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HAAR BASIS METHOD TO SOLVE SOME INVERSE PROBLEMS FOR TWO-DIMENSIONAL PARABOLIC AND HYPERBOLIC EQUATIONS

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Abstract. A numerical method consists of combining Haar basis method and Tikhonov regularization method. We apply the method to solve some inverse problems for two-dimensional parabolic and hyperbolic equations using noisy data. In this paper, a stable numerical solution of these problems is presented. This method uses a sensor located at a point inside the body and measures the $u(x, y, t)$ at a point $x = a$, $0 < a < 1$. We also show that the rate of convergence of the method is as exponential. Numerical results show that a good estimation on the unknown functions of the inverse problems can be obtained within a couple of minutes CPU time at Pentium IV-2.53 GHz PC.

Keywords: Inverse problems, Haar basis method; Error analysis, Tikhonov regularization method, Noisy data.

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Inverse problems are applied in many important scientific and technological fields. Hence, analysis, design implementation and testing of inverse algorithms are also the great scientific and technological interest.

The inverse heat conduction problem in a one-dimensional composite slab with rate-dependent pyrolysis chemical reaction and outgassing flow effects is investigated using the iterative regularization approach. The thermal properties of the composites are considered to be temperature-dependent, [32].

Cheng-Hung Huanga, Chun-Ying Yeha, and Helcio R.B. Orlande presented an iterative regularization method based on an inverse algorithm. The algorithm is applied to simultaneously determine the unknown temperature, concentration-dependent heat, and mass production rates for a chemically reacting fluid. This work is done using interior measurements of temperature and concentration [17].

Kim et al. [27] solved an inverse heat conduction problem to estimate the surface temperature from temperature readings. Su and Neto [28] solved a two-dimensional inverse heat conduction problem to estimate the radial and circumferential transient dependence of the strength of a volumetric heat source in a cylindrical rod. Huang and Tsai [16] solved a three-dimensional inverse heat conduction problem to estimate the local time-dependent surface heat transfer coefficients for plate finned-tube heat exchangers.

However, only few works have been done on the two-dimensional problems because of the complicated interaction and reflection of the thermal wave [9, 8]. Yang Ching-yu,

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[9] developed the two-dimensional hyperbolic heat conduction equations in an arbitrary body-fitted coordinate grid and used non-oscillatory numerical schemes to approach the problem. Chen and Lin [8] formulated a numerical scheme involving the Laplace transform technique and the control volume method for the problem.

Mathematically, the inverse problems belong to the class of problems called the ill-posed problems, i.e. small errors in the measured data can lead to large deviations in the estimated quantities. As a consequence, their solution does not satisfy the general requirement of existence, uniqueness, and stability under small changes to the input data. To overcome such difficulties, a variety of techniques for solving inverse problems have been proposed [32]-[29] and among the various methods such as: Tikhonov regularization [31], iterative regularization [2], mollification [22], BFM (Base Function Method) [25], SFDM (Semi Finite Difference Method) [21], and the FSM (Function Specification Method) [3].

Beck et al. [3] compared the FSM, the Tikhonov regularization and the iterative regularization, using experimental data. Beck and Murio [5] presented a new method that combines the function specification method of Beck with the regularization technique of Tikhonov. Murio and Paloschi [23] proposed a combined procedure based on a data filtering interpretation of the mollification method and FSM.

Haar functions have been used from 1910 when they were introduced by the Hungarian mathematician Haar [12]. The Haar transform is one of the earliest of what is known now as a compact, dyadic, orthonormal wavelet transform. The Haar function, being an odd rectangular pulse pair, is the simplest and oldest orthonormal wavelet with compact support. In the mean time, several definitions of the Haar functions and various generalizations have been published and used. They were intended to adopt this concept to some practical applications as well as to extend its in applications to different classes of signals. Haar functions appear very attractive in many applications as for example, image coding, edge extraction and binary logic design.

Recently, Haar wavelets, [14], have been applied extensively for signal processing in communications and physics research, and have proved to be a wonderful mathematical tool. After discretizing the differential equations in a conventional way like the finite difference approximation, wavelets can be used for algebraic manipulations in the system of equations obtained which lead to better condition number of the resulting system.

The previous work, [14], in the system analysis via Haar wavelets was led by Chen and Hsiao [7], who first derived a Haar operational matrix for the integrals of the Haar functions vector and put the application for the Haar analysis into the dynamical systems. Then, the pioneer work in state analysis of linear time delayed systems via Haar wavelets was laid down by Hsiao [15], who first proposed a Haar product matrix and a coefficient matrix. Hsiao and Wang proposed a key idea to transform the time-varying function and its product with states into a Haar product matrix. Kalpana and Raja Balachandar [18] presented Haar wavelet based method of analysis for observer design in the generalized state space or singular system of transistor circuits.

In this paper, a numerical method is presented based on Haar wavelet method and 0th, 1st, and 2nd Tikhonov regularization.

The organization of this paper is as follows: 1 is divided to two subsection. In subsection 1.1, we formulate and solve an inverse problem for the two-dimensional heat equation. Solution of an inverse problem for the two-dimensional wave equation will be discussed in subsection 2.1. Furthermore, in subsections 1.1 and 2.1, to regularize the resultant ill-conditioned linear system of equations, we apply the Tikhonov (of 0th, 1st and 2nd orders) regularization method to obtain the stable numerical approximation to the solution. Finally some numerical experiments will be given in section 3.

1. Mathematical formulation

Definition 1.1. *The Haar wavelet family for $x \in [0,1)$ is defined as follows,* [14],

$$h_i(x) = \begin{cases} 1, & x \in [\frac{k}{m}, \frac{k+0.5}{m}), \\ -1, & x \in [\frac{k+0.5}{m}, \frac{k+1}{m}), \\ 0, & elsewhere. \end{cases} \tag{1.1}$$

integer $m = 2^j$, $(j = 0, 1, \ldots, J)$ indicates the level of the wavelet;
$k = 0, 1, \ldots, m-1$ is the translation parameter. Maximal level of resolution is J. The index i is calculated according the formula $i = m + k + 1$, such that

$$\int_0^1 h_i(x)h_l(x)\ dx = \frac{1}{2^j}\ \delta_{il},$$

where δ_{il} is Kronecker delta.

In the case of minimal values $m = 1$, $k = 0$ we have $i = 2$, the maximal value of i is $i = 2^{J+1} = M$. It is assumed that the value $i = 1$ corresponds to the scaling function for which $h_1 \equiv 1$ in $[0,1)$. let us defined collocation points $x_l = \frac{l-0.5}{M}$, $(l = 1, 2, \ldots, M)$ and discretis the Haar function $h_i(x)$; in this way we get the coefficient matrix H and the operational matrices of integration P, Q, which are M square matrices, are defined by the equations

$$\begin{aligned} (H)_{il} &= (h_i(x_l)), && (1.2)\\ (PH)_{il} &= \int_0^{x_l} h_i(x)\, dx, && (1.3)\\ (QH)_{il} &= \int_0^{x_l}\int_0^x h_i(s)\, ds\, dx. && (1.4) \end{aligned}$$

The elements of the matrices H, P and Q can be evaluted according to (1.2), (1.3) *and* (1.4). *For example when $M = 2, 4$ we have,*

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},\ P_2 = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix},\ Q_2 = \frac{1}{32}\begin{pmatrix} 5 & -4 \\ 4 & -3 \end{pmatrix},$$

$$H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix},\ P_4 = \frac{1}{16}\begin{pmatrix} 8 & -4 & -2 & -2 \\ 4 & 0 & -2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix},$$

$$Q_4 = \frac{1}{128}\begin{pmatrix} 21 & -16 & -4 & -12 \\ 16 & -11 & -4 & -4 \\ 6 & -2 & -3 & 0 \\ 2 & -2 & 0 & -3 \end{pmatrix}.$$

Remark 1.1. *Any function $\Upsilon \in L^2([0,1) \times [0,1))$ can be decomposed as*

$$\Upsilon(x,y) = \sum_{l=1}^{\infty}\sum_{i=1}^{\infty} c_{il} h_i(x) h_l(y),$$

where the coefficients c_{il} are determined by

$$c_{il} = 2^{j_1+j_2}\int_0^1\int_0^1 \Upsilon(x,y) h_i(x) h_l(y)\, dxdy,$$

where

$$i = 2^{j_1} + k_1 + 1,\ l = 2^{j_2} + k_2 + 1,\ j_1, j_2 \geqslant 0,\ 0 \leqslant k_1 < 2^{j_1},\ 0 \leqslant k_2 < 2^{j_2}.$$

The series expansion of $\Upsilon(x,y)$ *contains an infinite terms. If* $\Upsilon(x,y)$ *is piecewise constant by itself, or may be approximated as piecewise constant during each subinterval, then* $\Upsilon(x,y)$ *will be terminated at finite terms, that is,*

$$\Upsilon(x,y) = \sum_{l=1}^{M2}\sum_{i=1}^{M1} c_{il}h_i(x)h_l(y) = H_{M1}^T(x)C_{M1\times M2}H_{M2}(y),$$

where the coefficients $C_{M1\times M2}$ *and the Haar functions vectors* $H_{M1}^T(x)$ *and* $H_{M2}(y)$ *are defined as,*

$$H_{M1}^T(x) = \begin{pmatrix} h_1(x) & h_2(x) & \dots & h_{M1}(x)\end{pmatrix},$$

$$H_{M2}(y) = \begin{pmatrix} h_1(y) & h_2(y) & \dots & h_{M2}(y)\end{pmatrix}^T,$$

$$C_{M1\times M2} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1(M2)} \\ c_{21} & c_{22} & \cdots & c_{2(M2)} \\ \vdots & \vdots & & \vdots \\ c_{(M1)1} & c_{(M1)2} & \cdots & c_{(M1)(M2)} \end{pmatrix}.$$

Where $'T'$ *means transpose and* $M1 = 2^{J_1+1}$, $M2 = 2^{J_2+1}$.

1.1. **Inverse Problem for the Two-Dimensional Heat Equation.** In this section, we consider the following inverse parabolic problem in the two-dimensionlal form

$$u_t = u_{xx} + u_{yy}, \qquad 0 < x < 1, \quad 0 < y < 1, \quad 0 < t < t_f, \tag{1.5a}$$

$$u(x,y,0) = f(x,y), \qquad 0 \le x \le 1, \quad 0 \le y \le 1, \tag{1.5b}$$

$$u(0,y,t) = g(y,t), \qquad 0 \le y \le 1, \quad 0 \le t \le t_f, \tag{1.5c}$$

$$u(1,y,t) = h(y,t), \qquad 0 \le y \le 1, \quad 0 \le t \le t_f, \tag{1.5d}$$

$$u(x,0,t) = p(x,t), \qquad 0 \le x \le 1, \quad 0 \le t \le t_f, \tag{1.5e}$$

$$u(x,1,t) = q(x,t), \qquad 0 \le x \le 1, \quad 0 \le t \le t_f, \tag{1.5f}$$

and the overspecified condition

$$u(a,y,t) = \phi(y,t), \qquad 0 \le y \le 1 \quad 0 \le t \le t_f, \tag{1.5g}$$

where $0 < a < 1$ is a fixed point, $f(x,y)$ is a continuous known function, $h(y,t)$, $p(x,t)$, $q(x,t)$ and $\phi(y,t)$ are infinitely differentiable known functions and t_f represents the final time of interest for the time evolution of the problem; while function $g(y,t)$ is unknown which should be determined from some interior temperature measurements.

Now, let us divide the interval $[0,t_f]$ into N equal parts of length $\Delta t = \frac{t_f}{N}$ and denote $t_s = (s-1)\Delta t,\ s = 1,2,...,N$. We assume that $\dot{u}''^{\circ\circ}$ can be expanded in terms of Haar basis as,

$$\dot{u}''^{\circ\circ}(x,y,t) = \sum_{j=1}^{M2}\sum_{i=1}^{M1} c_s(ij)h_i(x)h_j(y) = H_{M1}^T(x)C_{M1\times M2}H_{M2}(y) \tag{1.6}$$

where ., $'$ and $\circ$ mean differentiation with respect to t, x and y respectively. the vector $C_{M1\times M2}$ is constant in the subinterval $t \in [t_s, t_{s+1}]$.

Integrating equation (1.6) twice with respect to x from a to x and twice with respect to y from 0 to y we obtain,

$$\dot{u}^{\circ\circ}(x,y,t) = \dot{u}^{\circ\circ}(a,y,t) + (x-a)\dot{u}'^{\circ\circ}(a,y,t) + [(Q_{M1}H_{M1})^T(x)$$
$$-(Q_{M1}H_{M1})^T(a) - (x-a)(P_{M1}H_{M1})^T(a)]C_{M1\times M2}H_{M2}(y), \tag{1.7}$$
$$\dot{u}''(x,y,t) = \dot{u}''(x,0,t) + y\dot{u}''^{\circ}(x,0,t) + H^T_{M1}(x)C_{M1\times M2}(Q_{M2}H_{M2})(y), \tag{1.8}$$

By the boundary conditions, we obtain,

$$\dot{u}^{\circ\circ}(a,y,t) = \dot{\phi}^{\circ\circ}(y,t),\ \dot{u}^{\circ\circ}(1,y,t) = \dot{h}^{\circ\circ}(y,t),$$
$$\dot{u}''(x,0,t) = \dot{p}''(x,t),\ \dot{u}''(x,1,t) = \dot{q}''(x,t).$$

Putting $x = 1$ in equation (1.7) and $y = 1$ in equation (1.8), we obtain,

$$\dot{u}'^{\circ\circ}(a,y,t) = \frac{1}{1-a}\dot{h}^{\circ\circ}(y,t) - \frac{1}{1-a}\dot{\phi}^{\circ\circ}(y,t) + [\frac{1}{1-a}(Q_{M1}H_{M1})^T(a)$$
$$-\frac{1}{1-a}(Q_{M1}H_{M1})^T(1) + (P_{M1}H_{M1})^T(a)]C_{M1\times M2}H_{M2}(y), \tag{1.9}$$
$$\dot{u}''^{\circ}(x,0,t) = \dot{q}''(x,t) - \dot{p}''(x,t) - H^T_{M1}(x)C_{M1\times M2}(Q_{M2}H_{M2})(1). \tag{1.10}$$

Substituting equations (1.9) and (1.10) into equations (1.7) and (1.8), we obtain,

$$\dot{u}^{\circ\circ}(x,y,t) = \frac{1-x}{1-a}\dot{\phi}^{\circ\circ}(y,t) + \frac{x-a}{1-a}\dot{h}^{\circ\circ}(y,t) + [(Q_{M1}H_{M1})^T(x)$$
$$-\frac{x-a}{1-a}(Q_{M1}H_{M1})^T(1) + \frac{x-1}{1-a}(Q_{M1}H_{M1})^T(a)]C_{M1\times M2}H_{M2}(y), \tag{1.11}$$
$$\dot{u}''(x,y,t) = (1-y)\dot{p}''(x,t) + y\dot{q}''^{\circ}(x,t)$$
$$+H^T_{M1}(x)C_{M1\times M2}[(Q_{M2}H_{M2})(y) - y(Q_{M2}H_{M2})(1)]. \tag{1.12}$$

Integrating equations (1.11) and (1.12) with respect to t from t_s to t and twice with respect to y from 0 to y from equation (1.11) we obtain,

$$u''(x,y,t) = (t-t_s)H^T_{M1}(x)C_{M1\times M2}[(Q_{M2}H_{M2})(y) - y(P_{M2}F_2)] + u''(x,y,t_s)$$
$$+y[q''(x,t) - q''(x,t_s)] + (1-y)[p''(x,t) - p''(x,t_s)], \tag{1.13}$$
$$u^{\circ\circ}(x,y,t) = u^{\circ\circ}(x,y,t_s) + \frac{1-x}{1-a}[\phi^{\circ\circ}(y,t) - \phi^{\circ\circ}(y,t_s)]$$
$$+\frac{x-a}{1-a}[h^{\circ\circ}(y,t) - h^{\circ\circ}(y,t_s)] + (t-t_s)[(Q_{M1}H_{M1})^T(x)$$
$$-\frac{x-a}{1-a}(P_{M1}F_1)^T + \frac{x-1}{1-a}(Q_{M1}H_{M1})^T(a)]C_{M1\times M2}H_{M2}(y), \tag{1.14}$$
$$\dot{u}(x,y,t) = [(Q_{M1}H_{M1})^T(x) - \frac{x-a}{1-a}(P_{M1}F_1)^T + \frac{x-1}{1-a}(Q_{M1}H_{M1})^T(a)]C_{M1\times M2}$$
$$[(Q_{M2}H_{M2})(y) - y(P_{M2}F_2)]$$
$$+\frac{x-a}{1-a}[\dot{h}(y,t) - y\dot{h}(1,t) + (y-1)\dot{h}(0,t)]$$
$$+\frac{1-x}{1-a}[\dot{\phi}(y,t) - y\dot{\phi}(1,t) + (y-1)\dot{\phi}(0,t)]$$
$$+(1-y)\dot{p}(x,t) + y\dot{q}(x,t), \tag{1.15}$$
$$u(x,y,t) = (t-t_s)[(Q_{M1}H_{M1})^T(x) - \frac{x-a}{1-a}(P_{M1}F_1)^T + \frac{x-1}{1-a}(Q_{M1}H_{M1})^T(a)]$$
$$C_{M1\times M2}[(Q_{M2}H_{M2})(y) - y(P_{M2}F_2)]$$
$$+\frac{1-x}{1-a}[\phi(y,t) - \phi(y,t_s) - y\{\phi(1,t) - \phi(1,t_s)\} + (y-1)\{\phi(0,t) - \phi(0,t_s)\}]$$

$$+\frac{x-a}{1-a}[h(y,t)-h(y,t_s)-y\{h(1,t)-h(1,t_s)\}+(y-1)\{h(0,t)-h(0,t_s)\}]$$
$$+u(x,y,t_s)+(1-y)[p(x,t)-p(x,t_s)]+y[q(x,t)-q(x,t_s)]. \tag{1.16}$$

Discretizising the results by assuming $x \to x_l, y \to y_k, t \to t_{s+1}$ we obtain,

$$u''(x_l,y_k,t_{s+1}) = (t_{s+1}-t_s)H^T_{M1}(x_l)C_{M1\times M2}[(Q_{M2}H_{M2})(y_k)-y_k(P_{M2}F_2)]$$
$$+u''(x_l,y_k,t_s)+y_k[q''(x_l,t_{s+1})-q''(x_l,t_s)]$$
$$+(1-y_k)[p''(x_l,t_{s+1})-p''(x_l,t_s)], \tag{1.17}$$

$$u^{\circ\circ}(x_l,y_k,t_{s+1}) = u^{\circ\circ}(x_l,y_k,t_s)+\frac{1-x_l}{1-a}[\phi^{\circ\circ}(y_k,t_{s+1})-\phi^{\circ\circ}(y_k,t_s)]$$
$$+\frac{x_l-a}{1-a}[h^{\circ\circ}(y_k,t_{s+1})-h^{\circ\circ}(y_k,t_s)]+(t_{s+1}-t_s)[(Q_{M1}H_{M1})^T(x_l)$$
$$-\frac{x_l-a}{1-a}(P_{M1}F_1)^T+\frac{x_l-1}{1-a}(Q_{M1}H_{M1})^T(a)]C_{M1\times M2}H_{M2}(y_k), \tag{1.18}$$

$$\dot{u}(x_l,y_k,t_{s+1}) = [(Q_{M1}H_{M1})^T(x_l)-\frac{x_l-a}{1-a}(P_{M1}F_1)^T+\frac{x_l-1}{1-a}(Q_{M1}H_{M1})^T(a)]$$
$$C_{M1\times M2}[(Q_{M2}H_{M2})(y_k)-y_k(P_{M2}F_2)]$$
$$+\frac{x_l-a}{1-a}[\dot{h}(y_k,t_{s+1})-y_k\dot{h}(1,t_{s+1})+(y_k-1)\dot{h}(0,t_{s+1})]$$
$$+\frac{1-x_l}{1-a}[\dot{\phi}(y_k,t_{s+1})-y_k\dot{\phi}(1,t_{s+1})+(y_k-1)\dot{\phi}(0,t_{s+1})]$$
$$+(1-y_k)\dot{p}(x_l,t_{s+1})+y_k\dot{q}(x_l,t_{s+1}), \tag{1.19}$$

$$u(x_l,y_k,t_{s+1}) = (t_{s+1}-t_s)[(Q_{M1}H_{M1})^T(x_l)-\frac{x_l-a}{1-a}(P_{M1}F_1)^T$$
$$+\frac{x_l-1}{1-a}(Q_{M1}H_{M1})^T(a)]C_{M1\times M2}[(Q_{M2}H_{M2})(y_k)-y_k(P_{M2}F_2)]$$
$$+\frac{1-x_l}{1-a}[\phi(y_k,t_{s+1})-\phi(y_k,t_s)-y_k\{\phi(1,t_{s+1})-\phi(1,t_s)\}$$
$$+(y_k-1)\{\phi(0,t_{s+1})-\phi(0,t_s)\}]+\frac{x_l-a}{1-a}[h(y_k,t_{s+1})-h(y_k,t_s)$$
$$-y_k\{h(1,t_{s+1})-h(1,t_s)\}+(y_k-1)\{h(0,t_{s+1})-h(0,t_s)\}]$$
$$+u(x_l,y_k,t_s)+(1-y_k)[p(x_l,t_{s+1})-p(x_l,t_s)]+y_k[q(x_l,t_{s+1})-q(x_l,t_s)], \tag{1.20}$$

where vectors F_1 and F_2 are defined as

$$F_1 = [1,\underbrace{0,\ldots,0}_{(M1-1)}]^T, \quad F_2 = [1,\underbrace{0,\ldots,0}_{(M2-1)}]^T,$$

and H, P, Q are obtained from equations (1.2), (1.3), (1.4).
In the following scheme

$$\dot{u}(x_l,y_k,t_{s+1}) = u''(x_l,y_k,t_{s+1})+u^{\circ\circ}(x_l,y_k,t_{s+1}), \tag{1.21}$$

which leads us from the time layer t_s to t_{s+1} is used, where,

$$x_l = \frac{l-0.5}{M1},\ l=1,2,\ldots,(M1=2^{J_1+1}),$$
$$y_k = \frac{k-0.5}{M2},\ k=1,2,\ldots,(M2=2^{J_2+1}),$$

are collocation points.

Substituting equations (1.17), (1.18), (1.19) into equation (1.21), we obtain

$$\begin{aligned}
&[(Q_{M1}H_{M1})^T(x_l) - \frac{x_l-a}{1-a}(P_{M1}F_1)^T + \frac{x_l-1}{1-a}(Q_{M1}H_{M1})^T(a) - TH_{M1}^T(x_l)] \\
&C_{M1\times M2}[(Q_{M2}H_{M2})(y_k) - y_k(P_{M2}F_2)] - T[(Q_{M1}H_{M1})^T(x_l) \\
&-\frac{x_l-a}{1-a}(P_{M1}F_1)^T + \frac{x_l-1}{1-a}(Q_{M1}H_{M1})^T(a)]C_{M1\times M2}H_{M2}(y_k) \\
&= \frac{1-x_l}{1-a}[\phi^{\circ\circ}(y_k,t_{s+1}) - \phi^{\circ\circ}(y_k,t_s) - \dot{\phi}(y_k,t_{s+1}) + y_k\dot{\phi}(1,t_{s+1}) + (1-y_k)\dot{\phi}(0,t_{s+1})] \\
&+\frac{x_l-a}{1-a}[h^{\circ\circ}(y_k,t_{s+1}) - h^{\circ\circ}(y_k,t_s) - \dot{h}(y_k,t_{s+1}) + y_k\dot{h}(1,t_{s+1}) + (1-y_k)\dot{h}(0,t_{s+1})] \\
&+y_k[q''(x_l,t_{s+1}) - q''(x_l,t_s) - \dot{q}(x_l,t_{s+1})] \\
&+(1-y_k)[p''(x_l,t_{s+1}) - p''(x_l,t_s) - \dot{p}(x_l,t_{s+1})] \\
&+u''(x_l,y_k,t_s) + u^{\circ\circ}(x_l,y_k,t_s). \qquad (1.22)
\end{aligned}$$

The wavelet coefficient $C_{M1\times M2}$ can be calculated from the equation (1.22).

Thus the linear system corresponding to the wavelet coefficient $C_{M1\times M2}$ can be expressed as

$$\Lambda\Theta = B. \qquad (1.23)$$

The Matrix Λ is ill-conditioned. On the other hand, as $\phi(y,t)$ is affected by measurement errors, the estimate of Θ by (1.23) will be unstable so that the Tikhonov regularization method must be used to control this measurement errors. The Tikhonov regularized solution [13, 19, 30, 31] to the system of linear algebraic equation (1.23) is given by

$$F_\alpha(\Theta) = \|\Lambda\Theta - B\|_2^2 + \alpha\|R^{(s)}\Theta\|_2^2.$$

On the case of the zeroth-, first-, and second-order Tikhonov regularization method the matrix $R^{(s)}$, for $s = 0, 1, 2$, is given by, see e.g. [20]:

$$R^{(0)} = I_{M_1\times M_1} \in \mathbb{R}^{M_1\times M_1},$$

$$R^{(1)} = \begin{pmatrix} -1 & 1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 1 & 0 \\ 0 & 0 & \dots & 0 & -1 & 1 \end{pmatrix} \in \mathbb{R}^{(M_1-1)\times M_1},$$

$$R^{(2)} = \begin{pmatrix} 1 & -2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -2 & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 & -2 & 1 \end{pmatrix} \in \mathbb{R}^{(M_1-2)\times M_1},$$

where $M_1 = (\gamma+1)\times(\iota+1)$.

Therefore, we obtain the Tikhonov regularized solution of the regularized equation as

$$\Theta_\alpha = \left[\Lambda^T\Lambda + \alpha(R^{(s)})^T R^{(s)}\right]^{-1}\Lambda^T B.$$

In our computation, we use the gcv scheme to determine a suitable value of α [10, 11].

2. Error Analysis

In this section, the convergence of the Haar basis method is investigated for inverse problem with two-dimensional heat equation.

At the first, suppose that $\dot{u}''^{\circ\circ}(x,y,t) = \Upsilon(x,y,t)$. We show that if $\Upsilon(x,y,t)$ is continuous and it satisfies Lipschitz condition with respect to one of its variables and $M1 = M2 = 2^{J+1}$, then the method is convergence.

Now, assume that $\Upsilon(x,y,t)$ satisfies Lipschitz condition on $[0,1]\times[0,1]$, that is,

$$\exists\ L>0,\ \forall\ (x,y_1),\ (x,y_2)\in[0,1]\times[0,1]\ :\ |\Upsilon(x,y_1,t)-\Upsilon(x,y_2,t)|\le L|y_1-y_2|. \quad (2.1)$$

Also, suppose that $\Upsilon^*(x,y)$ is an approximation of $\Upsilon(x,y)$ as follows:

$$\Upsilon^*(x,y,t)=\sum_{i_2=1}^{M2}\sum_{i_1=1}^{M1} c_{i_1i_2}h_{i_1}(x)h_{i_2}(y),$$

and

$$\Upsilon^*(x,y,t)=\sum_{i_2=1}^{M}\sum_{i_1=1}^{M} c_{i_1i_2}h_{i_1}(x)h_{i_2}(y),$$

where the coefficients $c_{i_1i_2}$ are determined by

$$c_{i_1i_2}=2^{2j}\int_0^1\int_0^1 \Upsilon(x,y)h_{i_1}(x)h_{i_2}(y)dxdy,$$

where

$$i_1=i_2=2^j+k+1,\ j\geqslant 0,\ 0\leqslant k<2^j.$$

Therefore, we can compute the error as follows:

$$\begin{aligned}
e^*(x,y) &= \Upsilon(x,y,t)-\Upsilon^*(x,y,t)\\
&= \sum_{i_2=M+1}^{\infty}\sum_{i_1=M+1}^{\infty} c_{i_1i_2}h_{i_1}(x)h_{i_2}(y)\\
&= \sum_{i_1,i_2=M+1}^{\infty} c_{i_1i_2}h_{i_1}(x)h_{i_2}(y)
\end{aligned}$$

Hence, $\|e^*\|_2^2$ is as:

$$\begin{aligned}
\|e^*\|_2^2 &= \int_0^1\int_0^1 (\sum_{i_1,i_2=M+1}^{\infty} c_{i_1i_2}h_{i_1}(x)h_{i_2}(y))^2 dxdy\\
&= \int_0^1\int_0^1 (\sum_{i_1,i_2=M+1}^{\infty} c_{i_1i_2}h_{i_1}(x)h_{i_2}(y)\sum_{l_1,l_2=M+1}^{\infty} c_{l_1l_2}h_{l_1}(x)h_{l_2}(y))dxdy\\
&= \sum_{i_1,i_2=M+1}^{\infty}\sum_{l_1,l_2=M+1}^{\infty} c_{i_1i_2}c_{l_1l_2}(\int_0^1\int_0^1 h_{i_1}(x)h_{l_1}(x)h_{i_2}(y)h_{l_2}(y)dxdy)\\
&= \sum_{i_1,i_2=M+1}^{\infty} c_{i_1i_2}^2\frac{1}{2^{2j}}.
\end{aligned}$$

Since $c_{i_1i_2} = 2^{2j}\int_0^1\int_0^1 \Upsilon(x,y,t)h_{i_1}(x)h_{i_2}(y)dxdy$, according to (1.1), we can write

$$\begin{aligned} c_{i_1i_2} = 2^{2j}(&\int_{\frac{k}{2^j}}^{\frac{k+0.5}{2^j}}\int_{\frac{k}{2^j}}^{\frac{k+0.5}{2^j}} \Upsilon(x,y,t)dydx - \int_{\frac{k}{2^j}}^{\frac{k+0.5}{2^j}}\int_{\frac{k+0.5}{2^j}}^{\frac{k+1}{2^j}} \Upsilon(x,y,t)dydx \\ &- \int_{\frac{k+0.5}{2^j}}^{\frac{k+1}{2^j}}\int_{\frac{k}{2^j}}^{\frac{k+0.5}{2^j}} \Upsilon(x,y,t)dydx + \int_{\frac{k+0.5}{2^j}}^{\frac{k+1}{2^j}}\int_{\frac{k+0.5}{2^j}}^{\frac{k+1}{2^j}} \Upsilon(x,y,t)dydx). \end{aligned}$$

Now, using the mean value theorem, we can conclude

$$\exists x_1 \in [\frac{k}{2^j}, \frac{k+0.5}{2^j}], y_1 \in [\frac{k}{2^j}, \frac{k+0.5}{2^j}], s.t. \int_{\frac{k}{2^j}}^{\frac{k+0.5}{2^j}}\int_{\frac{k}{2^j}}^{\frac{k+0.5}{2^j}} \Upsilon(x,y,t)dydx = \frac{1}{2^{2j+2}} f(x_1,y_1,t),$$

$$\exists x_2 \in [\frac{k}{2^j}, \frac{k+0.5}{2^j}], y_2 \in [\frac{k+0.5}{2^j}, \frac{k+1}{2^j}], s.t. \int_{\frac{k}{2^j}}^{\frac{k+0.5}{2^j}}\int_{\frac{k+0.5}{2^j}}^{\frac{k+1}{2^j}} \Upsilon(x,y,t)dydx = \frac{1}{2^{2j+2}} \Upsilon(x_2,y_2,t),$$

$$\exists x_3 \in [\frac{k+0.5}{2^j}, \frac{k+1}{2^j}], y_3 \in [\frac{k}{2^j}, \frac{k+0.5}{2^j}], s.t. \int_{\frac{k+0.5}{2^j}}^{\frac{k+1}{2^j}}\int_{\frac{k}{2^j}}^{\frac{k+0.5}{2^j}} \Upsilon(x,y,t)dydx = \frac{1}{2^{2j+2}} \Upsilon(x_3,y_3,t),$$

$$\exists x_4 \in [\frac{k+0.5}{2^j}, \frac{k+1}{2^j}], y_4 \in [\frac{k+0.5}{2^j}, \frac{k+1}{2^j}], s.t. \int_{\frac{k+0.5}{2^j}}^{\frac{k+1}{2^j}}\int_{\frac{k+0.5}{2^j}}^{\frac{k+1}{2^j}} \Upsilon(x,y,t)dydx = \frac{1}{2^{2j+2}} \Upsilon(x_4,y_4,t),$$

Thus, we can compute $c_{i_1i_2}$ as follows:

$$\begin{aligned} c_{i_1i_2} &= 2^{2j}[\frac{1}{2^{2j+2}}(\Upsilon(x_1,y_1,t) - \Upsilon(x_2,y_2,t) - \Upsilon(x_3,y_3,t) + \Upsilon(x_4,y_4,t))] \\ &= \frac{1}{4}[\Upsilon(x_1,y_1,t) - \Upsilon(x_1,y_2,t) + \Upsilon(x_1,y_2,t) - \Upsilon(x_2,y_2,t) \\ &- \Upsilon(x_3,y_3,t) + \Upsilon(x_3,y_4,t) - \Upsilon(x_3,y_4,t) + \Upsilon(x_4,y_4,t)] \\ &\leq \frac{L}{4}[(y_1-y_2) + (x_1-x_2) + (y_3-y_4) + (x_3-x_4)] \\ &\leq \frac{L}{4}[\frac{1}{2^j} + \frac{1}{2^j} + \frac{1}{2^j} + \frac{1}{2^j}] \\ &= (\frac{L}{4})(\frac{4}{2^j}) = \frac{L}{2^j}. \end{aligned}$$

The first inequality is obtained with regard to relation (2.1). On the other hand, we have

$$\begin{aligned}
\|e^*\|_2^2 &= \sum_{i_1,i_2=M+1}^{\infty} c_{i_1 i_2}^2 \frac{1}{2^{2j}} \\
&\le \sum_{i_1,i_2=M+1}^{\infty} \frac{L^2}{2^{2j}} \frac{1}{2^{2j}} \\
&= \sum_{i_1,i_2=M+1}^{\infty} \frac{L^2}{2^{4j}} \\
&= L^2 \sum_{i_1,i_2=M+1}^{\infty} \frac{1}{2^{4j}} \\
&= L^2 \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} \frac{1}{2^{4j}} \\
&= L^2 \sum_{j=J+1}^{\infty} \frac{2^j - 1 + 1}{2^{4j}} \\
&= L^2 \sum_{j=J+1}^{\infty} \frac{1}{2^{3j}} \\
&= \frac{8}{7} L^2 (\frac{1}{2^{J+1}})^3
\end{aligned}$$

Since $M = 2^{J+1}$, we have

$$\|e^*\|_2^2 \le \frac{8}{7} L^2 (\frac{1}{M})^3,$$

and

$$\|e^*\|_2 \le \sqrt{\frac{8}{7}} L (\frac{1}{M})^{\frac{3}{2}}.$$

Therefore, the Haar basis method will be convergent, i.e.

$$\lim_{J\to\infty} e^* = 0$$

Moreover, the convergence is of order exponential, that is,

$$\|e^*\|_2 = O(\frac{1}{2^{J+1}})^{\frac{3}{2}} = O(\frac{1}{M})^{\frac{3}{2}}.$$

2.1. **Inverse Problem for the Two-Dimensional Wave Equation.** In this section, we consider the following inverse parabolic problem in the Two-dimensionlal form

$$u_{tt} = u_{xx} + u_{yy}, \quad 0 < x < 1, \quad 0 < y < 1, \quad 0 < t < t_f, \tag{2.2a}$$

$$u(x,y,0) = f_1(x,y), \quad 0 \le x \le 1, \quad 0 \le y \le 1, \tag{2.2b}$$

$$u_t(x,y,0) = f_2(x,y), \quad 0 \le x \le 1, \quad 0 \le y \le 1, \tag{2.2c}$$

$$u(0,y,t) = g(y,t), \quad 0 \le y \le 1, \quad 0 \le t \le t_f, \tag{2.2d}$$

$$u(1,y,t) = h(y,t), \quad 0 \le y \le 1, \quad 0 \le t \le t_f, \tag{2.2e}$$

$$u(x,0,t) = p(x,t), \quad 0 \le x \le 1, \quad 0 \le t \le t_f, \tag{2.2f}$$

$$u(x,1,t) = q(x,t), \quad 0 \le x \le 1, \quad 0 \le t \le t_f, \tag{2.2g}$$

and the overspecified condition

$$u(a,y,t)=\phi(y,t), \qquad 0\le y\le 1 \quad 0\le t\le t_f, \tag{2.2h}$$

where $0<a<1$ is a fixed point, $f_1(x,y)$ and $f_2(x,y)$ are continuous known functions, $h(y,t)$, $p(x,t)$, $q(x,t)$ and $\phi(y,t)$ are infinitely differentiable known functions and t_f represents the final time of interest for the time evolution of the problem; while the function $g(y,t)$ is unknown which should be determined from some interior temperature measurements.

Now, let us divide the interval $[0,t_f]$ into N equal parts of length $\Delta t=\frac{t_f}{N}$ and denote $t_s=(s-1)\Delta t,\ s=1,2,...,N$. We assume that $\ddot{u}''^{\circ\circ}$ can be expanded in terms of Haar basis as

$$\ddot{u}''^{\circ\circ}(x,y,t)=\sum_{j=1}^{M2}\sum_{i=1}^{M1}c_s(ij)h_i(x)h_j(y)=H^T_{M1}(x)C_{M1\times M2}H_{M2}(y), \tag{2.3}$$

where ., $'$ and $\circ$ means differentiation with respect to t, x and y respectively. the vector $C_{M1\times M2}$ is constant in the subinterval $t\in[t_s,t_{s+1}]$.

Integrating equation (2.3) twice with respect to x from a to x and twice with respect to y from 0 to y, we obtain the following equations

$$\ddot{u}^{\circ\circ}(x,y,t)=\ddot{u}^{\circ\circ}(a,y,t)+(x-a)\ddot{u}'^{\circ\circ}(a,y,t)+[(Q_{M1}H_{M1})^T(x)$$
$$-(x-a)(P_{M1}H_{M1})^T(a)-(Q_{M1}H_{M1})^T(a)]C_{M1\times M2}H_{M2}(y), \tag{2.4}$$
$$\ddot{u}''(x,y,t)=\ddot{u}''(x,0,t)+y\ddot{u}''^{\circ}(x,0,t)+H^T_{M1}(x)C_{M1\times M2}(Q_{M2}H_{M2})(y). \tag{2.5}$$

By the boundary conditions, the following equations are resulted

$$\ddot{u}^{\circ\circ}(a,y,t)=\ddot{\phi}^{\circ\circ}(y,t),\ \ddot{u}^{\circ\circ}(1,y,t)=\ddot{h}^{\circ\circ}(y,t),$$
$$\ddot{u}''(x,0,t)=\ddot{p}''(x,t),\ \ddot{u}''(x,1,t)=\ddot{q}''(x,t).$$

Putting $x=1$ in equation (2.4) and $y=1$ in equation (2.5), we obtain,

$$\ddot{u}'^{\circ\circ}(a,y,t)=\frac{1}{1-a}\ddot{h}^{\circ\circ}(y,t)-\frac{1}{1-a}\ddot{\phi}^{\circ\circ}(y,t)+[\frac{1}{1-a}(Q_{M1}H_{M1})^T(a)$$
$$+(P_{M1}H_{M1})^T(a)-\frac{1}{1-a}(Q_{M1}H_{M1})^T(1)]C_{M1\times M2}H_{M2}(y), \tag{2.6}$$
$$\ddot{u}''^{\circ}(x,0,t)=\ddot{q}''(x,t)-\ddot{p}''(x,t)+H^T_{M1}(x)C_{M1\times M2}(Q_{M2}H_{M2})(1). \tag{2.7}$$

Substituting equations (2.6) and (2.7) into equations (2.4) and (2.5), we obtain,

$$\ddot{u}^{\circ\circ}(x,y,t)=\frac{1-x}{1-a}\ddot{\phi}^{\circ\circ}(y,t)+\frac{x-a}{1-a}\ddot{h}^{\circ\circ}(y,t)+[(Q_{M1}H_{M1})^T(x)$$
$$-\frac{x-a}{1-a}(Q_{M1}H_{M1})^T(1)+\frac{x-1}{1-a}(Q_{M1}H_{M1})^T(a)]C_{M1\times M2}H_{M2}(y), \tag{2.8}$$
$$\ddot{u}''(x,y,t)=(1-y)\ddot{p}''(x,t)+y\ddot{q}''(x,t)$$
$$+H^T_{M1}(x)C_{M1\times M2}[(Q_{M2}H_{M2})(y)-y(Q_{M2}H_{M2}(1))]. \tag{2.9}$$

Integrating equations (2.8) and (2.9) twice with respect to t from t_s to t and twice with respect to y from 0 to y from equation (2.8) we obtain,

$$u^{\circ\circ}(x,y,t)=\frac{1-x}{1-a}[\phi^{\circ\circ}(y,t)-\phi^{\circ\circ}(y,t_s)-(t-t_s)\dot{\phi}^{\circ\circ}(y,t_s)]$$
$$+\frac{x-a}{1-a}[h^{\circ\circ}(y,t)-h^{\circ\circ}(y,t_s)-(t-t_s)\dot{h}^{\circ\circ}(y,t_s)]$$
$$+\frac{1}{2}(t^2+t_s^2-2tt_s)[(Q_{M1}H_{M1})^T(x)-\frac{x-a}{1-a}(P_{M1}F_1)^T$$

$$+\frac{x-1}{1-a}(Q_{M1}H_{M1})^T(a)]C_{M1\times M2}H_{M2}(y)$$
$$+u^{\circ\circ}(x,y,t_s)+(t-t_s)\dot{u}^{\circ\circ}(x,y,t_s), \tag{2.10}$$

$$u''(x,y,t)=\frac{1}{2}(t^2+t_s^2-2tt_s)H_{M1}^T(x)C_{M1\times M2}[(Q_{M2}H_{M2})(y)-y(P_{M2}F_2)]$$
$$+y[q''(x,t)-q''(x,t_s)-(t-t_s)\dot{q}''(x,t_s)]+(1-y)[p''(x,t)-p''(x,t_s)$$
$$-(t-t_s)\dot{p}''(x,t_s)]+u''(x,y,t_s)+(t-t_s)\dot{u}''(x,y,t_s), \tag{2.11}$$

$$\ddot{u}(x,y,t)=\frac{x-a}{1-a}[\ddot{h}(y,t)-y\ddot{h}(1,t)+(y-1)\ddot{h}(0,t)]$$
$$+\frac{1-x}{1-a}[\ddot{\phi}(y,t)-y\ddot{\phi}(1,t)+(y-1)\ddot{\phi}(0,t)]$$
$$+[(Q_{M1}H_{M1})^T(x)-\frac{x-a}{1-a}(P_{M1}F_1)^T+\frac{x-1}{1-a}(Q_{M1}H_{M1})^T(a)]$$
$$C_{M1\times M2}[(Q_{M2}H_{M2})(y)-y(P_{M2}F_2)]+(1-y)\ddot{p}(x,t)+y\ddot{q}(x,t), \tag{2.12}$$

$$u(x,y,t)=(1-y)[p(x,t)-p(x,t_s)-(t-t_s)\dot{p}(x,t_s)]$$
$$+y[q(x,t)-q(x,t_s)-(t-t_s)\dot{q}(x,t_s)]$$
$$+\frac{1-x}{1-a}[\phi(y,t)-\phi(y,t_s)-(t-t_s)\dot{\phi}(y,t_s)-y\{\phi(1,t)-\phi(1,t_s)-(t-t_s)\dot{\phi}(1,t_s)\}$$
$$+(y-1)\{\phi(0,t)-\phi(0,t_s)-(t-t_s)\dot{\phi}(0,t_s)\}]$$
$$+\frac{x-a}{1-a}[h(y,t)-h(y,t_s)-(t-t_s)\dot{h}(y,t_s)-y\{h(1,t)-h(1,t_s)-(t-t_s)\dot{h}(1,t_s)\}$$
$$+(y-1)\{h(0,t)-h(0,t_s)-(t-t_s)\dot{h}(0,t_s)\}]+\frac{1}{2}(t^2+t_s^2-2tt_s)$$
$$[(Q_{M1}H_{M1})^T(x)-\frac{x-a}{1-a}(P_{M1}F_1)^T+\frac{x-1}{1-a}(Q_{M1}H_{M1})^T(a)]C_{M1\times M2}$$
$$[(Q_{M2}H_{M2})(y)-y(P_{M2}F_2)]+u(x,y,t_s)+(t-t_s)\dot{u}(x,y,t_s), \tag{2.13}$$

where,

$$\dot{u}^{\circ\circ}(x,y,t_s)=(t-t_s)[(Q_{M1}H_{M1})^T(x)-\frac{x-a}{1-a}(P_{M1}F_1)^T$$
$$+\frac{x-1}{1-a}(Q_{M1}H_{M1})^T(a)]C_{M1\times M2}H_{M2}(y)+\frac{1-x}{1-a}[\dot{\phi}^{\circ\circ}(y,t)-\dot{\phi}^{\circ\circ}(y,t_s)]$$
$$+\frac{x-a}{1-a}[\dot{h}^{\circ\circ}(y,t)-\dot{h}^{\circ\circ}(y,t_s)]+\dot{u}^{\circ\circ}(x,y,t_s), \tag{2.14}$$

$$\dot{u}''(x,y,t)=(t-t_s)H_{M1}^T(x)C_{M1\times M2}[(Q_{M2}H_{M2})(y)-y(P_{M2}F_2)]$$
$$+y[\dot{q}''(x,t)-\dot{q}''(x,t_s)]+(1-y)[\dot{p}''(x,t)-\dot{p}(x,t_s)]+\dot{u}''(x,y,t_s), \tag{2.15}$$

$$\dot{u}(x,y,t)=(t-t_s)[(Q_{M1}H_{M1})^T(x)-\frac{x-a}{1-a}(P_{M1}F_1)^T+\frac{x-1}{1-a}(Q_{M1}H_{M1})^T(a)]$$
$$C_{M1\times M2}[(Q_{M2}H_{M2})(y)-y(P_{M2}F_2)]$$
$$+\frac{1-x}{1-a}[\dot{\phi}(y,t)-\dot{\phi}(y,t_s)-y\{\dot{\phi}(1,t)-\dot{\phi}(1,t_s)\}+(y-1)\{\dot{\phi}(0,t)-\dot{\phi}(0,t_s)\}]$$
$$+\frac{x-a}{1-a}[\dot{h}(y,t)-\dot{h}(y,t_s)-y\{\dot{h}(1,t)-\dot{h}(1,t_s)\}+(y-1)\{\dot{h}(0,t)-\dot{h}(0,t_s)\}]$$
$$+y[\dot{q}(x,t)-\dot{q}(x,t_s)]+(1-y)[\dot{p}(x,t)-\dot{p}(x,t_s)]+\dot{u}(x,y,t_s). \tag{2.16}$$

Discretizising the results by assuming $x\to x_l, y\to y_k, t\to t_{s+1}$ we obtain,

$$u^{\circ\circ}(x_l,y_k,t_{s+1})=\frac{1-x_l}{1-a}[\phi^{\circ\circ}(y_k,t_{s+1})-\phi^{\circ\circ}(y_k,t_s)-(t_{s+1}-t_s)\dot{\phi}^{\circ\circ}(y_k,t_s)]$$

$$
\begin{aligned}
&+\frac{x_l-a}{1-a}[h^{\circ\circ}(y_k,t_{s+1})-h^{\circ\circ}(y_k,t_s)-(t_{s+1}-t_s)\dot{h}^{\circ\circ}(y_k,t_s)]\\
&+\frac{1}{2}(t_{s+1}^2+t_s^2-2t_{s+1}t_s)[(Q_{M1}H_{M1})^T(x_l)-\frac{x_l-a}{1-a}(P_{M1}F_1)^T\\
&+\frac{x_l-1}{1-a}(Q_{M1}H_{M1})^T(a)]C_{M1\times M2}H_{M2}(y_k)\\
&+u^{\circ\circ}(x_l,y_k,t_s)+(t_{s+1}-t_s)\dot{u}^{\circ\circ}(x_l,y_k,t_s), \qquad (2.17)
\end{aligned}
$$

$$
\begin{aligned}
\dot{u}^{\circ\circ}(x_l,y_k,t_s)&=(t_{s+1}-t_s)[(Q_{M1}H_{M1})^T(x_l)-\frac{x_l-a}{1-a}(P_{M1}F_1)^T\\
&+\frac{x_l-1}{1-a}(Q_{M1}H_{M1})^T(a)]C_{M1\times M2}H_{M2}(y_k)+\frac{1-x_l}{1-a}[\dot{\phi}^{\circ\circ}(y_k,t_{s+1})\\
&-\dot{\phi}^{\circ\circ}(y_k,t_s)]+\frac{x_l-a}{1-a}[\dot{h}^{\circ\circ}(y_k,t_{s+1})-\dot{h}^{\circ\circ}(y_k,t_s)]+\dot{u}^{\circ\circ}(x_l,y_k,t_s), \qquad (2.18)
\end{aligned}
$$

$$
\begin{aligned}
u''(x_l,y_k,t_{s+1})&=\frac{1}{2}(t_{s+1}^2+t_s^2-2t_{s+1}t_s)H_{M1}^T(x_l)C_{M1\times M2}[(Q_{M2}H_{M2})(y_k)\\
&-y_k(P_{M2}F_2)]+y_k[q''(x_l,t_{s+1})-q''(x_l,t_s)-(t_{s+1}-t_s)\dot{q}''(x_l,t_s)]\\
&+(1-y_k)[p''(x_l,t_{s+1})-p''(x_l,t_s)-(t_{s+1}-t_s)\dot{p}''(x_l,t_s)]\\
&+u''(x_l,y_k,t_s)+(t_{s+1}-t_s)\dot{u}''(x_l,y_k,t_s), \qquad (2.19)
\end{aligned}
$$

$$
\begin{aligned}
\dot{u}''(x_l,y_k,t_{s+1})&=(t_{s+1}-t_s)H_{M1}^T(x_l)C_{M1\times M2}[(Q_{M2}H_{M2})(y_k)\\
&-y_k(P_{M2}F_2)]+y_k[\dot{q}''(x_l,t_{s+1})-\dot{q}''(x_l,t_s)]\\
&+(1-y_k)[\dot{p}''(x_l,t_{s+1})-\dot{p}(x_l,t_s)]+\dot{u}''(x_l,y_k,t_s), \qquad (2.20)
\end{aligned}
$$

$$
\begin{aligned}
\ddot{u}(x_l,y_k,t_{s+1})&=\frac{x_l-a}{1-a}[\ddot{h}(y_k,t_{s+1})-y_k\ddot{h}(1,t_{s+1})+(y_k-1)\ddot{h}(0,t_{s+1})]\\
&+\frac{1-x_l}{1-a}[\ddot{\phi}(y_k,t_{s+1})-y_k\ddot{\phi}(1,t_{s+1})+(y_k-1)\ddot{\phi}(0,t_{s+1})]\\
&+[(Q_{M1}H_{M1})^T(x_l)-\frac{x_l-a}{1-a}(P_{M1}F_1)^T+\frac{x_l-1}{1-a}(Q_{M1}H_{M1})^T(a)]\\
&C_{M1\times M2}[(Q_{M2}H_{M2})(y_k)-y_k(P_{M2}F_2)]\\
&+(1-y_k)\ddot{p}(x_l,t_{s+1})+y_k\ddot{q}(x_l,t_{s+1}), \qquad (2.21)
\end{aligned}
$$

$$
\begin{aligned}
u(x_l,y_k,t_{s+1})&=(1-y_k)[p(x_l,t_{s+1})-p(x_l,t_s)-(t_{s+1}-t_s)\dot{p}(x_l,t_s)]\\
&+y_k[q(x_l,t_{s+1})-q(x_l,t_s)-(t_{s+1}-t_s)\dot{q}(x_l,t_s)]+\frac{1-x_l}{1-a}[\phi(y_k,t_{s+1})-\phi(y_k,t_s)\\
&-(t_{s+1}-t_s)\dot{\phi}(y_k,t_s)-y_k\{\phi(1,t_{s+1})-\phi(1,t_s)-(t_{s+1}-t_s)\dot{\phi}(1,t_s)\}\\
&+(y_k-1)\{\phi(0,t_{s+1})-\phi(0,t_s)-(t_{s+1}-t_s)\dot{\phi}(0,t_s)\}]\\
&+\frac{x_l-a}{1-a}[h(y_k,t_{s+1})-h(y_k,t_s)-(t_{s+1}-t_s)\dot{h}(y_k,t_s)\\
&-y_k\{h(1,t_{s+1})-h(1,t_s)-(t_{s+1}-t_s)\dot{h}(1,t_s)\}\\
&+(y_k-1)\{h(0,t_{s+1})-h(0,t_s)-(t_{s+1}-t_s)\dot{h}(0,t_s)\}]+\frac{1}{2}(t_{s+1}^2\\
&+t_s^2-2t_{s+1}t_s)[(Q_{M1}H_{M1})^T(x_l)-\frac{x_l-a}{1-a}(P_{M1}F_1)^T+\frac{x_l-1}{1-a}(Q_{M1}H_{M1})^T(a)]\\
&C_{M1\times M2}[(Q_{M2}H_{M2})(y_k)-y_k(P_{M2}F_2)]\\
&+u(x_l,y_k,t_s)+(t_{s+1}-t_s)\dot{u}(x_l,y_k,t_s), \qquad (2.22)
\end{aligned}
$$

$$
\begin{aligned}
\dot{u}(x_l,y_k,t_{s+1})&=(t_{s+1}-t_s)[(Q_{M1}H_{M1})^T(x_l)-\frac{x_l-a}{1-a}(P_{M1}F_1)^T\\
&+\frac{x_l-1}{1-a}(Q_{M1}H_{M1})^T(a)]C_{M1\times M2}[(Q_{M2}H_{M2})(y_k)-y_k(P_{M2}F_2)]
\end{aligned}
$$

$$+\frac{1-x_l}{1-a}[\dot{\phi}(y_k,t_{s+1})-\dot{\phi}(y_k,t_s)-y_k\{\dot{\phi}(1,t_{s+1})-\dot{\phi}(1,t_s)\}$$
$$+(y_k-1)\{\dot{\phi}(0,t)-\dot{\phi}(0,t_s)\}]+\frac{x_l-a}{1-a}[\dot{h}(y_k,t_{s+1})$$
$$-\dot{h}(y_k,t_s)-y_k\{\dot{h}(1,t_{s+1})-\dot{h}(1,t_s)\}+(y_k-1)\{\dot{h}(0,t_{s+1})-\dot{h}(0,t_s)\}]$$
$$+y_k[\dot{q}(x_l,t_{s+1})-\dot{q}(x_l,t_s)]+(1-y_k)[\dot{p}(x_l,t_{s+1})-\dot{p}(x_l,t_s)]+\dot{u}(x_l,y_k,t_s), \tag{2.23}$$

where vectors F_1 and F_2 are defined as

$$F_1=[1,\underbrace{0,\ldots,0}_{(M1-1)}]^T,\quad F_2=[1,\underbrace{0,\ldots,0}_{(M2-1)}]^T,$$

and H, P, Q are obtained from equations (1.2), (1.3), and (1.4).
In the following scheme

$$\ddot{u}(x_l,y_k,t_{s+1})=u''(x_l,y_k,t_{s+1})+u^{\circ\circ}(x_l,y_k,t_{s+1}), \tag{2.24}$$

which leads us from the time layer t_s to t_{s+1} is used, where,

$$x_l=\frac{l-0.5}{M1},\ l=1,2,\ldots,(M1=2^{J_1+1}),$$
$$y_k=\frac{k-0.5}{M2},\ k=1,2,\ldots,(M2=2^{J_2+1}),$$

are collocation points.

Substituting equations (2.17), (2.19), (2.21) into equation (2.24), we obtain

$$[(Q_{M1}H_{M1})^T(x_l)-\frac{x_l-a}{1-a}(P_{M1}F_1)^T+\frac{x_l-1}{1-a}(Q_{M1}H_{M1})^T(a)$$
$$-\frac{1}{2}(t_{s+1}^2+t_s^2-2t_{s+1}t_s)H_{M1}^T(x_l)]C_{M1\times M2}[(Q_{M2}H_{M2})(y_k)-y_k(P_{M2}F_2)]$$
$$-\frac{1}{2}(t_{s+1}^2+t_s^2-2t_{s+1}t_s)[(Q_{M1}H_{M1})^T(x_l)-\frac{x_l-a}{1-a}(P_{M1}F_1)^T$$
$$+\frac{x_l-1}{1-a}(Q_{M1}H_{M1})^T(a)]C_{M1\times M2}H_{M2}(y_k)$$
$$=\frac{1-x_l}{1-a}[\phi^{\circ\circ}(y_k,t_{s+1})-\phi^{\circ\circ}(y_k,t_s)-T\dot{\phi}^{\circ\circ}(y_k,t_s)-\ddot{\phi}(y_k,t_{s+1})$$
$$+y_k\ddot{\phi}(1,t_{s+1})+(1-y_k)\ddot{\phi}(0,t_{s+1})]+\frac{x_l-a}{1-a}[h^{\circ\circ}(y_k,t_{s+1})-h^{\circ\circ}(y_k,t_s)$$
$$-T\dot{h}^{\circ\circ}(y_k,t_s)-\ddot{h}(y_k,t_{s+1})+y_k\ddot{h}(1,t_{s+1})+(1-y_k)\ddot{h}(0,t_{s+1})]$$
$$+y_k[q''(x_l,t_{s+1})-q''(x_l,t_s)-T\dot{q}''(x_l,t_s)-\ddot{q}(x_l,t_{s+1})]$$
$$+(1-y_k)[p''(x_l,t_{s+1})-p''(x_l,t_s)-T\dot{p}''(x_l,t_s)-\ddot{p}(x_l,t_{s+1})]$$
$$+u''(x_l,y_k,t_s)+u^{\circ\circ}(x_l,y_k,t_s)+T[\dot{u}''(x_l,y_k,t_s)+\dot{u}^{\circ\circ}(x_l,y_k,t_s). \tag{2.25}$$

The wavelet coefficient $C_{M1\times M2}$ can be calculated from the equation (2.25).

In matrix form, the wavelet coefficient $C_{M1\times M2}$ can be obtained resolution of the following matrix equation

$$A\lambda=b. \tag{2.26}$$

Similarly, Tikhonov's regularized solution [30, 13, 19] to the system of linear algebraic equation (2.26) is given by

$$\lambda_\alpha=\left[A^TA+\alpha(R^{(s)})^TR^{(s)}\right]^{-1}A^Tb.$$

3. Numerical Results and Discussion

In this section, we are going to demonstrate numerically, some of results for the unknown boundary condition in the two inverse problems (1.5) and (2.2). As we know, the inverse problems are ill-posed and therefore it is necessary to investigate the stability of the present method by giving a test problem.

Remark 3.1. *In an inverse problem there are two sources of error in the estimation; the first source is the unavoidable bias deviation, and the second source of error is the variance due to the amplification of measurement errors,* [6].

Therefore, we compare exact and approximate solutions by considering total error S *defined by*

$$S = \Big[\frac{1}{N-1}\sum_{i=1}^{N}(\widehat{\Phi_i} - \Phi_i)^2\Big]^{\frac{1}{2}}, \tag{3.1}$$

where N *,*Φ *and* $\widehat{\Phi}$ *are the number of estimated values , the estimated values and the exact values, respectively.*

Example 1. *In this example we solve the problem* (1.5) *with given data,*

$$\begin{aligned}
u(x,y,0) &= \sin(x) + \cos(y), && 0 \le x \le 1, 0 \le y \le 1,\\
u(1,y,t) &= e^{-t}(\sin 1 + \cos(y)), && 0 \le y \le 1, 0 \le t \le t_f,\\
u(x,0,t) &= e^{-t}(\sin(x) + 1), && 0 \le x \le 1, 0 \le t \le t_f,\\
u(x,1,t) &= e^{-t}(\sin(x) + \cos 1), && 0 \le x \le 1, 0 \le t \le t_f,\\
u(0.1,y,t) &= e^{-t}(\sin 0.1 + \cos(y)), && 0 \le y \le 1, 0 \le t \le t_f.
\end{aligned}$$

The exact solution of this problem is $u(x,y,t) = e^{-t}(\sin(x) + \cos(y))$ *and* $g(y,t) = e^{-t}cos(y)$*. The results obtained for* $u(0,y,t)$ *with* $t_f = 1$*,* $\Delta t = 0.1, 0.01$*,* $y = 0.625$ *and* $M1 = M2 = 4$ *with noisy data (noisy data=input data+(0.01)rand(1)) are presented in Tables 1, 2 and Figures 1-6.*

t	*Exact*	*0th order Tikhonov*	*1st order Tikhonov*	*2nd order Tikhonov*
0.1	0.733790	0.733754	0.733675	0.733675
0.2	0.663960	0.663824	0.663665	0.663835
0.3	0.600776	0.600489	0.600501	0.600456
0.4	0.543605	0.543430	0.543430	0.543314
0.5	0.491874	0.491800	0.491649	0.491684
0.6	0.445066	0.444882	0.444944	0.444996
0.7	0.402712	0.402423	0.402514	0.402475
0.8	0.364389	0.364217	0.364052	0.364142
0.9	0.329713	0.329665	0.329381	0.329643
1	0.298337	0.298281	0.298181	0.298283
S		$1.369e-004$	$1.933e-004$	$1.599e-004$

Table 1. The comparison between exact solution and Tikhonov's solutions for $g(0.625, t)$ with the noisy data when $\Delta t = 0.1$.

t	*Exact*	*0th order Tikhonov*	*1st order Tikhonov*	*2nd order Tikhonov*
0.01	0.802894	0.802883	0.802794	0.802749
0.02	0.794905	0.794792	0.794795	0.794750
0.1	0.733790	0.733543	0.733569	0.733683
0.11	0.726488	0.726247	0.726349	0.726286
0.5	0.491874	0.491679	0.491646	0.491723
0.51	0.486980	0.486713	0.486739	0.486785
0.8	0.364389	0.364160	0.364127	0.364190
0.81	0.360763	0.360536	0.360607	0.360566
0.9	0.329713	0.329528	0.329526	0.329567
0.91	0.326432	0.326244	0.326178	0.326194
1	0.298337	0.298255	0.298105	0.298215
S		$1.600e-004$	$1.615e-004$	$1.471e-004$

Table 2. The comparison between exact solution and Tikhonov's solutions for $g(0.625, t)$ with the noisy data when $\Delta t = 0.01$.

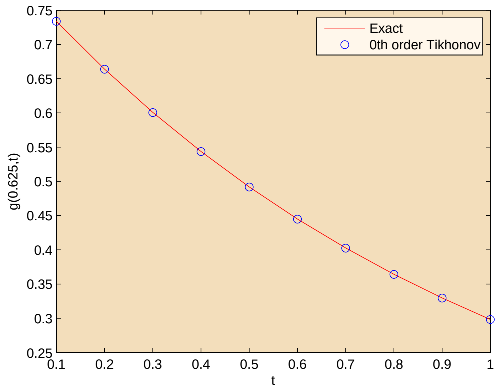

Figure 1. The comparison between the exact solution and 0th order Tikhonov solution for $g(0.625, t)$ with noisy data when $\Delta t = 0.1$.

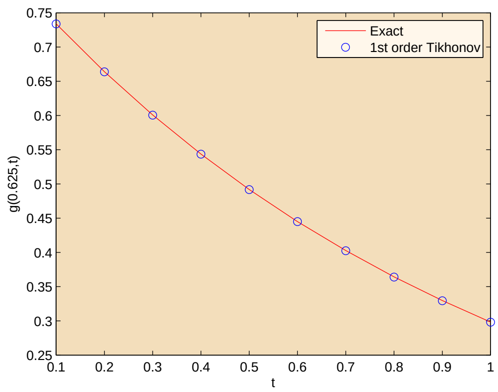

Figure 2. The comparison between the exact solution and 1st order Tikhonov solution for $g(0.625, t)$ with noisy data when $\Delta t = 0.1$.

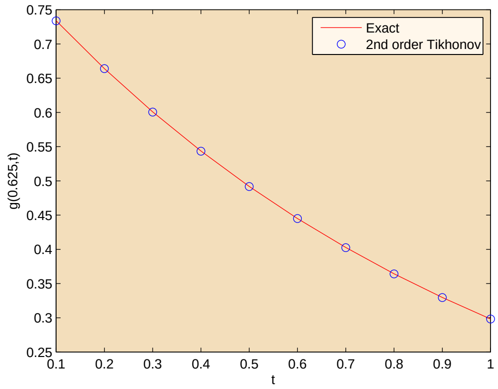

Figure 3. The comparison between the exact solution and 2nd order Tikhonov solution for $g(0.625, t)$ with noisy data when $\Delta t = 0.1$.

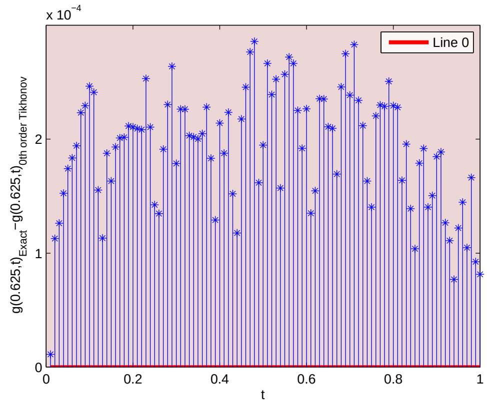

Figure 4. Difference between the the exact solution and 0th order Tikhonov solution for $g(0.625, t)$ of problem (1.5) with noisy data when $\Delta t = 0.01$.

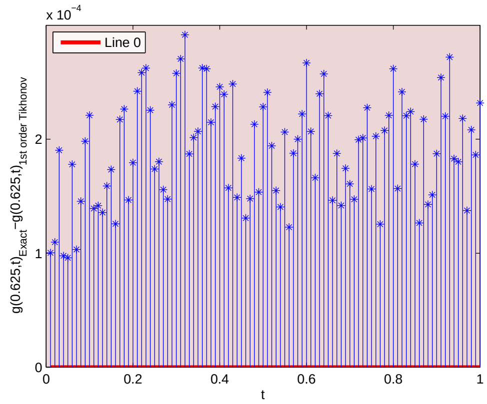

Figure 5. Difference between the the exact solution and 1st order Tikhonov solution for $g(0.625, t)$ of problem (1.5) with noisy data when $\Delta t = 0.01$.

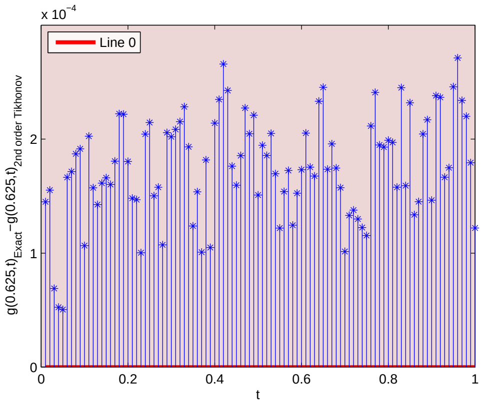

Figure 6. Difference between the the exact solution and 2nd order Tikhonov solution for $g(0.625, t)$ of problem (1.5) with noisy data when $\Delta t = 0.01$.

Example 2. *In this example we solve the problem* (2.2) *with given data,*

$$\begin{aligned}
u(x, y, 0) &= \sinh(x) + \cosh(y), && 0 \le x \le 1, 0 \le y \le 1,\\
u_t(x, y, 0) &= \sinh(x) + \cosh(y), && 0 \le x \le 1, 0 \le y \le 1,\\
u(1, y, t) &= e^t(\sinh 1 + \cosh(y)), && 0 \le y \le 1, 0 \le t \le t_f,\\
u(x, 0, t) &= e^t(\sinh(x) + 1), && 0 \le x \le 1, 0 \le t \le t_f,\\
u(x, 1, t) &= e^t(\sinh(x) + \cosh 1), && 0 \le x \le 1, 0 \le t \le t_f,\\
u(0.1, y, t) &= e^t(\sinh 0.1 + \cosh(y)), && 0 \le y \le 1, 0 \le t \le t_f.
\end{aligned}$$

The exact solution of this problem is $u(x, y, t) = e^t(\sinh(x) + \cosh(y))$ *and* $g(y, t) = e^t \cosh(y)$. *The results obtained for* $u(0, y, t)$ *with* $t_f = 1$, $\Delta t = 0.1, 0.01$, $y = 0.625$ *and* $M1 = M2 = 4$ *with noisy data (noisy data=input data+(0.01)rand(1)) are presented in Tables 3, 4 and Figures 7-12.*

t	*Exact*	*0th order Tikhonov*	*1st order Tikhonov*	*2nd order Tikhonov*
0.1	1.328143	1.328061	1.328121	1.328117
0.2	1.467825	1.467701	1.467750	1.467738
0.3	1.622198	1.621092	1.622116	1.622079
0.4	1.792806	1.792647	1.792716	1.792670
0.5	1.981357	1.981102	1.981207	1.981103
0.6	2.189738	2.189501	2.189565	2.189370
0.7	2.420035	2.419809	2.419820	2.419651
0.8	2.674552	2.674289	2.674297	2.674155
0.9	2.955837	2.955653	2.955595	2.955440
1	3.266705	3.266575	3.266431	3.266361
S		$1.617e-004$	$1.500e-004$	$2.350e-004$

Table 3. The comparison between exact solution and Tikhonov's solutions for $g(0.625, t)$ with noisy data when $\Delta t = 0.1$.

t	*Exact*	*0th order Tikhonov*	*1st order Tikhonov*	*2nd order Tikhonov*
0.01	1.213832	1.213829	1.213831	1.213830
0.02	1.226031	1.226024	1.226028	1.226026
0.1	1.328143	1.328038	1.328051	1.328086
0.11	1.341491	1.341375	1.341386	1.341426
0.5	1.981357	1.981082	1.981011	1.981148
0.51	2.001270	2.000993	2.000920	2.001058
0.8	2.674552	2.674240	2.674216	2.674274
0.81	2.701432	2.701130	2.701090	2.701157
0.9	2.955837	2.955637	2.955527	2.955637
0.91	2.985544	2.985344	2.985246	2.985352
1	3.266705	3.266459	3.266480	3.266522
S		$1.763e-004$	$1.998e-004$	$1.551e-004$

Table 4. The comparison between exact solution and Tikhonov's solutions for $g(0.625, t)$ with noisy data when $\Delta t = 0.01$.

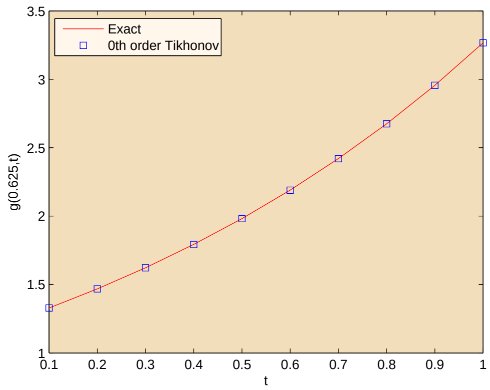

Figure 7. The comparison between the exact solution and 0th order Tikhonov solution for $g(0.625, t)$ with noisy data when $\Delta t = 0.1$.

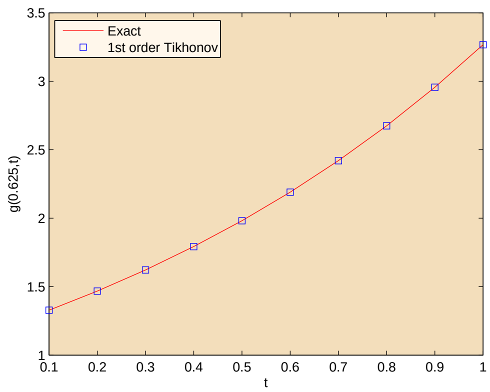

Figure 8. The comparison between the exact solution and 1st order Tikhonov solution for $g(0.625, t)$ with noisy data when $\Delta t = 0.1$.

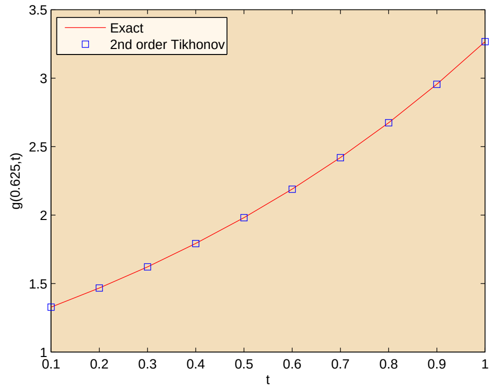

Figure 9. The comparison between the exact solution and 2nd order Tikhonov solution for $g(0.625,t)$ with noisy data when $\Delta t = 0.1$.

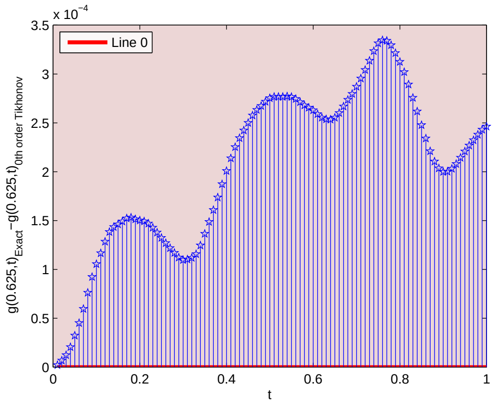

Figure 10. Difference between the the exact solution and 0th order Tikhonov solution for $g(0.625,t)$ of problem (2.2) with noisy data when $\Delta t = 0.01$.

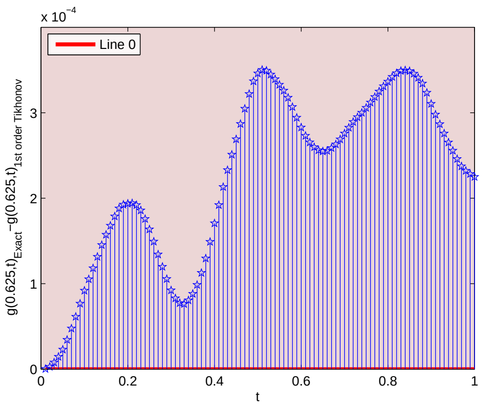

Figure 11. Difference between the the exact solution and 1st order Tikhonov solution for $g(0.625,t)$ of problem (2.2) with noisy data when $\Delta t = 0.01$.

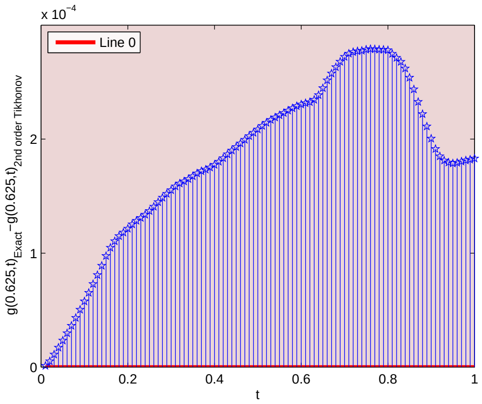

Figure 12. Difference between the the exact solution and 2nd order Tikhonov solution for $g(0.625, t)$ of problem (2.2) with noisy data when $\Delta t = 0.01$.

4. Conclusion

A numerical method, is proposed to estimate unknown boundary condition for these kinds of inverse problems in two-dimensional parabolic and hyperbolic equations and the following results are obtained.

1. The present study successfully applies the numerical method to inverse problems for two-dimensional parabolic and hyperbolic equations.

2. Numerical results show that a good estimation can be obtained within a couple of minutes CPU time at pentium IV-2.53 GHz PC.

3. The present method has been found stable with respect to small perturbation in the input data.

4. Numerical results show that, unknown function, evolutions estimated by the 0th order, 1st order and 2nd order Tikhonov regularization methods give very similar results.

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