problem.

Remark 4.1. *Errors on the set of sinc grid points*

$$s = \{x_i\}_{-M_x}^{N_x} \times \{t_j\}_{-M_t}^{N_t},$$

where

$$x_i = \Phi^{-1}(ih) = \frac{a + e^{ih}}{1 + e^{ih}}, \; i = -M_x, \cdots, N_x,$$

$$t_j = \Upsilon^{-1}(jh) = e_{jh}, \; i = -M_t, \cdots, N_t,$$

are reported as follows:

$$\| E_s(h) \| = | \, numeric - exact \, | \, .$$

Example 4.1. *In this example, we solve the problem* (2.1) *with given data,*

$$u(x,0) = 0, \; 0 \leq x \leq 1,$$

$$u(1,t) = 0.9te^{-t} sin(1), \; 0 \leq t \leq \infty,$$

$$u(0.1,t) = 0, \; 0 \leq t \leq \infty.$$

The exact solution of this problem is $u(x,t) = (x - 0.1)te^{-t} sinx$ *and* $f(x,t) = ((x - 0.1)sinx - 2tcosx)e^{-t}$. *The results are presented for* $u(0,t)$ *when* $\alpha = \beta = \gamma = \frac{1}{2}$ *and* $\delta = 1$ *with noisy data (noisy data=input data+(0.01)rand(1)) and* $M_x = 4, 8, 16$ *in Tables 1-9,respectively and Figures 3-11.*

t	$numeric$	$exact$	$\|\|E_s(h)(x,t)\|\|$
0.001	$-1.6189e-004$	0	$1.6189e-004$
0.008	$-7.5420e-004$	0	$7.5420e-004$
0.040	$-3.4875e-003$	0	$3.4875e-003$
0.200	$-1.4214e-002$	0	$1.4214e-002$
1.000	$-3.0961e-002$	0	$3.0961e-002$
4.810	$-3.3016e-003$	0	$3.3016e-003$

Table 1. The comparison between exact and numeric solution for $p(t)_{0th\ order\ Tikhonov}$ with noisy data when $M_x = 4$.

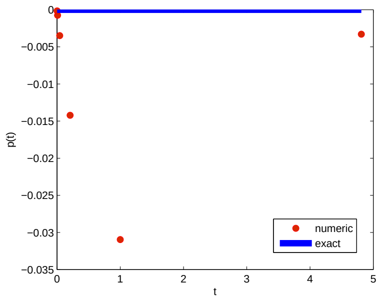

FIGURE 3. The comparison between the exact results and 0th order Tikhonov of the problem (2.1) with noisy data when $M_x = 4$.

t	$numeric$	$exact$	$\|\|E_s(h)(x,t)\|\|$
0.0001	$-1.1692e-005$	0	$1.1692e-005$
0.001	$-1.0725e-004$	0	$1.0725e-004$
0.01	$-9.7821e-004$	0	$9.7821e-004$
0.1	$-8.1880e-003$	0	$8.1880e-003$
0.3	$-1.9935e-002$	0	$1.9935e-002$
1.0	$-3.0956e-002$	0	$3.0956e-002$
3.0	$-1.2264e-002$	0	$1.2264e-002$
9.2	$-7.6848e-005$	0	$7.6848e-005$

Table 2. The comparison between exact and numeric solution for $p(t)_{0th\ order\ Tikhonov}$ with noisy data when $M_x = 8$.

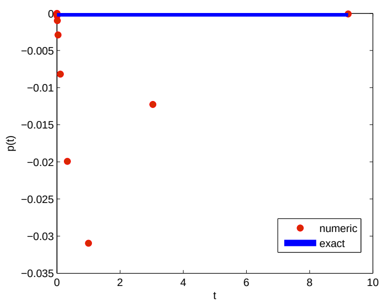

FIGURE 4. The comparison between the exact results and 0th order Tikhonov of the problem (2.1) with noisy data when $M_x = 8$.

t	$numeric$	$exact$	$\|\|E_s(h)(x,t)\|\|$
0.000003	$-2.9757e-007$	0	$2.9875e-007$
0.00001	$-1.4157e-006$	0	$1.4159e-006$
0.00008	$-6.7942e-006$	0	$6.7942e-006$
0.0001	$-1.4895e-005$	0	$1.4895e-005$
0.001	$-1.5685e-004$	0	$1.5685e-004$
0.01	$-1.6255e-003$	0	$1.6255e-003$
0.09	$-7.2542e-003$	0	$7.2542e-003$
0.2	$-1.4209e-002$	0	$1.4209e-002$
0.4	$-2.4318e-002$	0	$2.4318e-002$
1.0	$-3.0955e-002$	0	$3.0955e-002$
2.1	$-2.0587e-002$	0	$2.0587e-002$
4.8	$-3.2965e-003$	0	$3.2965e-003$
10.5	$-2.3242e-005$	0	$2.3242e-005$

Table 3. The comparison between exact and numeric solution for $p(t)_{0th\ order\ Tikhonov}$ with noisy data when $M_x = 16$.

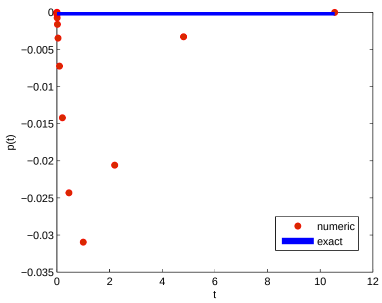

FIGURE 5. The comparison between the exact results and 0th order Tikhonov of the problem (2.1) with noisy data when $M_x = 16$.

t	$numeric$	$exact$	$\|\|E_s(h)(x,t)\|\|$
0.001	$-1.6580e-004$	0	$1.6676e-004$
0.008	$-7.5811e-004$	0	$7.5832e-004$
0.040	$-3.4914e-003$	0	$3.4915e-003$
0.200	$-1.4218e-002$	0	$1.4218e-002$
1.000	$-3.0964e-002$	0	$3.0964e-002$
4.810	$-3.3055e-003$	0	$3.3055e-003$

Table 4. The comparison between exact and numeric solution for $p(t)_{1st\ order\ Tikhonov}$ with noisy data when $M_x = 4$.

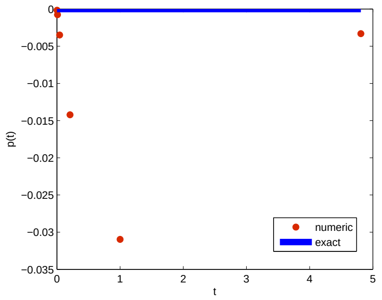

FIGURE 6. The comparison between the exact results and 1th order Tikhonov of the problem (2.1) with noisy data when $M_x = 4$.

t	$numeric$	$exact$	$\|\|E_s(h)(x,t)\|\|$
0.0001	$-1.2376e-005$	0	$1.2416e-005$
0.001	$-1.0794e-004$	0	$1.0794e-004$
0.01	$-9.7890e-004$	0	$9.7890e-004$
0.1	$-8.1887e-003$	0	$8.1887e-003$
0.3	$-1.9936e-002$	0	$1.9936e-002$
1.0	$-3.0956e-002$	0	$3.0956e-002$
3.0	$-1.2265e-002$	0	$1.2265e-002$
9.2	$-7.7533e-005$	0	$7.7539e-005$

Table 5. The comparison between exact and numeric solution for $p(t)_{1st\ order\ Tikhonov}$ with noisy data when $M_x = 8$.

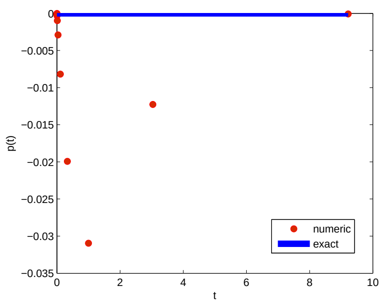

FIGURE 7. The comparison between the exact results and 1th order Tikhonov of the problem (2.1) with noisy data when $M_x = 8$.

t	$numeric$	$exact$	$\|\|E_s(h)(x,t)\|\|$
0.000003	$-2.0809e-006$	0	$.2727e-006$
0.00001	$-3.1991e-006$	0	$3.3270e-006$
0.00008	$-8.5776e-006$	0	$8.6261e-006$
0.0001	$-1.6678e-005$	0	$1.6703e-005$
0.001	$-1.5863e-004$	0	$1.5863e-004$
0.01	$-1.6273e-003$	0	$1.6273e-003$
0.09	$-7.2560e-003$	0	$7.2560e-003$
0.2	$-1.4211e-002$	0	$1.4211e-002$
0.4	$-2.4320e-002$	0	$2.4320e-002$
1.0	$-3.0957e-002$	0	$3.0957e-002$
2.1	$-2.0589e-002$	0	$2.0589e-002$
4.8	$-3.2983e-003$	0	$3.2983e-003$
10.5	$-2.5025e-005$	0	$2.5042e-005$

Table 6. The comparison between exact and numeric solution for $p(t)_{1st\ order\ Tikhonov}$ with noisy data when $M_x = 16$.

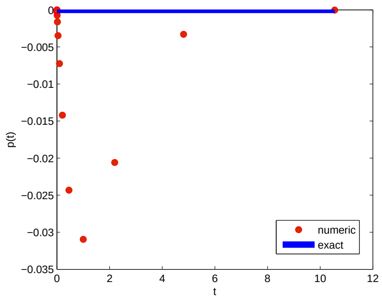

FIGURE 8. The comparison between the exact results and 1th order Tikhonov of the problem (2.1) with noisy data when $M_x = 16$.

t	$numeric$	$exact$	$\|\|E_s(h)(x,t)\|\|$
0.0001	$-1.151e-005$	0	$1.151e-005$
0.001	$-1.050e-004$	0	$1.050e-004$
0.01	$-9.580e-004$	0	$9.580e-004$
0.1	$-8.018e-003$	0	$8.018e-003$
0.3	$-1.952e-002$	0	$1.952e-002$
1.0	$-3.031e-002$	0	$3.031e-002$
3.0	$-1.201e-002$	0	$1.201e-002$
9.2	$-7.531e-005$	0	$7.531e-005$

Table 7. The comparison between exact and numeric solution for $p(t)_{2nd\ order\ Tikhonov}$ with noisy data when $M_x = 4$.

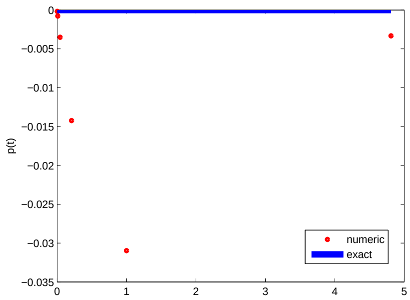

FIGURE 9. The comparison between the exact results and 2th order Tikhonov of the problem (2.1) with noisy data when $M_x = 4$.

t	$numeric$	$exact$	$\|\|E_s(h)(x,t)\|\|$
0.0001	$-1.2255e-005$	0	$1.2274e-005$
0.001	$-1.0781e-004$	0	$1.0782e-004$
0.01	$-9.7877e-004$	0	$9.7878e-004$
0.1	$-8.1886e-003$	0	$8.1886e-003$
0.3	$-1.9936e-002$	0	$1.9936e-002$
1.0	$-3.0956e-002$	0	$3.0956e-002$
3.0	$-1.2265e-002$	0	$1.2265e-002$
9.2	$-7.7412e-005$	0	$7.7415e-005$

Table 8. The comparison between exact and numeric solution for $p(t)_{2nd\ order\ Tikhonov}$ with noisy data when $M_x = 8$.

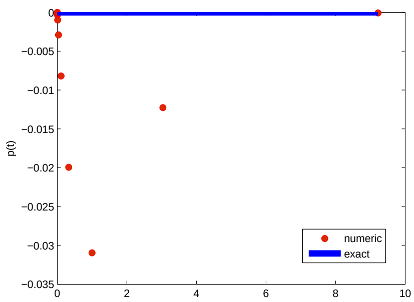

FIGURE 10. The comparison between the exact results and 2th order Tikhonov of the problem (2.1) with noisy data when $M_x = 8$.

t	$numeric$	$exact$	$\|\|E_s(h)(x,t)\|\|$
0.000003	$-4.6133e-005$	0	$4.8685e-005$
0.00001	$-4.7251e-005$	0	$4.9746e-005$
0.00008	$-5.2630e-005$	0	$5.4880e-005$
0.0001	$-6.0731e-005$	0	$6.2691e-005$
0.001	$-2.0268e-004$	0	$2.0328e-004$
0.01	$-1.6714e-003$	0	$1.6715e-003$
0.09	$-7.3001e-003$	0	$7.3001e-003$
0.2	$-1.4255e-002$	0	$1.4255e-002$
0.4	$-2.4364e-002$	0	$2.4364e-002$
1.0	$-3.1001e-002$	0	$3.1001e-002$
2.1	$-2.0633e-002$	0	$2.0633e-002$
4.8	$-3.3424e-003$	0	$3.3424e-003$
10.5	$-6.9078e-005$	0	$7.0807e-005$

Table 9. The comparison between exact and numeric solution for $p(t)_{2nd\ order\ Tikhonov}$ with noisy data when $M_x = 16$.

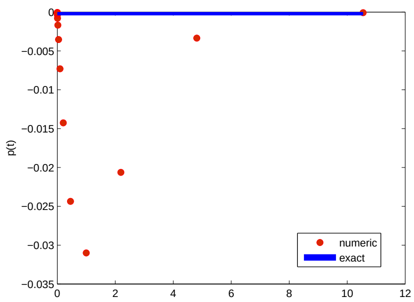

FIGURE 11. The comparison between the exact results and 2th order Tikhonov of the problem (2.1) with noisy data when $M_x = 16$.

5. CONCLUSIONS

In this paper, a numerical approach was proposed to estimate unknown boundary condition for the inverse heat conduction problem using noisy data. The approach is based on Sinc-Galerkin method. The obtained solutions by this approach are stable. The preferences of our approach with respect to the other done works in literature are as follows:
1. In the proposed approach, the solutions are obtained in a more extensive time range, i.e., the whole domain.
2. The obtained results are often based on noiseless data.
This difficulty was overcame in this paper and the results were completed based on noisy data. Finally, it was shown that the rate of convergence of the approach is exponential.

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