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Strong convergence of a multi-step implicit iterative scheme with errors for common fixed points of uniformly L–Lipschitzian total asymptotically strict pseudocontractive mappings

Austine Efut Ofem[a]

[a]*Department of Mathematics, University of Uyo, Uyo, Nigeria.*

Abstract

In this paper, we introduce a three step implicit iteration process with errors and prove strong convergence theorem of the new iterative scheme for finite family of uniformly L–Lipschitzian total asymptotically strict pseudocontractive mappings in Banach spaces. The results in this paper extend, generalize and unify well known results in the existing literature.

Keywords: Fixed point, Banach space, Lipschitzian, total asymptotically strict pseudocontractive mapping, strong convergence.
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1. Introduction

Let E be a real Banach space with dual E^*. We denote by J the *normalized duality* mapping from E into 2^{E^*} defined by

$$J(\zeta) = \{f^* \in E^* : \langle \zeta, f^* \rangle = \|\zeta\|^2 = \|f^*\|^2\}, \ \forall \zeta \in E, \tag{1.1}$$

where $\langle ., . \rangle$ denotes the generalized duality pairing. The single-valued-normalized duality mapping is denoted by j and $F(G)$ denotes the set of fixed points of mapping G, i.e., $F(G) = \{\zeta \in E : G\zeta = \zeta\}$.

In the sequel, we give the following definitions which will be useful in this study.

Definition 1.1. Let K be a nonempty subset of real Banach space E. A mapping $G : K \to K$ is said to be:

Email address: ofemaustine@gmail.com (Austine Efut Ofem)

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- *nonexpansive* if

$$\|G\zeta - G\eta\| \leq \|\zeta - \eta\|, \ \forall\, \zeta, \eta \in K; \tag{1.2}$$

- *k-strictly pseudocontractive* if there exists $j(\zeta - \eta) \in J(\zeta - \eta)$ and a constant $\lambda \in (0,1)$ such that

$$\langle G\zeta - G\eta, j(\zeta - \eta)\rangle \leq \|\zeta - \eta\|^2 - \lambda\|(I - T)\zeta - (I - T)\eta\|^2, \ \forall\, \zeta, \eta \in K. \tag{1.3}$$

  It is easy to see that such mappings are Lipschitz with constant $L = \frac{\lambda+1}{\lambda}$;
- *uniformly L–Lipschitzian* if there exists a constant $L > 0$ such that

$$\|G^n\zeta - G^n\eta\| \leq L\|\zeta - \eta\|, \ \forall\, \zeta, \eta \in K \text{ and } n \geq 1; \tag{1.4}$$

- *asymptotically λ-strictly pseudocontractive* with sequence $\{h_n\} \subset [1, \infty)$ and $h_n \to 1$ as $n \to \infty$. If there exists a constant $\lambda \in (0,1)$ and for any given $\zeta, \eta \in K$ there exists $j(\zeta - \eta) \in J(\zeta - \eta)$ such that

$$\langle G^n\zeta - G^n\eta, j(\zeta - \eta)\rangle \leq h_n\|\zeta - \eta\|^2 - \lambda\|(I - G^n)\zeta - (I - G^n)\eta\|^2, \tag{1.5}$$

  $\forall n \geq 1$.
- *asymptotically λ-strictly pseudocontractive in the intermediate sense* if there exists a constant $\lambda \in (0,1)$ and sequences $\mu_n \in [0, \infty)$ and $\xi_n \in [0, \infty)$ with $\mu_n \to 0$, $\xi_n \to 0$ as $n \to \infty$. For any $\zeta, \eta \in K$, there exists $j(x - y) \in J(\zeta - \eta)$ such that

$$\begin{aligned}\langle G^n\zeta - G^n\eta, j(\zeta - \eta)\rangle \ &\leq \ (1 + \mu_n)\|\zeta - \eta\|^2 \\ &\quad -\lambda\|(I - G^n)\zeta - (I - G^n)\eta\|^2 + \xi_n \ \forall\, n \geq 1,\end{aligned} \tag{1.6}$$

- *total asymptotically strictly pseudocontractive* if there exists a constant $\lambda \in (0,1)$ and sequences $\mu_n \in [0, \infty)$ and $\xi_n \in [0, \infty)$ with $\mu_n \to 0$, $\xi_n \to 0$ as $n \to \infty$. For any $\zeta, \eta \in K$, there exists $j(x-y) \in J(\zeta-\eta)$ such that

$$\begin{aligned}\langle G^n\zeta - G^n\eta, j(\zeta - \eta)\rangle \ &\leq \ \|\zeta - \eta\|^2 - \lambda\|(I - G^n)\zeta - (I - G^n)\eta\|^2 \\ &\quad +\mu_n\phi(\|\zeta - \eta\|) + \xi_n \ \forall\, n \geq 1,\end{aligned} \tag{1.7}$$

  where $\phi : [0, \infty) \to [0, \infty)$ is continuous and strictly increasing function with $\phi(0) = 0$.

Remark 1.2. If $\phi(\lambda) = \lambda^2$, then total asymptotically strictly pseudocontractive mapping reduces to asymptotically λ–strictly pseudocontractive mapping in the intermediate sense. If $\xi_n = 0$, then asymptotically λ–strictly pseudocontractive mapping in the intermediate sense reduces to asymptotically λ–strictly pseudocontractive mapping and if $k_n = 1$, $n = 1$, then an asymptotically λ–strictly pseudocontractive mapping reduces to strictly pseudocontractive mapping. Hence, the class of total asymptotically strictly pseudocontractive mappings properly includes all the classes of mappings defined above.

These class of mappings have been studied by several authors (see for example, [3], [43], [44], [38] and the references there in).

In 1974, Ishikawa [17] introduced an iteration process $\{\zeta_n\}$ defined by

$$\begin{cases} \zeta_0 \in K, \\ \zeta_{n+1} = (1 - \alpha_n)\zeta_n + \alpha_n G\eta_n, \qquad \forall n \geq 1, \\ \eta_n = (1 - \delta_n)\zeta_n + \delta_n G\zeta_n, \end{cases} \tag{1.8}$$

where $\{\alpha_n\}$ and $\{\delta_n\}$ are sequences in [0,1]. This iteration process reduces to Mann iteration [20] if $\delta_n = 0$ for all $n \geq 1$ as follows:

$$\begin{cases} \zeta_0 \in K, \\ \zeta_{n+1} = (1 - \alpha_n)\zeta_n + \alpha_n G\zeta_n, \end{cases} \qquad \forall n \geq 1, \tag{1.9}$$

where $\{\alpha_n\}$ is a sequence in [0,1].
In 1991, Schu [30] introduced the following Mann-type iterative process for an asymptotically nonexpansive mapping in Hilbert spaces

$$\begin{cases} \zeta_0 \in K, \\ \zeta_{n+1} = (1-\alpha_n)x_n + \alpha_n G^n \zeta_n, \end{cases} \quad \forall n \geq 1, \tag{1.10}$$

where $\{\alpha_n\}$ is a sequence in [0,1].

In 2014, Saluja [29] improved the modified explicit scheme (1.10) of Schu [30] as follows:

$$\begin{cases} \zeta_0 \in K, \\ \zeta_{n+1} = \alpha_n \zeta_n + (1-\alpha_n) G_{i(n)}^{h(n)} \zeta_n, \end{cases} \quad \forall n \geq 1, \tag{1.11}$$

where $\{\alpha_n\}$ is a sequence in [0,1], $n = (h-1)N + i$, $i = n(i) \in I = \{1, 2, ..., N\}$, $h = h(n) \geq 1$ is some positive integers and $h(n) \to \infty$ as $n \to \infty$.

In 2001, Xu and Ori [40] introduced the following implicit iteration process for finite family of nonexpansive self-mapping in Hilbert spaces.

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = \alpha_n \zeta_{n-1} + (1-\alpha_n) G_n \zeta_n, \end{cases} \quad \forall n \geq 1, \tag{1.12}$$

where $\{\alpha_n\}$ is a sequence in [0,1] and $G_n = G_{n(\,mod\, N)}$.

In 2003, Sun [35] modified the implicit iteration of Xu and Ori [40] aa follows:

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = \alpha_n \zeta_{n-1} + (1-\alpha_n) G_{i(n)}^{h(n)} \zeta_n, \end{cases} \quad \forall n \geq 1, \tag{1.13}$$

where $\{\alpha_n\}$ is a sequence in [0,1], $n = (h-1)N + i$, $i = n(i) \in I = \{1, 2, ..., N\}$, $h = h(n) \geq 1$ is some positive integers and $h(n) \to \infty$ as $n \to \infty$.

In 2006, Su and Li [34] introduced the following implicit Ishikwa-type iteration scheme and called it composite implicit iteration process and applied the iteration process for the approximation of common fixed points of a finite family of strictly pseudocontractive maps:

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = \alpha_n \zeta_{n-1} + (1-\alpha_n) G_n \eta_n, \\ \eta_n = \delta_n \zeta_{n-1} + (1-\delta_n) G_n \zeta_n, \end{cases} \quad \forall n \geq 1, \tag{1.14}$$

where $\{\alpha_n\}$ and $\{\delta_n\}$ are sequences in [0,1] and $G_n = G_{n\,(mod)\,N}$.

In 2011, Igbokwe and Ini [16] modified and improved the composite implicit iteration process of Su and Li [34] for the approximation of common fixed points of finite family of $\lambda-$strictly asymptotically pseudocontractive mappings in Banach spaces. Precisely, they considered the following modified averaging composite iteration process:

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = \alpha_n \zeta_{n-1} + (1-\alpha_n) G_{i(n)}^{h(n)} \eta_n, \\ \eta_n = \delta_n \zeta_{n-1} + (1-\delta_n) G_{i(n)}^{h(n)} \zeta_n, \end{cases} \quad \forall n \geq 1, \tag{1.15}$$

where $\{\alpha_n\}$ and $\{\delta_n\}$ are sequences in [0,1] and $n = (h-1)N + i$, $i = i(n) \in \{1, 2, ..., N\}$, $h = h(n) \geq 1$ is some positive integers and $h(n) \to \infty$ as $n \to \infty$.

In 2010, Gu [13] introduced a composite implicit iteration process with errors for a finite family of strictly pseudocontractive mappings in Banach spaces as follows:

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = (1-\alpha_n-\beta_n)\zeta_{n-1} + \alpha_n G_n \eta_n + \beta_n u_n, & \forall n \geq 1, \\ \eta_n = (1-\delta_n-\gamma_n)\zeta_{n-1} + \delta_n G_n \zeta_n + \gamma_n v_n, \end{cases} \tag{1.16}$$

where $G_n = G_{n(mod\, n)}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, are four real sequences in $[0,1]$, $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K.

In 2012, Jim et al. [19] improved and modified the composite implicit iteration process of Gu [13] for a finite family of asymptotically ϕ-demicontractive maps in Banach spaces as follows:

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = (1-\alpha_n-\beta_n)\zeta_{n-1} + \alpha_n G_{i(n)}^{h(n)} \eta_n + \beta_n u_n, & \forall n \geq 1, \\ \eta_n = (1-\delta_n-\gamma_n)\zeta_{n-1} + \delta_n G_{i(n)}^{h(n)} \zeta_n + \gamma_n v_n, \end{cases} \tag{1.17}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, $\{\gamma_n\}$, are four real sequences in $[0,1]$, $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K and $n = (h-1)N + i$, $i = i(n) \in \{1, 2, ..., N\}$, $h = h(n) \geq 1$ is some positive integers and $h(n) \to \infty$ as $n \to \infty$.

Noor et al. [21] introduced and studied a three-step iteration process for solving non-linear operator equations in real Banach spaces. Since then, Noor iteration scheme has been applied to study the strong and weak convergence of several mappings (see, e.g., [8], [41], [36]). It was proved by Bnouhachem et al. [1] that three-step method performs better than two-step and one-step methods for solving variational inequalities. Moreover, three-step schemes are natural generalizations of the splitting methods to solve partial differential equations, (see [31], [33], [36]).

On the other hand, Glowinski and Le-Tallec [9] used a three-step iterative method to solve elasto-viscoplasticity, liquid crystal and eigenvalue problems. They also established that three-step iterative scheme performs better than one-step (Mann) and two-step (Ishikawa) iterative schemes. Haubruge et al. [10] studied the convergence analysis of the three-step iterative processes of Glowinski and Le-Tallec [9] and used the three-step iteration to obtain some new splitting type algorithms for solving variational inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iteration also lead to highly parallelized algorithms under certain conditions.

Hence, we can conclude by observing that three-step iterative schemes play an important role in solving various problems in pure and applied sciences. (one-step) and Ishikawa.

Implicit iterative schemes have been studied recently by several authors (see for example, [13], [24], [34], [35], [40], [6] and the references there in).

Motivated and inspired by the above results, we introduce a new modified three-steps composite implicit iteration process with errors for a finite family of N uniformly L-Lipschitzian total asymptotically strictly pseudocontractive mappings in Banach spaces as follows:

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = (1-\alpha_n-\beta_n)\zeta_{n-1} + \alpha_n G_{i(n)}^{h(n)} \eta_n + \beta_n u_n, \\ \eta_n = (1-\delta_n-\gamma_n)\zeta_{n-1} + \delta_n G_{i(n)}^{h(n)} z_n + \gamma_n v_n, & \forall n \geq 1, \\ z_n = (1-e_n-f_n)\zeta_n + e_n G_{i(n)}^{h(n)} \zeta_n + f_n w_n, \end{cases} \tag{1.18}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\delta_n\}$, $\{\gamma_n\}$, $\{e_n\}$, $\{f_n\}$, are real sequences in $[0,1]$ satisfying $e_n + f_n \leq 1$, $\alpha_n + \beta_n \leq 1$ and $\delta_n + \gamma_n \leq 1$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in K and $n = (h-1)N + i$, $i = i(n) \in \{1, 2, ..., N\}$, $h = h(n) \geq 1$ is some positive integers and $h(n) \to \infty$ as $n \to \infty$.

On the other hand, it is of high importance to check if any constructed iteration process is well defined so as to know if it can be employed to approximate the fixed points of some mappings. Now, we show that (1.18) can be employed to approximate the fixed point of asymptotically total pseodocontractive mapping which is Lipschitz continuous. Let G_i be L_i-Lipschitz total assymptotically pseudocontractive mappings with sequences μ_{in} and $\xi_{in} \subset [0,\infty)$ such that μ_{in} and $\xi_{in} \to \infty \to 0$ as $n \to \infty$.

Define a mapping $\Psi_n : K \to K$ by

$$\begin{aligned}\Psi_n(\zeta) &= (1-\alpha_n-\beta_n)\zeta_{n-1} + \alpha_n G_{i(n)}^{h(n)}\{(1-\delta_n-\gamma_n)\zeta_{n-1} \\ &\quad +\delta_n G_{i(n)}^{h(n)}[(1-e_n-f_n)\zeta + e_n G_{i(n)}^{h(n)}\zeta + f_n w_n] + \gamma_n v_n\} + \beta_n u_n,\end{aligned}$$

for all $n \geq 1$. It follows that

$$\begin{aligned}\|\Psi_n(\zeta)-\Psi_n(\eta)\| &= \alpha_n\|G_{i(n)}^{h(n)}\{(1-\delta_n-\gamma_n)\zeta_{n-1} + \delta_n G_{i(n)}^{h(n)}[(1-e_n-f_n)\zeta \\ &\quad +e_n G_{i(n)}^{h(n)}\zeta + f_n w_n] + \gamma_n v_n\} - G_{i(n)}^{h(n)}\{(1-\delta_n-\gamma_n)\zeta_{n-1} \\ &\quad +\delta_n G_{i(n)}^{h(n)}[(1-e_n-f_n)\eta + e_n G_{i(n)}^{h(n)}\eta + f_n w_n] + \gamma_n v_n\}\| \\ &\leq \alpha_n L\|\delta_n G_{i(n)}^{h(n)}[(1-e_n-f_n)\zeta + e_n G_{i(n)}^{h(n)}\zeta + f_n w_n] \\ &\quad -\delta_n G_{i(n)}^{h(n)}[(1-e_n-f_n)\eta + e_n G_{i(n)}^{h(n)}\eta + f_n w_n]\| \\ &\leq \alpha_n\delta_n L^2\|(1-e_n-f_n)(\zeta-\eta) + e_n(G_{i(n)}^{h(n)}\zeta - G_{i(n)}^{h(n)}\eta)\| \\ &\leq \alpha_n\delta_n L^2[(1-e_n-f_n)\|\zeta-\eta\| + e_n\|G_{i(n)}^{h(n)}\zeta - G_{i(n)}^{h(n)}\eta\|] \\ &\leq \alpha_n\delta_n L^2[(1-e_n-f_n)\|\zeta-\eta\| + e_n L\|\zeta-\eta\|] \\ &\leq \alpha_n\delta_n L^2[1-e_n+e_n L]\|\zeta-\eta\| \\ &= \alpha_n\delta_n L^2[1+e_n(L-1)]\|\zeta-\eta\|, \ \forall \zeta,\eta \in K,\end{aligned}$$

where $L = \max\{L_i : 1 \leq i \leq N\}$.

If $\alpha_n\delta_n L^2[1+e_n(L-1)] < 1$ for all $n \geq 1$, then Ψ_n is a contraction. By Banach contraction mapping principle, we can see that there exists a unique fixed point $\zeta_n \in K$ such that

$$\begin{aligned}\Psi_n(\zeta) &= (1-\alpha_n-\beta_n)\zeta_{n-1} + \alpha_n G_{i(n)}^{h(n)}\{(1-\delta_n-\gamma_n)\zeta_{n-1} \\ &\quad +\delta_n G_{i(n)}^{h(n)}[(1-e_n-f_n)\zeta + e_n G_{i(n)}^{h(n)}\zeta + f_n w_n] + \gamma_n v_n\} + \beta_n u_n,\end{aligned}$$

for all $n \geq 1$. This shows that the implicit iteration sequence (1.18) is well defined.

Remark 1.3. It is actually fascinating to see that the iteration process (1.18) reduces to:

- (1.9) when $\delta_n = \gamma_n = e_n = f_n = 0$, $G^n = G$, $N = 1$ (Mann iterative scheme [20]).
- (1.10) when $\delta_n = \gamma_n = e_n = f_n = 0$, $N = 1$ (Schu iterative scheme [30]).
- (1.11) when $\delta_n = \gamma_n = e_n = f_n = 0$ (Saluja iteraive scheme [29]).
- (1.14) when $\beta_n = \gamma_n = e_n = f_n = 0$, $G^n = G$ (Su and Li iterative scheme [34]).
- (1.15) when $\beta_n = \gamma_n = e_n = f_n = 0$ (Igbokwe and Ini ietative scheme [16]).
- (1.16) when $e_n = f_n = 0$, $G^n = G$ (Gu iterative scheme [13]).
- (1.17) when $e_n = f_n = 0$, (Jim et al. iterative scheme [19]).

Interestingly, our new iterative scheme properly includes those mentioned above and several other explicit and implicit iterative schemes in the existing literature. And hence, our results will generalize, extend, complement and unify the results of Jim et al. iterative scheme [19], Gu [13], Mann [20], Xu and Ori [40], Osilike [24], Su and Li[34], Chang [2], Schu [30], Saluja [29] and several other well known results in the existing literature.

It is our purpose in this paper to use a simple and quite different method to study the strong convergence of our new implicit iterative sequence $\{\zeta_n\}$ defined by (1.18) to a common fixed points of finite family of uniformly L-Lipschitzian total asymptotically strictly pseudocontractive mappings in a Banach space. Our results extend and improve some recent results in [24], [13], [34], [16], [40], [6] and several others in the existing literature.

2. Preliminaries

In order to prove our main results, we also need the following Lemmas.

Lemma 2.1 (see [2]). *Let $J : E \to 2^E$ be the normalized duality mapping. Then for any $\zeta, \eta \in E$, one has*

$$\|\zeta + \eta\|^2 \leq \|\zeta\|^2 + 2\langle \eta, j(\zeta + \eta)\rangle, \ \ \forall j(\zeta + \eta) \in J(\zeta + \eta).$$

Lemma 2.2 (see [22]). *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be nonnegative real sequences satisfying the following conditions:*

$$a_{n+1} \leq (1 + b_n)a_n + c_n \ \forall n \geq n_0$$

If $\sum_{n=0}^{\infty} c_n < \infty$, $\sum_{n=0}^{\infty} b_n < \infty$. Then,

(i) the $\lim_{n\to\infty} a_n$ exists.

(ii) In addition if there exists a subsequence $\{a_{n_i}\} \subset \{a_n\}$such that $a_{n_i} \to 0$, then $a_n \to 0$ (as $n \to \infty$).

3. Main results

Lemma 3.1. *Let E be a real Banach space and let K be a closed convex subset of E. Let $\{G_i\}_{i=1}^N : K \to K$ be a finite family of uniformly L_i-Lipschitzian $(\lambda_i, \{\mu_{in}\}, \{\xi_{in}\}, \phi_i)$–total asymptotically strictly pseudocontractive mappings such that $\Im = \bigcap_{i=1}^N F(G_i) \neq \emptyset$. Let $\{u_n\}, \{v_n\}$ and $\{w_n\}$ be bounded sequences in K. Let $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\}, \{\gamma_n\}, \{e_n\}$ and $\{f_n\}$ be six real sequences in [0,1] such that $\alpha_n + \beta_n \leq 1$, $\delta_n + \gamma_n \leq 1$ and $e_n + f_n \leq 1$. Assume that the following conditions are satisfied:*

(i) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $\sum\limits_{n=1}^{\infty} \alpha_n^2 < \infty, \sum_{n=1}^{\infty} \alpha_n\beta_n < \infty, \sum\limits_{n=1}^{\infty} \alpha_n\delta_n < \infty, \sum_{n=1}^{\infty} \alpha_n\gamma_n < \infty,$
$\sum\limits_{n=1}^{\infty} \alpha_n\delta_n f_n < \infty, \sum\limits_{n=1}^{\infty} \alpha_n\mu_n < \infty, \sum\limits_{n=1}^{\infty} \alpha_n\xi_n < \infty$;

(iii) $\sum\limits_{n=1}^{\infty} \beta_n < \infty$;

(iv) $\alpha_n\delta_n L^2[1 + e_n(L - 1)] < 1$, *where* $L = \max\{L_i : 1 \leq i \leq N\}$.

Let $\{\zeta_n\}$ be the iteration process generated by (1.18), then for arbitrary $x_0 \in K$ we obtain that

$$\lim_{n\to} \|\zeta_n - G_t\zeta_n\| = 0, \ \forall \, t \in \{1, 2, ..., N\}. \tag{3.1}$$

.

Proof. Since for each $G_i : K \to K$, $1 \leq i \leq N$ is total asymptotically strictly pseudocontractive mapping, then we have for all $\zeta, \eta \in K$, there exists a constant $\lambda_i \in (0, 1)$, $L_i \geq 1$ and sequences $\{\mu_{in}\}, \{\xi_{in}\} \subset [0, \infty)$ with $\mu_{in} \to 0$ and $\xi_{in} \to 0$ as $n \to \infty$ such that

$$\begin{aligned} \langle G_i^n\zeta - G_i^n\eta, j(\zeta - \eta)\rangle \ &\leq \ \|\zeta - \eta\|^2 - \lambda_i\|(I - G_i^n)\zeta - (I - G_i^n)\eta\|^2 \\ &\quad +\mu_{in}\phi_i(\|\zeta - \eta\|) + \xi_{in}, \ \forall \ n \geq 1, \end{aligned} \tag{3.2}$$

where $\phi_i : [0, \infty) \to [0, \infty)$ is continuous and strictly increasing function with $\phi_i(0) = 0$, and

$$\|G_i^n\zeta - G_i^n\eta\| \leq L_i\|\zeta - \eta\|, \ n \geq 1. \tag{3.3}$$

Let $\lambda = \min\{\lambda_i : 1 \leq i \leq N\}, \mu_n = \max\{\mu_{in} : 1 \leq i \leq N\}, \xi_n = \max\{\xi_{in} : 1 \leq i \leq N\}, \phi = \max\{\phi_i : 1 \leq i \leq N\}$.

For $\zeta,\eta\in K, p\in\Im$ with $1\leq i\leq N$, then we have

$$\begin{aligned}\langle G_i^n\zeta-G_i^n\eta, j(\zeta-\eta)\rangle &\leq \|\zeta-\eta\|^2-\lambda\|(I-G_i^n)\zeta-(I-G_i^n)\eta\|^2\\ &\quad+\mu_n\phi(\|\zeta-\eta\|)+\xi_n,\ \forall\ n\geq 1,\end{aligned}\tag{3.4}$$

where $\phi:[0,\infty)\to[0,\infty)$ is continuous and strictly increasing function with $\phi(0)=0$, and

$$\|G_i^n\zeta-G_i^n\eta\|\leq L\|\zeta-\eta\|,\ n\geq 1.\tag{3.5}$$

It follows from (1.18) that

$$\begin{aligned}\|z_n-p\| &= \|(1-e_n-f_n)\zeta_n+e_nG_{i(n)}^{h(n)}\zeta_n+f_nw_n-p\|\\ &= \|(1-e_n-f_n)(\zeta_n-p)+e_n(G_{i(n)}^{h(n)}\zeta_n-p)+f_n(w_n-p)\|\\ &\leq \|(1-e_n-f_n)\|\zeta_n-p\|+e_n\|G_{i(n)}^{h(n)}\zeta_n-p\|+f_n\|w_n-p\|\\ &\leq \|\zeta_n-p\|+e_nL\|\zeta_n-p\|+f_n\|w_n-p\|\\ &\leq (1+e_nL)\|\zeta_n-p\|+f_n\|w_n-p\|\\ &\leq (1+L)\|\zeta_n-p\|+f_n\|w_n-p\|.\end{aligned}\tag{3.6}$$

Using (1.18) and (3.6), we obtain,

$$\begin{aligned}\|\eta_n-p\| &= \|(1-\delta_n-\gamma_n)\zeta_{n-1}+\delta_nG_{i(n)}^{h(n)}z_n+\gamma_nv_n-p\|\\ &= \|(1-\delta_n-\gamma_n)(\zeta_{n-1}-p)+\delta_n(G_{i(n)}^{h(n)}z_n-p)+\gamma_n(v_n-p)\|\\ &\leq (1-\delta_n-\gamma_n)\|\zeta_{n-1}-p\|+\delta_n\|G_{i(n)}^{h(n)}z_n-p\|+\gamma_n\|v_n-p\|\\ &\leq \|\zeta_{n-1}-p\|+\delta_nL\|z_n-p\|+\gamma_n\|v_n-p\|\\ &= \|\zeta_{n-1}-p\|+\delta_nL\{(1+L)\|\zeta_n-p\|+f_n\|w_n-p\|\}+\gamma_n\|v_n-p\|\\ &= \|\zeta_{n-1}-p\|+\delta_nL(1+L)\|\zeta_n-p\|\\ &\quad+\delta_nf_nL\|w_n-p\|+\gamma_n\|v_n-p\|.\end{aligned}\tag{3.7}$$

Again from (1.18) and (3.7) we obtain

$$\begin{aligned}\|\eta_n-\zeta_n\| &= \|\eta_n-\zeta_{n-1}+\zeta_{n-1}-\zeta_n\|\leq\|\eta_n-\zeta_{n-1}\|+\|\zeta_{n-1}-\zeta_n\|\\ &= \|(1-\delta_n-\gamma_n)\zeta_{n-1}+\delta_nG_{i(n)}^{h(n)}z_n+\gamma_nv_n-\zeta_{n-1}\|\\ &\quad+\|\zeta_{n-1}-[(1-\alpha_n-\beta_n)\zeta_{n-1}+\alpha_nG_{i(n)}^{h(n)}\eta_n+\beta_nu_n]\|\\ &= \|\delta_n(G_{i(n)}^{h(n)}z_n-\zeta_{n-1})+\gamma_n(v_n-\zeta_{n-1})\|\\ &\quad+\|\alpha_n(\zeta_{n-1}-G_{i(n)}^{h(n)}\eta_n)+\beta_n(\zeta_{n-1}-u_n)\|\\ &= \|\delta_n(G_{i(n)}^{h(n)}z_n-p+p-\zeta_{n-1})+\gamma_n(v_n-p+p-\zeta_{n-1})\|\\ &\quad+\|\alpha_n(\zeta_{n-1}-p+p-G_{i(n)}^{h(n)}\eta_n)+\beta_n(\zeta_{n-1}-p+p-u_n)\|\\ &\leq \delta_n\|G_{i(n)}^{h(n)}z_n-p\|+\delta_n\|p-\zeta_{n-1}\|+\gamma_n\|v_n-p\|+\gamma_n\|p-\zeta_{n-1}\|\\ &\quad+\alpha_n\|\zeta_{n-1}-p\|+\alpha_n\|p-G_{i(n)}^{h(n)}\eta_n\|+\beta_n\|\zeta_{n-1}-p\|+\beta_n\|p-u_n\|\\ &\leq \delta_nL\|z_n-p\|+\delta_n\|p-\zeta_{n-1}\|+\gamma_n\|v_n-p\|+\gamma_n\|p-\zeta_{n-1}\|\\ &\quad+\alpha_n\|\zeta_{n-1}-p\|+\alpha_nL\|p-\eta_n\|+\beta_n\|\zeta_{n-1}-p\|+\beta_n\|p-u_n\|\\ &= \alpha_nL\|\eta_n-p\|+\delta_nL\|z_n-p\|+(\alpha_n+\beta_n+\delta_n+\gamma_n)\|\zeta_{n-1}-p\|\\ &\quad+\beta_n\|u_n-p\|+\gamma_n\|v_n-p\|.\end{aligned}\tag{3.8}$$

Substituting (3.6) and (3.7) into (3.8) we obtain

$$\begin{aligned}
\|\eta_n-\zeta_n\| &\leq \alpha_n L\{\|\zeta_{n-1}-p\|+\delta_n L(1+L)\|\zeta_n-p\|+\delta_n f_n L\|w_n-p\|+\gamma_n\|v_n-p\|\} \\
&\quad +\delta_n L\{(1+L)\|\zeta_n-p\|+f_n\|w_n-p\|\}+(\alpha_n+\beta_n+\delta_n+\gamma_n)\|\zeta_{n-1}-p\| \\
&\quad +\beta_n\|u_n-p\|+\gamma_n\|v_n-p\| \\
&= \alpha_n L\|\zeta_{n-1}-p\|+\alpha_n\delta_n L^2(1+L)\|\zeta_n-p\|+\alpha_n\delta_n f_n L^2\|w_n-p\| \\
&\quad +\alpha_n\gamma_n L\|v_n-p\|+\delta_n L(1+L)\|\zeta_n-p\|+\delta_n f_n L\|w_n-p\| \\
&\quad +(\alpha_n+\beta_n+\delta_n+\gamma_n)\|\zeta_{n-1}-p\|+\beta_n\|u_n-p\|+\gamma_n\|v_n-p\| \\
&= (\alpha_n L+\alpha_n+\beta_n+\delta_n+\gamma_n)\|\zeta_{n-1}-p\|+(\alpha_n\delta_n L^2+\delta_n L)(1+L)\|\zeta_n-p\| \\
&\quad +(\alpha_n\delta_n f_n L^2+\delta_n f_n L)\|w_n-p\|+(\alpha_n\gamma_n L+\gamma_n)\|v_n-p\|+\beta_n\|u_n-p\| \\
&= [\alpha_n(L+1)+\beta_n+\delta_n+\gamma_n]\|\zeta_{n-1}-p\|+\delta_n L(\alpha_n L+1)(1+L)\|\zeta_n-p\| \\
&\quad +\delta_n f_n L(\alpha_n L+1)\|w_n-p\|+\gamma_n(\alpha L+1)\|v_n-p\|+\beta_n\|u_n-p\| \\
&\leq [\alpha_n(L+1)+\beta_n+\delta_n+\gamma_n]\|\zeta_{n-1}-p\|+\delta_n L(L+1)^2\|\zeta_n-p\| \\
&\quad +\delta_n f_n L(L+1)\|w_n-p\|+\gamma_n(L+1)\|v_n-p\|+\beta_n\|u_n-p\|. \qquad (3.9)
\end{aligned}$$

Using (1.18), we obtain

$$\begin{aligned}
\|\zeta_n-p\|^2 &= \|(1-\alpha_n-\beta_n)\zeta_{n-1}+\alpha_n G_{i(n)}^{h(n)}\eta_n+\beta_n u_n-p\|^2 \\
&= \|(1-\alpha_n-\beta_n)(\zeta_{n-1}-p)+\alpha_n(G_{i(n)}^{h(n)}\eta_n-p)+\beta_n(u_n-p)\|^2 \\
&\leq (1-\alpha_n-\beta_n)^2\|\zeta_{n-1}-p\|^2+2\langle\alpha_n(G_{i(n)}^{h(n)}\eta_n-p)+\beta_n(u_n-p),j(\zeta_n-p)\rangle \\
&= (1-\alpha_n-\beta_n)^2\|\zeta_{n-1}-p\|^2+2\alpha_n\langle G_{i(n)}^{h(n)}\eta_n-p,j(\zeta_n-p)\rangle \\
&\quad +2\beta_n\langle u_n-p,j(\zeta_n-p)\rangle \\
&= (1-\alpha_n-\beta_n)^2\|\zeta_{n-1}-p\|^2+2\alpha_n\langle G_{i(n)}^{h(n)}\eta_n-G_{i(n)}^{h(n)}\zeta_n \\
&\quad +G_{i(n)}^{h(n)}\zeta_n-p,j(\zeta_n-p)\rangle+2\beta_n\langle u_n-p,j(\zeta_n-p)\rangle \\
&= (1-\alpha_n-\beta_n)^2\|\zeta_{n-1}-p\|^2+2\alpha_n\langle G_{i(n)}^{h(n)}\eta_n-G_{i(n)}^{h(n)}\zeta_n,j(\zeta_n-p)\rangle \\
&\quad +2\alpha_n\langle G_{i(n)}^{h(n)}\zeta_n-p,j(\zeta_n-p)\rangle+2\beta_n\langle u_n-p,j(\zeta_n-p)\rangle \\
&\leq (1-\alpha_n-\beta_n)^2\|\zeta_{n-1}-p\|^2+2\alpha_n\|G_{i(n)}^{h(n)}\eta_n-G_{i(n)}^{h(n)}\zeta_n\|\|\zeta_n-p\| \\
&\quad +2\beta_n\|u_n-p\|\|\zeta_n-p\|+2\alpha_n\langle G_{i(n)}^{h(n)}\zeta_n-p,j(\zeta_n-p)\rangle \\
&\leq (1-\alpha_n)^2\|\zeta_{n-1}-p\|^2+2\alpha_n L\|\eta_n-\zeta_n\|\|\zeta_n-p\| \\
&\quad +2\beta_n\|u_n-p\|\|\zeta_n-p\|+2\alpha_n\langle G_{i(n)}^{h(n)}\zeta_n-p,j(\zeta_n-p)\rangle. \qquad (3.10)
\end{aligned}$$

Substituting (3.9) into (3.10) we obtain

$$\begin{aligned}
\|\zeta_n-p\|^2 &\leq (1-\alpha_n)^2\|\zeta_{n-1}-p\|^2+2\alpha_nL\{[\alpha_n(L+1)+\beta_n+\delta_n+\gamma_n]\|\zeta_{n-1}-p\| \\
&+\delta_nL(L+1)^2\|\zeta_n-p\|+\delta_nf_nL(L+1)\|w_n-p\|+\gamma_n(L+1)\|v_n-p\| \\
&+\beta_n\|u_n-p\|\}\|\zeta_n-p\|+2\beta_n\|u_n-p\|\|\zeta_n-p\|+2\alpha_n\langle G_{i(n)}^{h(n)}\zeta_n-\zeta_n \\
&+\zeta_n-p,j(\zeta_n-p)\rangle \\
&= (1-\alpha_n)^2\|\zeta_{n-1}-p\|^2+2\alpha_nL[\alpha_n(L+1)+\beta_n+\delta_n+\gamma_n]\|\zeta_{n-1}-p\|\|\zeta_n-p\| \\
&+2\alpha_n\delta_nL^2(L+1)^2\|\zeta_n-p\|^2+2\alpha_n\delta_nf_nL^2(L+1)\|w_n-p\|\|\zeta_n-p\| \\
&+2\alpha_n\gamma_nL(L+1)\|v_n-p\|\|\zeta_n-p\|+2\alpha_n\beta_nL\|u_n-p\|\|\zeta_n-p\| \\
&+2\beta_n\|u_n-p\|\|\zeta_n-p\|+2\alpha_n\langle G_{i(n)}^{h(n)}\zeta_n-\zeta_n,j(\zeta_n-p)\rangle \\
&+2\alpha_n\langle\zeta_n-p,j(\zeta_n-p)\rangle. \\
&= (1-\alpha_n)^2\|\zeta_{n-1}-p\|^2+2\alpha_nL[\alpha_n(L+1)+\beta_n+\delta_n+\gamma_n]\|\zeta_{n-1}-p\|\|\zeta_n-p\| \\
&+2\alpha_n\delta_nL^2(L+1)^2\|\zeta_n-p\|^2+2\alpha_n\delta_nf_nL^2(L+1)\|w_n-p\|\|\zeta_n-p\| \\
&+2\alpha_n\gamma_nL(L+1)\|v_n-p\|\|\zeta_n-p\|+2\beta_n(\alpha_nL+1)\|u_n-p\|\|\zeta_n-p\| \\
&+2\alpha_n\langle G_{i(n)}^{h(n)}\zeta_n-\zeta_n,j(\zeta_n-p)\rangle+2\alpha_n\langle\zeta_n-p,j(\zeta_n-p)\rangle \\
&\leq (1-\alpha_n)^2\|\zeta_{n-1}-p\|^2+2\alpha_nL[\alpha_n(L+1)+\beta_n+\delta_n+\gamma_n]\|\zeta_{n-1}-p\|\|\zeta_n-p\| \\
&+2\alpha_n\delta_nL^2(L+1)^2\|\zeta_n-p\|^2+2\alpha_n\delta_nf_nL^2(L+1)\|w_n-p\|\|\zeta_n-p\| \\
&+2\alpha_n\gamma_nL(L+1)\|v_n-p\|\|\zeta_n-p\|+2\beta_n(L+1)\|u_n-p\|\|\zeta_n-p\| \\
&+2\alpha_n\langle G_{i(n)}^{h(n)}\zeta_n-\zeta_n,j(\zeta_n-p)\rangle+2\alpha_n\langle\zeta_n-p,j(\zeta_n-p)\rangle. \qquad (3.11)
\end{aligned}$$

Setting $M=\max\{\sup\{u_n-p\|\},\sup\{v_n-p\|\},\sup\{w_n-p\|\},n\geq 1\}$ and noting that

$$\left.\begin{aligned}
&\|\zeta_{n-1}-p\|\|\zeta_n-p\|\leq\tfrac{1}{2}(\|\zeta_{n-1}-p\|^2+\|\zeta_n-p\|^2), \\
&\|u_n-p\|\|\zeta_n-p\|\leq\tfrac{1}{2}(\|u_n-p\|^2+\|\zeta_n-p\|^2), \\
&\|v_n-p\|\|\zeta_n-p\|\leq\tfrac{1}{2}(\|v_n-p\|^2+\|\zeta_n-p\|^2), \\
&\|w_n-p\|\|\zeta_n-p\|\leq\tfrac{1}{2}(\|w_n-p\|^2+\|\zeta_n-p\|^2),
\end{aligned}\right\}\forall n\geq 1, \qquad (3.12)$$

we obtain

$$\begin{aligned}
\|\zeta_n-p\|^2 &\leq (1-\alpha_n)^2\|\zeta_{n-1}-p\|^2 \\
&+2\alpha_nL[\alpha_n(L+1)+\beta_n+\delta_n+\gamma_n]\times\frac{1}{2}(\|\zeta_{n-1}-p\|^2+\|\zeta_n-p\|^2) \\
&+2\alpha_n\delta_nL^2(L+1)^2\|\zeta_n-p\|^2+2\alpha_n\delta_nf_nL^2(L+1) \\
&\times\frac{1}{2}(\|w_n-p\|^2+\|\zeta_n-p\|^2)+2\alpha_n\gamma_nL(L+1)\times\frac{1}{2}(\|v_n-p\|^2+\|\zeta_n-p\|^2) \\
&+2\beta_n(L+1)\times\frac{1}{2}(\|u_n-p\|^2+\|\zeta_n-p\|^2) \\
&+2\alpha_n\|\zeta_n-p\|^2+2\alpha_n\langle G_{i(n)}^{h(n)}\zeta_n-\zeta_n,j(\zeta_n-p)\rangle \\
&\leq \{(1-\alpha_n)^2+\alpha_nL[\alpha_n(L+1)+\beta_n+\delta_n+\gamma_n]\}\|\zeta_{n-1}-p\|^2 \\
&+\{\alpha_nL[\alpha_n(L+1)+\beta_n+\delta_n+\gamma_n]+2\alpha_n\delta_nL^2(L+1)^2 \\
&+\alpha_n\delta_nf_nL^2(L+1)+\alpha_n\gamma_nL(L+1)+\beta_n(L+1)+2\alpha_n\}\|\zeta_n-p\|^2 \\
&+\alpha_n\delta_nf_nL^2(L+1)M_1+\alpha_n\gamma_nL(L+1)M_1+\beta_n(L+1)M_1 \\
&+2\alpha_n\langle G_{i(n)}^{h(n)}\zeta_n-\zeta_n,j(\zeta_n-p)\rangle. \qquad (3.13)
\end{aligned}$$

By (3.4), for the point $\zeta_n \in K$ and $p \in \Im$, there exists $j(\zeta_n - p) \in J(\zeta_n - p)$ such that

$$\langle G_{i(n)}^{h(n)}\zeta_n - \zeta_n, j(\zeta_n - p)\rangle \le -\lambda\|G_{i(n)}^{h(n)}\zeta_n - \zeta_n\| + \mu_n\phi(\|\zeta_n - p\|) + \xi_n, \ \forall n \ge 1.$$

Since ϕ is a strictly increasing function, it follows that $\phi(\zeta) \le \phi(M)$, if $\zeta \le M$; $\phi(\zeta) \le M^*\zeta^2$, if $\zeta \ge M$. In either case, we can obtain

$$\phi(\zeta) \le \phi(M) + M^*\zeta^2. \tag{3.14}$$

Hence from (3.13) and (3.14), we have

$$\begin{aligned}\|\zeta_n - p\|^2 &\le \{(1-\alpha_n)^2 + 2\alpha_n L[\alpha_n(L+1) + \beta_n + \delta_n + \gamma_n]\}\|\zeta_{n-1} - p\|^2\\ &\quad + \{\alpha_n L[\alpha_n(L+1) + \beta_n + \delta_n + \gamma_n] + 2\alpha_n\delta_n L^2(L+1)^2\\ &\quad + \alpha_n\delta_n f_n L^2(L+1) + \alpha_n\gamma_n L(L+1) + \beta_n(L+1) + 2\alpha_n\}\|\zeta_n - p\|^2\\ &\quad + \alpha_n\delta_n f_n L^2(L+1)M_1 + \alpha_n\gamma_n L(L+1)M_1 + \beta_n(L+1)M_1\\ &\quad + 2\alpha_n\{-\lambda\|G_{i(n)}^{h(n)}\zeta_n - \zeta_n\|^2 + \mu_n\phi(\|\zeta_n - p\|) + \xi_n\}.\end{aligned}$$

Using (3.14) in (3.15), then we obtain

$$\begin{aligned}\|\zeta_n - p\|^2 &\le \{(1-\alpha_n)^2 + 2\alpha_n L[\alpha_n(L+1) + \beta_n + \delta_n + \gamma_n]\}\|\zeta_{n-1} - p\|^2\\ &\quad + \{\alpha_n L[\alpha_n(L+1) + \beta_n + \delta_n + \gamma_n] + 2\alpha_n\delta_n L^2(L+1)^2\\ &\quad + 2\alpha_n\delta_n f_n L^2(L+1) + \alpha_n\gamma_n L(L+1) + \beta_n(L+1) + 2\alpha_n\}\|\zeta_n - p\|^2\\ &\quad + \alpha_n\delta_n f_n L^2(L+1)M_1 + \alpha_n\gamma_n L(L+1)M_1 + \beta_n(L+1)M_1 + 2\alpha_n\mu_n\phi(M)\\ &\quad + 2\alpha_n\mu_n M^*\|\zeta_n - p\|^2 + 2\alpha_n\xi_n - 2\alpha_n\lambda\|G_{i(n)}^{h(n)}\zeta_n - \zeta_n\|^2\\ &\le \{(1-\alpha_n)^2 + 2\alpha_n L[\alpha_n(L+1) + \beta_n + \delta_n + \gamma_n]\}\|\zeta_{n-1} - p\|^2\\ &\quad + \{\alpha_n L[\alpha_n(L+1) + \beta_n + \delta_n + \gamma_n] + 2\alpha_n\delta_n L^2(L+1)^2\\ &\quad + \alpha_n\delta_n f_n L^2(L+1) + \alpha_n\gamma_n L(L+1) + 2\alpha_n\mu_n M^* + \beta_n(L+1) + 2\alpha_n\}\|\zeta_n - p\|^2\\ &\quad + \alpha_n\delta_n f_n L^2(L+1)M_1 + \alpha_n\gamma_n L(L+1)M_1 + \beta_n(L+1)M_1 + 2\alpha_n\mu_n\phi(M)\\ &\quad + 2\alpha_n\xi_n - 2\alpha_n\lambda\|G_{i(n)}^{h(n)}\zeta_n - \zeta_n\|^2\\ &= \tau_n\|\zeta_{n-1} - p\|^2 + \nu_n\|\zeta_n - p\|^2 + \varpi_n - 2\alpha_n\lambda\|G_{i(n)}^{h(n)}\zeta_n - \zeta_n\|^2,\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}\tau_n &= (1-\alpha_n)^2 + 2\alpha_n L[\alpha_n(L+1) + \beta_n + \delta_n + \gamma_n],\\ \nu_n &= \alpha_n L[\alpha_n(L+1) + \beta_n + \delta_n + \gamma_n] + 2\alpha_n\delta_n L^2(L+1)^2\\ &\quad + \alpha_n\delta_n f_n L^2(L+1) + \alpha_n\gamma_n L(L+1) + 2\alpha_n\mu_n M^* + \beta_n(L+1) + 2\alpha_n \text{ and}\\ \varpi_n &= \alpha_n\delta_n f_n L^2(L+1)M_1 + \alpha_n\gamma_n L(L+1)M_1 + \beta_n(L+1)M_1 + 2\alpha_n\mu_n\phi(M)\\ &\quad + 2\alpha_n\xi_n.\end{aligned}$$

From (3.15) we have,

$$\begin{aligned}\|\zeta_n - p\|^2 &\le \left[\frac{\tau_n}{1-\nu_n}\right]\|\zeta_{n-1} - p\|^2 + \frac{\varpi_n}{1-\nu_n} - \left[\frac{2\alpha_n\lambda}{1-\nu_n}\right]\|G_{i(n)}^{h(n)}\zeta_n - \zeta_n\|^2\\ &= \left[1 + \frac{\vartheta_n}{1-\nu_n}\right]\|x_{n-1} - p\|^2 + \frac{\varpi_n}{1-\nu_n} - \left[\frac{2\alpha_n\lambda}{1-\nu_n}\right]\|G_{i(n)}^{h(n)}\zeta_n - \zeta_n\|^2,\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}\vartheta_n = \tau_n + \nu_n - 1 &= \alpha_n^2 + 3\alpha_n L[\alpha_n(L+1) + \beta_n + \delta_n + \gamma_n]\\ &\quad + 2\alpha_n\delta_n L^2(L+1)^2 + \alpha_n\delta_n f_n L^2(L+1)\\ &\quad + \alpha_n\gamma_n L(L+1) + 2\alpha_n\mu_n M^* + \beta_n(L+1).\end{aligned}$$

Since $\lim_{n\to\infty} \alpha_n = 0$, it follows from the conditions (ii) and (iii) that

$$\begin{aligned}\nu_n &= \alpha_n L(\alpha_n(L+1)+\beta_n+\delta_n+\gamma_n)+2\alpha_n\delta_n L^2(L+1)^2\\ &\quad+\alpha_n\delta_n f_n L^2(L+1)+\alpha_n\gamma_n L(L+1)+2\alpha_n\mu_n M^*+\beta_n(L+1)+2\alpha_n \to 0 \text{ as } n\to\infty,\end{aligned}$$

therefore, there exists a number n_0 such that $1-\nu_n \geq \frac{1}{2}$, for any $n \geq n_0$.

Hence, we have

$$\begin{aligned}\|\zeta_n-p\|^2 &\leq [1+2\vartheta_n]\|\zeta_{n-1}-p\|^2+2\varpi_n-2\alpha_n\lambda\|G_{i(n)}^{h(n)}\zeta_n-\zeta_n\|^2\\ &= [1+b_n]\|\zeta_{n-1}-p\|^2+c_n-2\alpha_n\lambda\|G_{i(n)}^{h(n)}\zeta_n-\zeta_n\|^2 \qquad (3.17)\\ &\leq [1+b_n]\|\zeta_{n-1}-p\|^2+c_n,\end{aligned}$$

where $b_n = 2\vartheta_n$ and $c_n = 2\varpi_n$. From the conditions (ii) and (iii), it easy to see that $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. It follows from Lemma 2.2 that
$\lim_{n\to\infty} \|\zeta_n-p\|^2$ exists and so also $\lim_{n\to\infty} \|\zeta_n-p\|$ exists, therefore, $\{\zeta_n\}$ is bounded, hence there exists a constant $M_2 > 0$ such that $\|\zeta_n-p\| \leq M_2$, $\forall n \geq 1$. It follows from (3.17) that

$$\begin{aligned}2\alpha_n\lambda\|G_{i(n)}^{h(n)}\zeta_n-\zeta_n\|^2 &\leq \|\zeta_{n-1}-p\|^2-\|\zeta_n-p\|^2+b_n\|\zeta_{n-1}-p\|^2+c_n\\ &\leq \|\zeta_{n-1}-p\|^2-\|\zeta_n-p\|^2+b_nM_2+c_n,\ \forall n\geq n_0.\end{aligned}$$

Thus,

$$2\lambda\sum_{j=n_0+1}^{\infty}\alpha_j\|G_{j(n)}^{h(n)}\zeta_j-\zeta_j\| \leq \|\zeta_{n_0}-p\|^2+M_2\sum_{j=n_0+1}^{\infty}b_j+\sum_{j=n_0+1}^{\infty}c_j,$$

and hence,

$$2\lambda\sum_{n=1}^{\infty}\alpha_n\|G_{i(n)}^{h(n)}\zeta_n-\zeta_n\| \leq \|\zeta_{n_0}-p\|^2+M_2\sum_{n=1}^{\infty}b_n+\sum_{n=1}^{\infty}c_n. \qquad (3.18)$$

Since $\sum_{j=n_0+1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, it follows from (3.18) that

$$\sum_{n=1}^{\infty}\alpha_n\|G_{i(n)}^{h(n)}\zeta_n-\zeta_n\|^2 < \infty. \qquad (3.19)$$

Since $\sum_{n=1}^{\infty}\alpha_n = \infty$, then from (3.19), we must have that

$$\liminf_{n\to\infty}\|\zeta_n-G_{i(n)}^{h(n)}\zeta_n\| = 0. \qquad (3.20)$$

Notice from (1.18) that

$$\begin{aligned}\|\zeta_n-\zeta_{n-1}\| &= \|(1-\alpha_n-\beta_n)\zeta_{n-1}+\alpha_nG_{i(n)}^{h(n)}\eta_n+\beta_nu_n-\zeta_{n-1}\|\\ &= \|\alpha_n(G_{i(n)}^{h(n)}\eta_n-\zeta_{n-1})+\beta_n(u_n-\zeta_{n-1})\|\\ &\leq \alpha_n(\|G_{i(n)}^{h(n)}\eta_n-p\|+\|\zeta_{n-1}-p\|)+\beta_n(u_n-\zeta_{n-1})\|\\ &\leq \alpha_n(L\|\eta_n-p\|+\|\zeta_{n-1}-p\|)+\beta_n(u_n-\zeta_{n-1})\|. \qquad (3.21)\end{aligned}$$

Since $\lim_{n\to\infty} \|\zeta_n - p\|$ exists for all $p \in \Im$ and therefore $\{\|\zeta_n - p\|\}$ is bounded. Since follows from $\lim_{n\to\infty} \alpha_n = 0$, then using (3.7) and the restrictions (ii), that we obtain

$$\lim_{n\to\infty} \|\zeta_n - \zeta_{n-1}\| = 0. \tag{3.22}$$

This implies that

$$\lim_{n\to\infty} \|\zeta_n - \zeta_{n+l}\| = 0, \ \forall\ l = 1, 2, ...N. \tag{3.23}$$

Since for each $n > N$, $n = (h(n)-1)N+i(n)$, where $i(n) \in \{1, 2, ..., N\}$, then $n-N = (k(n)-1)N+i(n)-N = [(h(n)-1)-1]N + i(n) = (h(n-N)-1)N + i(n-N)$, thus $h(n-N) = h(n)-1$ and $i(n-N) = i(n)$, hence, we see that

$$\begin{aligned} \|\zeta_n - G_{i(n)}\zeta_n\| &\leq \|\zeta_n - G_{i(n)}^{h(n)}\zeta_n\| + \|G_{i(n)}^{h(n)}\zeta_n - G_{i(n)}\zeta_n\| \\ &\leq \|\zeta_n - G_{i(n)}^{h(n)}\zeta_n\| + L\|G_{i(n)}^{h(n)-1}\zeta_n - \zeta_n\| \\ &\leq \|\zeta_n - G_{i(n)}^{h(n)}\zeta_n\| + L(\|G_{i(n)}^{h(n)-1}\zeta_n - G_{i(n-N)}^{h(n)-1}\zeta_{n-N}\| \\ &\quad +\|G_{i(n-N)}^{h(n)-1}\zeta_{n-N} - \zeta_{n-N}\| + \|\zeta_{n-N} - \zeta_n\|). \end{aligned} \tag{3.24}$$

Notice that $h(n-N) = h(n)-1$ and $i(n-N) = i(n)$. Thus, it implies that

$$\begin{aligned} \|G_{i(n)}^{h(n)-1}\zeta_n - G_{i(n-N)}^{h(n)-1}\zeta_{n-N}\| &= \|G_{i(n)}^{h(n)-1}\zeta_n - G_{i(n)}^{h(n)-1}\zeta_{n-N}\| \\ &\leq L\|\zeta_n - \zeta_{n-N}\| \end{aligned} \tag{3.25}$$

and

$$\|G_{i(n-N)}^{h(n)-1}\zeta_{n-N} - \zeta_{n-N}\| = \|G_{i(n-N)}^{h(n-N)}\zeta_{n-N} - \zeta_{n-N}\|. \tag{3.26}$$

Substituting (3.25) and (3.26) into (3.24)

$$\begin{aligned} \|\zeta_n - G_{i(n)}\zeta_n\| &\leq \|\zeta_n - G_{i(n)}^{h(n)}\zeta_n\| + L(L\|\zeta_n - \zeta_{n-N}\| \\ &\quad +|G_{i(n-N)}^{h(n-N)}\zeta_{n-N} - \zeta_{n-N}\| + \|\zeta_{n-N} - \zeta_n\|). \end{aligned}$$

It follows from (3.20) and (3.23) that

$$\lim_{n\to\infty} \|\zeta_n - G_{i(n)}\zeta_n\| = 0. \tag{3.27}$$

In particular, we see that

$$\begin{cases} \lim_{h\to\infty} \|\zeta_{hN+1} - G_1\zeta_{hN+1}\| = 0, \\ \lim_{h\to\infty} \|\zeta_{hN+2} - G_2\zeta_{hN+2}\| = 0, \\ \quad\vdots \\ \lim_{h\to\infty} \|\zeta_{hN+N} - G_N\zeta_{hN+N}\| = 0. \end{cases} \tag{3.28}$$

For any $t, s = 1, 2, ...N$, we obtain that

$$\begin{aligned} \|\zeta_{hN+s} - G_t\zeta_{hN+s}\| &\leq \|\zeta_{hN+s} - \zeta_{hN+t}\| + \|\zeta_{hN+t} - G_t\zeta_{hN+t}\| \\ &\quad +\|G_t\zeta_{hN+t} - G_t\zeta_{hN+s}\| \\ &\leq (1+L)\|\zeta_{hN+s} - \zeta_{hN+t}\| + \|\zeta_{hN+t} - G_t\zeta_{hN+t}\|. \end{aligned}$$

Letting $h \to \infty$, we obtain

$$\lim_{h\to\infty} \|\zeta_{hN+s} - G_t\zeta_{hN+s}\| = 0, \tag{3.29}$$

which is equivalent to

$$\lim_{n\to\infty} \|\zeta_n - G_t\zeta_n\| = 0. \tag{3.30}$$

This completes the proof of Lemma 3.1.

Theorem 3.2. *Let E be a real Banach space and let K be a closed convex subset of E. Let $\{G_i\}_{i=1}^N : K \to K$ be a finite family of uniformly L_i-Lipschitzian $(\lambda_i, \{\mu_{in}\}, \{\xi_{in}\}, \phi_i)$–total asymptotically strictly pseudocontractive mappings such that $\Im = \bigcap_{i=1}^N F(G_i) \neq \emptyset$. Let $\{u_n\}, \{v_n\}$ and $\{w_n\}$ be bounded sequences in K. Let $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\}, \{\gamma_n\}, \{e_n\}$ and $\{f_n\}$ be six real sequences in [0,1] such that $\alpha_n + \beta_n \leq 1$, $\delta_n + \gamma_n \leq 1$ and $e_n + f_n \leq 1$. Assume that the following conditions are satisfied:*

(i) $\sum_{n=1}^\infty \alpha_n = \infty$;
(ii) $\sum_{n=1}^\infty \alpha_n^2 < \infty, \sum_{n=1}^\infty \alpha_n\beta_n < \infty, \sum_{n=1}^\infty \alpha_n\delta_n < \infty, \sum_{n=1}^\infty \alpha_n\gamma_n < \infty,$
$\sum_{n=1}^\infty \alpha_n\delta_n f_n < \infty, \sum_{n=1}^\infty \alpha_n\mu_n < \infty, \sum_{n=1}^\infty \alpha_n\xi_n < \infty$;
(iii) $\sum_{n=1}^\infty \beta_n < \infty$;
(iv) $\alpha_n\delta_n L^2[1 + e_n(L-1)] < 1$, *where* $L = \max\{L_i : 1 \leq i \leq N\}$.

Let $\{\zeta_n\}$ be the iteration process generated by (1.18), for arbitrary $\zeta_0 \in K$. If one mapping in $\{G_1, G_2, ..., G_N\}$ is semicompact, then the sequence $\{\zeta_n\}$ converges strongly to some point in $\Im$.

Proof. Without loss of generality, we may assume that G_1 is semicompact. It implies from (3.30) that

$$\lim_{n\to\infty} \|\zeta_n - G_1\zeta_n\| = 0. \tag{3.31}$$

Since G_1 is semicompct, then definitely there exist a subsequence $\{\zeta_{n_q}\}$ of $\{\zeta_n\}$ such that $\{\zeta_{n_q}\} \to g \in K$ strongly. From (3.30), we obtain

$$\|g - G_t g\| \leq \|g - \zeta_{n_q}\| + \|\zeta_{n_q} - G_t\zeta_{n_q}\| + \|G_t\zeta_{n_q} - G_t g\|.$$

Since G_t is Lipschitz continuous, we have that $g \in \Im$. From Lemma 3.1, we know that $\lim_{n\to\infty} \|\zeta_n - p\|$ exists for each $p \in \Im$. This immediately implies that $\lim_{n\to\infty} \|\zeta_n - g\|$ exists. Notice that from $\{\zeta_{n_q}\} \to g \in K$, we finally obtain

$$\lim_{n\to\infty} \|\zeta_n - g\| = 0. \tag{3.32}$$

This completes the proof.

The following results can be obtain immediately from Theorem 3.2.

Corollary 3.3. *Let E be a real Banach space and let K be a closed convex subset of E. Let $\{G_i\}_{i=1}^N : K \to K$ be a finite family of uniformly L_i-Lipschitzian $(\lambda_i, \{\mu_{in}\}, \{\xi_{in}\}, \phi_i)$–total asymptotically strictly pseudocontractive mappings such that $\Im = \bigcap_{i=1}^N F(G_i) \neq \emptyset$. Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences in K. Let $\{\alpha_n\}, \{\beta_n\}, \{\delta_n\}$ and $\{\gamma_n\}$ be four real sequences in [0,1] such that $\alpha_n + \beta_n \leq 1$ and $\delta_n + \gamma_n \leq 1$. Assume that the following conditions are satisfied:*

(i) $\sum_{n=1}^\infty \alpha_n = \infty$;
(ii) $\sum_{n=1}^\infty \alpha_n^2 < \infty, \sum_{n=1}^\infty \alpha_n\beta_n < \infty, \sum_{n=1}^\infty \alpha_n\delta_n < \infty, \sum_{n=1}^\infty \alpha_n\gamma_n < \infty,$
$\sum_{n=1}^\infty \alpha_n\mu_n < \infty, \sum_{n=1}^\infty \alpha_n\xi_n < \infty$;

(iii) $\sum_{n=1}^{\infty} \beta_n < \infty$;
(iv) $\alpha_n \delta_n L^2 < 1$, *where* $L = \max\{L_i : 1 \leq i \leq N\}$.

For arbitrary $\zeta_0 \in K$, *let* $\{\zeta_n\}$ *be the iteration process generated by*

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = (1 - \alpha_n - \beta_n)\zeta_{n-1} + \alpha_n G_{i(n)}^{h(n)} \eta_n + \beta_n u_n, & \forall n \geq 1. \\ \eta_n = (1 - \delta_n - \gamma_n)\zeta_{n-1} + \delta_n G_{i(n)}^{h(n)} \zeta_n + \gamma_n v_n. \end{cases} \tag{3.33}$$

If one mapping in $\{G_1, G_2, ..., G_N\}$ *is semicompact, then the sequence* $\{\zeta_n\}$ *converges strongly to some point in* $\Im$.

Proof. Set $e_n = f_n = 0$ in Theorem 3.2.

Corollary 3.4. *Let* E *be a real Banach space and let* K *be a closed convex subset of* E. *Let* $\{G_i\}_{i=1}^N : K \to K$ *be a finite family of uniformly* L_i*-Lipschitzian* $(\lambda_i, \{\mu_{in}\}, \{\xi_{in}\}, \phi_i)$*–total asymptotically strictly pseudocontractive mappings such that* $\Im = \bigcap_{i=1}^N F(G_i) \neq \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\delta_n\}$ *be real sequences in [0,1]. Assume that the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty, \sum_{n=1}^{\infty} \alpha_n \delta_n < \infty, \sum_{n=1}^{\infty} \alpha_n \mu_n < \infty, \sum_{n=1}^{\infty} \alpha_n \xi_n < \infty$;
(iii) $\alpha_n \delta_n L^2 < 1$, *where* $L = \max\{L_i : 1 \leq i \leq N\}$.

For arbitrary $\zeta_0 \in K$, *let* $\{\zeta_n\}$ *be the iteration process generated by*

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = (1 - \alpha_n)\zeta_{n-1} + \alpha_n G_{i(n)}^{h(n)} \eta_n, & \forall n \geq 1. \\ \eta_n = (1 - \delta_n)\zeta_{n-1} + \delta_n G_{i(n)}^{h(n)} \zeta_n. \end{cases} \tag{3.34}$$

If one mapping in $\{G_1, G_2, ..., G_N\}$ *is semicompact, then the sequence* $\{\zeta_n\}$ *converges strongly to some point in* $\Im$.

Proof. Set $\beta_n = \gamma_n = 0$ in corollary 3.3.

Corollary 3.5. *Let* E *be a real Banach space and let* K *be a closed convex subset of* E. *Let* $\{G_i\}_{i=1}^N : K \to K$ *be a finite family of uniformly* L_i*-Lipschitzian* $(\lambda_i, \{\mu_{in}\}, \{\xi_{in}\}, \phi_i)$*–total asymptotically strictly pseudocontractive mappings such that* $\Im = \bigcap_{i=1}^N F(G_i) \neq \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be real sequences in [0,1]. Assume that the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty, \sum_{n=1}^{\infty} \alpha_n \beta_n < \infty, \sum_{n=1}^{\infty} \alpha_n \mu_n < \infty, \sum_{n=1}^{\infty} \alpha_n \xi_n < \infty$;
(iii) $\sum_{n=1}^{\infty} \beta_n < \infty$.

For arbitrary $\zeta_0 \in K$, *let* $\{\zeta_n\}$ *be the iteration process generated by*

$$\begin{cases} \zeta_0 \in K, \\ \zeta_n = (1 - \alpha_n - \beta_n)\zeta_{n-1} + \alpha_n G_{i(n)}^{h(n)} \zeta_{n-1} + \beta_n u_n. \end{cases} \quad \forall n \geq 1. \tag{3.35}$$

If one mapping in $\{G_1, G_2, ..., G_N\}$ *is semicompact, then the sequence* $\{\zeta_n\}$ *converges strongly to some point in* $\Im$.

Proof. Set $\delta_n = \gamma_n = 0$ in corollary 3.3.

Corollary 3.6. *Let E be a real Banach space and let K be a closed convex subset of E. Let $\{G_i\}_{i=1}^N : K \to K$ be a finite family of uniformly L_i-Lipschitzian $(\lambda_i, \{\mu_{in}\}, \{\xi_{in}\}, \phi_i)$–total asymptotically strictly pseudocontractive mappings such that $\Im = \bigcap_{i=1}^N F(G_i) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequences in [0,1]. Assume that the following conditions are satisfied:*

(i) $\sum_{n=1}^\infty \alpha_n = \infty$;
(ii) $\sum_{n=1}^\infty \alpha_n^2 < \infty, \sum_{n=1}^\infty \alpha_n\mu_n < \infty, \sum_{n=1}^\infty \alpha_n\xi_n < \infty$.

For arbitrary $\zeta_0 \in K$, let $\{\zeta_n\}$ be the iteration process generated by

$$\begin{cases} \zeta_0 \in K, & \\ \zeta_n = (1-\alpha_n)\zeta_{n-1} + \alpha_n G_{i(n)}^{h(n)}\zeta_{n-1}. & \forall n \geq 1. \end{cases} \tag{3.36}$$

If one mapping in $\{G_1, G_2, ..., G_N\}$ is semicompact, then the sequence $\{\zeta_n\}$ converges strongly to some point in $\Im$.

Proof. Set $\beta_n = 0$ in corollary 3.5.

This is just to state but a few of the numerous results that can be obtain from Theorem 3.2.

4. Conclusion

Since our new implicit iteration process properly includes the iterative schemes considered by Osilike [24], Gu [13], Su and Li [34], Igbokewe and Ini [16], Xu and Ori [40], Chen [6], Saluja [29] and owing to the fact that the class total asymptotically strictly pseudocontractive mapping is more general than the classes of nonexpasive, asymptotically λ– strictly pseudocontractive and asymptotically λ– strictly pseudocontractive mappings in the intermediate sense, then it follows that the results of Osilkie [24], Gu [13], Su and Li [34], Igbokewe and Ini [16], Xu and Ori [40], Chen [6], Saluja [29] are special cases of Theorem 3.2. Hence, our results generalize, extend, improve and complement their results and several other results in the literature relating to this class of mappings.

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References

[1] A. Bnouhachem, M. A. Noor and T. M. Rassias, Three-steps iterative algorithms for mixed variational inequalities, Appl. Math. Comput., **183** (2006), 436-446.
[2] S.S.Chang, Some results for asymptotically pseudocontractive mappings and asymptotically nonexpansive mappings, Proc. Amer. Math. Soc., 129 (2001), 845–853.
[3] S. S. Chang, L. Wang, Y. K. Tang, and L. Yang, The split common fixed point problem for total asymptotically strictly pseudocontractive mappings, Journal of Applied Mathematics, **2012** (2012) , Article ID 385638, 13 pages .
[4] C. E. Chidume and C .O. Chidume, Convergence theorems for fixed points of uniformly continuous generalized Φ–hemi-contractive mappings, J. Math. Anal. Appl., **303** (2005), 545-554.
[5] C. E. Chidume and C. O. Chidume, Convergence theorem for zeros of generalized Lipschitz generalized phi-quasi-accretive operators, Proc. Amer. Math. Soc., **134** (2006), 243–251.
[6] R. C. Chen, Y. S. Song, H. Zhou, Convergence theorems for implicit iteration process for a finite family continuous pseudo-contractive mappings, J. Math. Anal. Appl., **314** (2006), 701–706.
[7] F. Cianciaruso, G. Marino and X. Wang, Weak and strong convergence of the Ishikawa iterative process for a finite family of asymptotically nonexpansive mappings, Applied Mathematics and Computation, **216** (2010), 3558-3567.
[8] Y. J. Cho, H. Y. Zhou and G. Guo, Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings, Comput. Math. Appl., **47** (2004), 707-717.

[9] R. Glowinski and P. Le-Tallec, Augmented Lagrangian and operator-splitting methods in nonlinear mechanics, SIAM, Philadelphia, 1989.

[10] S. Haubruge, V. H. Nguyen and J. J. Strodiot, Convergence analysis and applications of the Glowinski-Le-Tallec splitting method for finding a zero of the sum of two maximal monotone oper- ators, J. Optim. Theory Appl., **97** (1998), 645-673.

[11] F.Gu, Convergence theorems for ϕ-pseudocontractive type mappings in normed linear spaces, Northeast Math. J., **17**(3) (2001), 340–346.

[12] F. Gu, The new composite implicit iterative process with errors for common fixed points of a finite family of strictly pseudocontractive mappings, *Journal of Mathematical Analysis and Applications*, **2**(329) (2007), 766–776.

[13] F. Gu, Implicit and explicit iterative process with errors for a common fixed points of a finite family of strictly pseudocontractive mappings, *An. St. Univ. Ovidius Constanta*, **18**(1) (2010), 139-154.

[14] T. L. Hicks and J. R. Kubicek, On the mann iteration process in hilbert spaces, *Journal of Mathematical Analysis and Applications*, **59** (1977), 498-504.

[15] D. I. Igbokwe, Approximation of fixed points of asymptotically demicontractive mappings in arbitrary Banach spaces, Journal of Inequality in Pure and Applied Mathematics, **3**(1) (2002), 1–11.

[16] D. Igbokwe and O. Ini, A modified averaging composite Implicit Iteration process for common fixed points of a finite family of $k-$ strictly asymptotically pseudocontractive mappings, Advances in Pure Mathematics, **1** (2011), 204–209.

[17] S. Ishikawa, Fixed points by a new iteration method, Proceeding of the America Mathematical society, **4**(1974) 157–150.

[18] G. A. Okeke and J. O. Olareju, Modifed Noor iterations with errors for nonlinear equations in Banach spaces, J. Nonlinear Sci. Appl., **7** (2014), 180- 187.

[19] U. S. Jim, Z. Ongodiebi and F. A. Efiong, A new modified averaging implicit iteration process with errors for common fixed points of a finite family of asymptotically ϕ-demicontractive maps in arbitrary real Banach spaces, International Journal of Pure and Applied Mathematics, 78(3)(2012), 309-321.

[20] W. R. Mann, Mean value methods in iteration, Proceedings of American Mathematical Science, **4** (2003), 506–510.

[21] M. A. Noor, T. M. Kassias and Z. Huang, Three-step iterations for nonlinear accretive operator equations, J. Math. Anal. Appl., **274** (2001), 59-68.

[22] M. O. Osilike, Implicit iteration process for common fixed point of a finite family of strictly pseudocontractive maps, J. Math. Anal. Appl., **294** (2004), 73-81.

[23] M. O. Osilike, Iterative Approximation of fixed points asymptotically demicontractive mappings, Indian J. of Pure Appl. Maths., **29**(12)(1998), 1291–1300.

[24] M. O. Osilike and B. G. Akuchu, Common fixed points of finite family of asymptotically pseudocontractive mappings. Fixed Point Theory and Application, 2004 **2004**, 81–88.

[25] M. O. Osilike, S. C. Aniagbosor and B. G. Akuchu, Fixed points of asymptotically demicontractive mappings in arbitrary Banach spaces, Pan American Mathematical Journal, **12**(2) (2002), 77–88.

[26] M. O. Osilike and F. U. Isiogugu, Fixed points of asymptotically ϕ-demicontractive mappings in arbitrary Banach spaces, Pan-American Mathematical Journal, **15** (3)(2005), 59–69.

[27] M.O. Osilike, A. Udomene, D.I. Igbokwe, B.G. Akuchu, Demiclosedness principle and convergence theorems for k-strictly asymptotically pseudocontractive maps, J. Math. Anal. Appl., **326** (2007), 1334-1345.

[28] L. Qihou, Convergence Theorems of the Sequence of Iterates for Asymptotically Demicontarcive and Hemicontractive Mappings, Nonlinear Analysis: Theory, Methods and Applications, 26 **26**, (1996), 1835-1842.

[29] G. S. Saluja, Convergence of the explicit iteration method for strictly asymptotically pseudocontractive mappings in the intermediate sense, Novi Sad J. Math., **44** (1)(2014), 75-90.

[30] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, Bull. Austr. Math. Soc. , **43**(1991), 153–159.

[31] H. F. Senter and W. G. Dotson: Approximating fixed points of nonexpansive mappings, Proc. Amer. Math. Soc., **44**(1974), 375-380.

[32] J. Schu, Iterative construction of fixed point of asymptotically nonexpansive mappings, J. Math. Anal. Appl., **158**(1991), 407-413.

[33] N. Shahzad and A. Udomene: Approximating common fixed points of two asymptotically quasinonexpansive mappings in Banach spaces, Fixed Point Theory Appl., (2006), article ID 18909, 10 pages.

[34] Y. Su and S. Li, Composite implicit process for common fixed points of a finite family of strictly pseudocontractive maps, J. Math. Anal. Appl., **320** (2006), 882–891.

[35] Z. H. Sun, Strong convergence of an implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings, J. Math. Anal. Appl., **286** (2003), 351–358.

[36] S. Suantai, Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings, J. Math. Anal. Appl., **311** (2005), 506-517.

[37] B. S. Thakur, Weak and strong convergence of composite implicit iteration process, Appl. Math. Comput., **190** (2007), 965–973.

[38] Y. Wang and C. Wang, convergence of a new modified Ishikawa type iteration for common Fixed points of total asymptotically strict pseudocontractive semigroups, *Abstract and Applied Analysis*, 2013, Article ID 319241, 7 pages.

[39] X. Weng, Fixed point iteration for local strictly pseudocontractive mapping, Proceedings of the American Mathematical Society, **133** (2)(1991), 727-731.

[40] H. K. Xu and R. G. Ori, An implicit iteration process for nonexpansive mapping, Num. Fun. Anal. Optim., **22**(2001),

767–773.

[41] Y. Yao, Convergence of three-step iterations for asymptotically nonexpansive mappings, Appl. Math. Comput., **187** (2007), 883-892.

[42] L. P. Yang, Convergence of the new composite implicit iteration process with random errors, Nonlinear Anal., **69**(10) (2008), 3591-3600.

[43] L. Yang and F. H. Zhao, Large strong convergence theorems for total asymptotically strict pseudocontractive semigroup in banach spaces, Fixed Point Theory and Applications, 2012, 2012:24

[44] L. Yang1, S. Chang and F. H. Zhao, Strong convergence theorems for a finite family of total asymptotically strict pseudo-contractive semigroups in Banach spaces, Fixed Point Theory and Applications 2013, 2013:178