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STABILITY RESULT FOR AN ABSTRACT TIME DELAYED EVOLUTION EQUATION WITH ARBITRARY DECAY OF VISCOELASTICITY

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Abstract. The paper is concerned with a second-order abstract semilinear evolution equation with infinite memory and time delay. With the help of the semigroup arguments and under suitable conditions on initial data and the kernel memory function, we state and prove the global existence of solution. Then, we establish the decay rates of the energy using the multiplier method by defining a suitable Lyapunov functional. This work extends previous works with time delay for a much wider class of kernels. We give also some applications to illustrate our results.

1. Introduction

Let H be a real Hilbert space with inner product and related norm denoted by $\langle .,. \rangle$ and $\| \, . \, \|$, respectively. Let $A : D(A) \longrightarrow H$ and $B : D(B) \longrightarrow H$ be a self-adjoint linear positive operator with domains $D(A) \subset D(B) \subset H$ such that the embeddings are dense and compact. Let $C : H \longrightarrow H$ is a self-adjoint linear operator and $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is the kernel of the memory term. $\tau > 0$ represents a time delay and $F \; : D(A^{\frac{1}{2}}) \to H$ is function satisfying some conditions to be specified later. We consider the following second-order abstract semilinear evolution equation with infinite memory and time delay

$$\begin{cases} u_{tt}(t) + Au(t) - \int_0^{+\infty} h(s)Bu(t-s)ds + Cu_t(t-\tau) = F(u(t)), & t \in (0, +\infty), \\ u_t(t-\tau) = f_0(t-\tau) & t \in (0, \tau), \\ u(-t) = u_0(t), \quad u_t(0) = u_1, & t \in \mathbb{R}_+, \end{cases} \tag{1.1}$$

where the initial datum (u_0, u_1, f_0) belongs to a suitable spaces.

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In absence of time delay term, a large number of works are available, where various decay estimates were obtained, see [7, 14, 21]. For the particular case of the wave equation with finite memory, see [2, 24].

In many cases, delay is a source of instability and even an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay. Nicaise and Pignotti in [15] considered a wave equation with a linear damping and delay term and they proved that the energy is exponentially stable and some instability results are also given by constructing some sequences of delays for which the energy of some solutions does not tend to zero, see also [3, 17].

When the memory term is replaced by a frictional damping $Bu_t(t)$:

$$u_{tt}(t) + Au(t) + Bu_t(t) + \mu u_t(t-\tau) = 0, \qquad t > 0,$$

where μ, τ are fixed constants and B is a given operator, there exist in the literature different stability results. These results show that the damping $Bu_t(t)$ is strong enough to stabilize the system in presence of a time delay provided that $|\mu|$ is small enough, see [10, 16, 17].

Guesmia in [11] considered the following second-order abstract linear problem with infinite memory and time delay terms

$$\begin{cases} u_{tt}(t) + Au(t) - \int_0^{+\infty} h(s)Au(t-s)ds + \mu u_t(t-\tau) = 0, & t > 0, \\ u(-t) = u_0(t), & t \in \mathbb{R}_+ \\ u_t(0) = u_1, \quad u_t(t-\tau) = f_0(t-\tau), & t \in (0,\tau), \end{cases}$$

He proved that the unique dissipation given by the memory term is strong enough to stabilize exponentially the system in presence of delay. In this work and others, the condition $h'(s) \leq -\delta h(s)$ for all $s \geq 0$ and some $\delta > 0$ is assumed to prove exponential decay of the energy, see [1, 4]. In [13], the previous condition is replaced by

$$h'(s) \leq -\zeta(t)h(s), \quad \forall s \geq 0, \tag{1.2}$$

where ζ is a positive nonincreasing differentiable function. The authors established the existence and the general decay results of the energy. Dai and Yang in [8] considered the same problem in [13] and solved the open problem proposed by Kirane and Said-Houari. Recently, Boukhatem and Benabderrahmane in [5] considered a variable coefficient viscoelastic equation with a time-varying delay in the boundary feedback and acoustic boundary conditions and nonlinear source term. They established a general decay results of the energy via suitable Lyapunov functionals and some properties of the convex functions where the kernel memory satisfies the equation (1.2). In [6], the same results have obtained in the case of constant delay.

Tatar in [23] introduced a new class of admissible kernels which lead to a wide range of possible decay rates. More precisely, He consider kernels satisfying

$$h(t-s) \geq \xi(t) \int_t^{+\infty} h(\pi - s)d\pi, \quad 0 \leq s \leq t,$$

for some $\xi(t) > 0$. This class contains the polynomial type functions and the exponential type. He proved that the last assumption on the relaxation in a viscoelastic problem ensuring uniform stability in an arbitrary rate.

For the case of distributed time delay, Guesmia and Tatar in [12] considered the following class of second-order linear hyperbolic equations

$$\begin{cases} u_{tt}(t) + Au(t) - \int_0^{+\infty} h(s)Bu(t-s)ds + \int_0^{+\infty} f(s)u_t(t-s)ds = 0, & t > 0, \\ u(-t) = u_0(t), & t \in \mathbb{R}_+ \\ u_t(0) = u_1, & t \in \mathbb{R}_+, \end{cases}$$

where the function f is of class $C^1(\mathbb{R}_+, \mathbb{R})$ and satisfies, for some positive constant α,

$$|f(s)| \leq \alpha h(s), \quad \text{and} \quad |f'(s)| \leq \alpha h(s), \quad \forall s \in \mathbb{R}_+.$$

They given well-posedness and stability of the system and they proved that the infinite memory alone guarantees the asymptotic stability of the system and the decay rate of solutions is found explicitly in terms of the growth at infinity of the infinite memory and the distributed time delay convolution kernels.

Nicaise and Pignotti in [18] considered the following system

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + F(U(t)) + k\mathcal{B}U(t-\tau), & t \in (0, +\infty), \\ U(0) = u_0, \mathcal{B}U(t-\tau) = f(t), & t \in (0, \tau), \end{cases}$$

where $\mathcal{A}$ generates a C_0-semigroup $(S(t))_{t\geq 0}$ that is exponentially stable, i.e., there exist two positive constants M and w such that

$$\|S(t)\|_{\mathcal{L}(H)} \leq Me^{-wt}, \quad \forall t \geq 0,$$

and $\mathcal{L}(H)$ denotes the space of bounded linear operators from H into itself. For a fixed delay parameter τ, a fixed bounded operator $\mathcal{B}$ from H into itself and for a real parameter k and $F : H \longrightarrow H$ satisfies some Lipschitz conditions, the initial datum U_0 belongs to H and $f \in C([0, \tau]; H)$. They showed that, if the C_0-semigroup describing the linear part of the model is exponentially stable, then the whole system retains this good property when a suitable smallness condition on the time-delay feedback is satisfied, see also [19].

Motivated by previous works, we study the well-posedness and the stability result of a semilinear abstract viscoelastic equation with infinite memory in presence of a time delayed damping and a nonlinear source term. Our results extend the decay results in previous works to kernels h which do not necessarily converge exponentially to zero at infinity. Moreover, our problem generalizes the linear problems to those with a nonlinear source term and to problems with more general time delayed damping term.

The paper is organized as follows. In Sect. 2, we prove the well-posedness by using the semigroup arguments under some assumptions on A, B, C, h and F. Then, we state and prove the stability result of solution by using the energy method to produce a suitable Lyapunov functional with arbitrary decay on h. Section 4 is devoted to some concrete examples in the aim to illustrate our abstract result.

2. Well-posedness

In this section, we state some assumptions on A, B, C and h and prove the well-posedness result by using semigroup theory.

For studying the problem (1.1), we introduce a new variable z as in [15]

$$z(\rho, t) = u_t(t - \rho\tau), \quad \rho \in (0, 1), \ t > 0.$$

Thus, we have

$$\tau z_t(\rho,t) + z_\rho(\rho,t) = 0, \quad \rho \in (0,1),\ t > 0.$$

Moreover, as in [9], we define

$$\eta^t(s) = u(t) - u(t-s), \quad t,s > 0.$$

Therefore, problem (1.1) takes the form

$$\begin{cases} u_{tt}(t) + Au(t) - h_0 Bu(t) + \int_0^{+\infty} h(s) B\eta^t(s) ds & \\ \qquad\qquad + Cz(1,t) = F(u(t)), & t \in (0,+\infty), \\ \tau z_t(\rho,t) + z_\rho(\rho,t) = 0, & \rho \in (0,1),\ t > 0, \\ \eta_t^t(s) = u_t(t) - \eta_s^t(s), & t,s > 0, \\ z(\rho,0) = f_0(-\rho\tau), & \rho \in (0,1), \\ z(0,t) = u_t(t), & t > 0, \\ u(-t) = u_0(t), \quad u_t(0) = u_1, & t \geq 0, \\ \eta^0(s) = u_0(0) - u_0(s), & s \geq 0. \end{cases} \tag{2.1}$$

We will need the following assumptions:

(A1) There exist positive constants a and b satisfying

$$b\|u\|^2 \leq \left\| B^{\frac{1}{2}} u \right\|^2 \leq a \left\| A^{\frac{1}{2}} u \right\|^2, \quad \forall u \in D(A^{\frac{1}{2}}). \tag{2.2}$$

(A2) The kernel function $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is of class C^1 nonincreasing function satisfying

$$h_0 = \int_0^{+\infty} h(s) ds < \frac{1}{a}. \tag{2.3}$$

(A3) There exists $\mu \in \mathbb{R}^*$ such that

$$\|Cu\|^2 \leq |\mu| \|u\|^2, \quad \forall u \in H. \tag{2.4}$$

(A4) $F : D(A^{\frac{1}{2}}) \to H$ is globally Lipschitz continuous, namely

$$\exists \gamma > 0 \text{ such that } \|F(u) - F(v)\| \leq \gamma \left\| A^{\frac{1}{2}}(u-v) \right\|, \quad \forall u,v \in H$$

Let us denote $U = (u, u_t, \eta^t, z)^T$, the problem (2.1) can be rewritten:

$$\begin{cases} U_t(t) = \mathcal{A}U(t) + \mathcal{F}(U(t)), \quad \forall t > 0, \\ U(0) = U_0 = \left(u_0, u_1, \eta^0, f_0(-\tau .)\right)^T, \end{cases} \tag{2.5}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} \phi_2 \\ -(A - h_0 B)\phi_1 - \int_0^{+\infty} h(s) B\phi_3(s) ds - C\phi_4(1) \\ \phi_2 - \dfrac{\partial \phi_3}{\partial s} \\ \dfrac{-1}{\tau} \dfrac{\partial \phi_4}{\partial \rho} \end{pmatrix}$$

and

$$\mathcal{F}(\phi_1, \phi_2, \phi_3, \phi_4)^T = (0, F(\phi_1), 0, 0)^T$$

The domain $D(\mathcal{A})$ is given by

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{c} (\phi_1, \phi_2, \phi_3, \phi_4)^T \in \mathcal{H},\ (A - h_0 B)\phi_1 + \int_0^{+\infty} h(s) B\phi_3(s) ds \in H, \\ \phi_2 \in D(A^{\frac{1}{2}}), \dfrac{\partial \phi_3}{\partial s} \in L_h^2(\mathbb{R}_+, D(B^{\frac{1}{2}})), \\ \dfrac{\partial \phi_4}{\partial \rho} \in L^2(0,1;H),\ \phi_3(0) = 0,\ \phi_4(0) = \phi_2 \end{array} \right\}$$

where

$$\mathcal{H} = D(A^{\frac{1}{2}}) \times H \times L_h^2(\mathbb{R}_+, D(B^{\frac{1}{2}})) \times L^2(0,1;H).$$

The sets $L_h^2(\mathbb{R}_+, D(B^{\frac{1}{2}}))$ and $L^2(0,1;H)$ are respectively defined by

$$L_h^2(\mathbb{R}_+, D(B^{\frac{1}{2}})) = \left\{ \phi : \mathbb{R}_+ \to D(B^{\frac{1}{2}}), \quad \int_0^{+\infty} h(s) \, \left\| B^{\frac{1}{2}} \phi(s) \right\|^2 ds < +\infty \right\},$$

equipped with the inner product

$$\langle \phi_1, \phi_2 \rangle_{L_h^2(\mathbb{R}_+, D(B^{\frac{1}{2}}))} = \int_0^{+\infty} h(s) \, \left\langle B^{\frac{1}{2}} \phi_1(s), B^{\frac{1}{2}} \phi_2(s) \right\rangle ds.$$

And

$$L^2(0,1;H) = \left\{ \phi : (0,1) \to H, \quad \int_0^1 \|\phi(\rho)\|^2 d\rho < +\infty \right\},$$

equipped with the inner product

$$\langle \phi_1, \phi_2 \rangle_{L^2(0,1;H)} = \int_0^1 \langle \phi_1(\rho), \phi_2(\rho) \rangle \, d\rho.$$

The Hilbert space $\mathcal{H}$ equipped with the following inner product. For all $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$ and $W = (w_1, w_2, w_3, w_4)^T$ in $\mathcal{H}$, we have

$$\begin{aligned} \langle \Phi, W \rangle_{\mathcal{H}} &= \langle \phi_1, w_1 \rangle_{D(A^{\frac{1}{2}})} - h_0 \langle \phi_1, w_1 \rangle_{D(B^{\frac{1}{2}})} + \langle \phi_2, w_2 \rangle \\ &\quad + \langle \phi_3, w_3 \rangle_{L_h^2(\mathbb{R}_+, D(B^{\frac{1}{2}}))} + \tau\mu \langle \phi_4, w_4 \rangle_{L^2(0,1;H)}. \end{aligned}$$

The well-posedness of problem (2.5) is ensured by the following theorem:

Theorem 2.1. *Under the assumptions (A1)-(A4), for an initial datum $U_0 \in \mathcal{H}$, the system (2.5) has a unique mild solution $U \in C(\mathbb{R}_+, \mathcal{H})$ satisfies the following formula,*

$$U(t) = S(t)U_0 + \int_0^t S(t-s)\mathcal{F}(U(s))ds.$$

Moreover, if $U_0 \in \mathcal{D}(\mathcal{A})$ and $\mathcal{F} \in C^1(\mathcal{H})$, then the solution of (2.5) satisfies (classical solution)

$$U \in C(\mathbb{R}_+, \mathcal{D}(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

Proof. To prove Theorem 2.1, we use the semigroup theory. The problem (2.5) can be seen as an inhomogeneous evolution problem. It's clear that $\mathcal{F}$ is globally lipschitz continuous, let show that the operator $\mathcal{A}$ generate a linear C_0-semigroup $(S(t))_{t\geq 0}$ on $\mathcal{H}$. Indeed,

• First, we prove that the linear operator $\mathcal{A}$ is dissipative.

Take $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \in \mathcal{D}(\mathcal{A})$, then

$$\begin{aligned} \left\langle \mathcal{A}\Phi, \Phi \right\rangle_{\mathcal{H}} &= \langle \phi_2, \phi_1 \rangle_{D(A^{\frac{1}{2}})} + \int_0^{+\infty} h(s) \Big\langle \phi_2 - \frac{\partial \phi_3}{\partial s}, \phi_3 \Big\rangle_{D(B^{\frac{1}{2}})} ds \\ &\quad - h_0 \langle \phi_2, \phi_1 \rangle_{D(B^{\frac{1}{2}})} + \tau|\mu| \int_0^1 \Big\langle \frac{-1}{\tau} \frac{\partial \phi_4}{\partial \rho}, \phi_4 \Big\rangle d\rho \\ &\quad - \Big\langle (A - h_0 B)\phi_1 + \int_0^{+\infty} h(s) B\phi_3(s) ds + C\phi_4(1), \phi_2 \Big\rangle. \end{aligned}$$

Using the definition of $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$ and the fact that H is a real Hilbert space, we conclude

$$\langle A - h_0 B\phi_1, \phi_2 \rangle = \langle A^{\frac{1}{2}} \phi_2, A^{\frac{1}{2}} \phi_1 \rangle - h_0 \langle B^{\frac{1}{2}} \phi_2, B^{\frac{1}{2}} \phi_1 \rangle \tag{2.6}$$

using the Cauchy-Schwarz and Young's inequalities and by (2.4), we have

$$-\langle C\phi_4(1), \phi_2\rangle \le \frac{|\mu|}{2}\left(\|\phi_4(1)\|^2 + \|\phi_2\|^2\right). \tag{2.7}$$

$$\left\langle \int_0^{+\infty} h(s)B\phi_3(s)ds, \phi_2 \right\rangle = \int_0^{+\infty} h(s)\langle \phi_2, \phi_3\rangle_{D(B^{\frac{1}{2}})} ds.$$

Integrating by parts and using the definition of $\mathcal{D}(\mathcal{A})$ ($\phi_3(0) = 0$), we obtain

$$\int_0^{+\infty} h(s)\Big\langle -\frac{\partial \phi_3}{\partial s}, \phi_3\Big\rangle_{D(B^{\frac{1}{2}})} ds \le \frac{1}{2}\int_0^{+\infty} h'(s)\|B^{\frac{1}{2}}\phi_3(s)\|^2 ds. \tag{2.8}$$

Also using the fact that $\phi_4(0) = \phi_2$, we obtain

$$\tau|\mu| \int_0^1 \Big\langle \frac{-1}{\tau}\frac{\partial \phi_4}{\partial \rho}, \phi_4\Big\rangle d\rho = \frac{|\mu|}{2}\Big(\|\phi_4(0)\|^2 - \|\phi_4(1)\|^2\Big) = \frac{|\mu|}{2}\Big(\|\phi_2\|^2 - \|\phi_4(1)\|^2\Big). \tag{2.9}$$

Consequently, inserting (2.6), (2.7), (2.8) and (2.9) in (2.6) and using the fact that h is nonincreasing, we find

$$\langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} \le \frac{1}{2}\int_0^{+\infty} h'(s)\left\|B^{\frac{1}{2}}\phi_3(s)\right\|^2 ds + |\mu|\|u_t\|^2 \le |\mu|\|\Phi\|^2, \tag{2.10}$$

which means that the operator $\mathcal{A} - |\mu|I$ is dissipative.

• Let us now prove that $\lambda I - \mathcal{A}$ is surjective. Indeed, let $(f_1, f_2, f_3, f_4)^T \in \mathcal{H}$, we show that there exists $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T \in \mathcal{D}(\mathcal{A})$ satisfying

$$(\lambda I - \mathcal{A})\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix} = \begin{pmatrix}f_1\\ f_2\\ f_3\\ f_4\end{pmatrix},$$

which is equivalent to

$$\begin{cases} \lambda\phi_1 - \phi_2 = f_1 \\ \lambda\phi_2 + (A - h_0 B)\phi_1 + \int_0^{+\infty} h(s)B\phi_3(s)ds + C\phi_4(1) = f_2 \\ \lambda\phi_3 - \phi_2 + \dfrac{\partial\phi_3}{\partial s} = f_3 \\ \lambda\phi_4 + \dfrac{1}{\tau}\dfrac{\partial \phi_4}{\partial\rho} = f_4. \end{cases} \tag{2.11}$$

Suppose that we have found ϕ_1 with the appropriate regularity. Then, we have

$$\phi_2 = \lambda\phi_1 - f_1. \tag{2.12}$$

We note that the third equation in (2.11) with $\phi_3(0) = 0$ has a unique solution

$$\phi_3(s) = e^{-\lambda s}\int_0^s e^{\lambda y}\left(f_3(y) - f_1 + \lambda\phi_1\right)dy. \tag{2.13}$$

On the other hand, the fourth equation in (2.11) with $\phi_4(0) = \phi_2 = \lambda\phi_1 - f_1$ has a unique solution

$$\phi_4(\rho) = \left(\lambda\phi_1 - f_1 + \tau\int_0^\rho f_4(y)e^{\lambda\tau y}dy\right)e^{-\lambda\tau\rho}, \quad \rho \in (0,1). \tag{2.14}$$

In particular,

$$\phi_4(1) = \left(\lambda\phi_1 - f_1 + \tau\int_0^1 f_4(y)e^{\lambda\tau y}dy\right)e^{-\lambda\tau}.$$

It remains only to determine ϕ_1.

Next, plugging (2.12) and (2.13) into the second equation in (2.11), we get

$$\left(A-\alpha B+\lambda e^{-\lambda\tau}C+\lambda^2 I\right)\phi_1=\tilde{f}, \tag{2.15}$$

where

$$\alpha=h_0-\lambda\int_0^\infty h(s)e^{-\lambda s}\left(\int_0^s e^{\lambda y}dy\right)ds=\int_0^\infty h(s)e^{-\lambda s}ds,$$

and

$$\begin{aligned}\tilde{f} &= f_2+\lambda f_1+e^{-\lambda\tau}C\left(f_1-\tau\int_0^1 f_4(y)e^{\tau y}dy\right)\\ &\quad -\int_0^\infty e^{-\lambda s}h(s)\int_0^s e^{-\lambda y}B\left(f_3(y)-f_1\right)dyds.\end{aligned}$$

We have just to prove that (2.15) has a solution $\phi_1\in D(A^{\frac{1}{2}})$ and replace in (2.12), (2.13) and (2.14) to obtain $\Phi\in\mathcal{D}(\mathcal{A})$ satisfying (2.11).
We have $\alpha<h_0$, by (2.3) and (2.2), we deduce that $A-\alpha B$ is a positive definite operator. Then, we take the duality brackets $\langle .,.\rangle_{D(A^{\frac{1}{2}})'\times D(A^{\frac{1}{2}})}$ with $w\in D(A^{\frac{1}{2}})$:

$$\left\langle\left(A-\alpha B+\lambda e^{-\lambda\tau}C+\lambda^2 I\right)\phi_1,w\right\rangle_{D(A^{\frac{1}{2}})'\times D(A^{\frac{1}{2}})}=\left\langle\tilde{f},w\right\rangle_{D(A^{\frac{1}{2}})'\times D(A^{\frac{1}{2}})}. \tag{2.16}$$

Consequently, the left-hand side of (2.16) is bilinear, continuous and coercive on $D(A^{\frac{1}{2}})$. Since, applying the Lax-Milgram theorem and classical regularity arguments, we conclude that (2.11) has a unique solution $\phi_1\in D(A^{\frac{1}{2}})$ satisfying. Using (2.13),

$$\left((A-h_0B)\phi_1+\int_0^{+\infty}h(s)B\phi_3(s)ds\right)\in H.$$

In conclusion, we have found $\Phi=(\phi_1,\phi_2,\phi_3,\phi_4)^T\in\mathcal{D}(\mathcal{A})$, which verifies (2.11), and thus $\lambda I-\mathcal{A}$ is surjective for all $\lambda>0$ and the same holds for the operator $\lambda I-(\mathcal{A}-|\mu|I)$.

Then, the Lumer-Phillips theorem implies that $|\mu|I-\mathcal{A}$ is a maximal monotone operator, $\mathcal{A}-|\mu|I$ is an infinitesimal generator of a strongly continuous semigroup of contraction in $\mathcal{H}$. Hence, the operator $\mathcal{A}$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ in $\mathcal{H}$. Consequently, by using Theorem 1.2, Ch. 6 of [22], the problem (2.5) has a unique solution $U\in C([0,+\infty),\mathcal{H})$. □

3. Stability result

The stability result of the solution of (2.1) holds under the following additional assumptions:

(A5) There exist a positive constant d satisfying

$$\left\|A^{\frac{1}{2}}u\right\|^2\leq d\left\|B^{\frac{1}{2}}u\right\|^2,\quad \forall u\in D(A^{\frac{1}{2}}). \tag{3.1}$$

(A6) Moreover, we assume that $F(0)=0$ and there exists a continuous and differentiable mapping $\psi:D(A^{\frac{1}{2}})\to\mathbb{R}$ satisfying

$$D_\psi=F\quad\text{and}\quad\langle F(u),u\rangle\geq 2\psi(u),\quad \forall u\in D(A^{\frac{1}{2}}). \tag{3.2}$$

(A7) The function h satisfies (A2) and there exists a positive function $\xi \in C(\mathbb{R}_+, \mathbb{R}_+^*)$ satisfying $\lim_{s\to+\infty} \xi(s)$ exists such that

$$\begin{cases} h(t-s) \geq \xi(t) \displaystyle\int_t^{+\infty} h(\pi - s)d\pi, & \forall t \in \mathbb{R}_+,\ \forall s \in [0,t], \\ h'(s) < 0, & \forall s \in \mathbb{R}_+. \end{cases} \tag{3.3}$$

The first inequality in (3.3), introduced in [25] and [23], implies that h converges to zero at least exponentially but it does not involve the derivative of h. This class contains the polynomial (or power) type ($h(t) = (1+t)^{-a}$, $a > 1$) functions and the exponential type ($h(t) = e^{-at}$, $a > 0$) functions.

Let establish some several Lemmas needed of our main result. We define the modified energy functional E associated to problem (2.1) by

$$\begin{aligned} E(t) &= \frac{1}{2}\bigg(\left\|A^{\frac{1}{2}}u\right\|^2 - h_0 \left\|B^{\frac{1}{2}}u\right\|^2 + \|u_t\|^2 + \int_0^{+\infty} h(s) \left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds \\ &\quad -2\psi(u) + \tau|\mu| \int_0^1 \|z(\rho,t)\|^2 d\rho \bigg). \end{aligned} \tag{3.4}$$

Lemma 3.1. *Assume that (A1)-(A4) hold and let $U_0 \in \mathcal{D}(\mathcal{A})$. Then, the energy functional defined by (3.4) satisfies*

$$E'(t) \leq \frac{1}{2}\int_0^{+\infty} h'(s) \left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds + |\mu|\|u_t\|^2. \tag{3.5}$$

Proof. Multiplying the first equation of (2.1) by u_t. Using (A6) and repeating exactly the same arguments to obtain (2.10). □

Remark. *Note that, from (3.5), the energy of solutions to problem (2.1) is not decreasing in general. Indeed, the second term in the right-hand side of (3.5), coming from the delay term, is nonnegative.*

Now, as in [20], for $n \in \mathbb{N}^*$, let consider the set

$$A_n = \{s \in \mathbb{R}_+,\ \ h(s) + nh'(s) \leq 0\},$$

and put $h_n = \int_{A_n^c} h(s)ds$. We have $h_n > 0$, otherwise, $A_n^c = \emptyset$. Furthermore, by the second inequality in (3.3), we have

$$\lim_{n\to+\infty} A_n^c = \cap_{n\in\mathbb{N}^*} A_n^c = \emptyset, \text{ and then } \lim_{n\to+\infty} h_n = 0.$$

In order to state our results, we need the following four lemmas.

Lemma 3.2. *Let U be solution of* (2.1)*. Then the functional*

$$I_1(t) = -\left\langle u_t(t), \int_0^{+\infty} h(s)\eta^t(s)ds \right\rangle, \tag{3.6}$$

satisfies, for $\varepsilon_1, \varepsilon_2 > 0$,

$$\begin{aligned}
I_1'(t) \quad \leq \quad & -(h_0-\varepsilon_1)\|u_t\|^2 + \left(\varepsilon_2 + \frac{\sqrt{dh_n}}{2}\right)\left\|A^{\frac{1}{2}}u\right\|^2 - \frac{h_0^2}{2}\left\|B^{\frac{1}{2}}u\right|^2 \\
& +\left(2h_n - \frac{h_0}{2} + \frac{\sqrt{dh_n}}{2}\right)\int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds \\
& +\frac{h_0}{2}\int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}u(t-s)\right\|^2 ds \\
& -\left(2nh_0 + \frac{dnh_0}{4\varepsilon_2} + \frac{h(0)}{4b\varepsilon_1}\right)\int_0^{+\infty} h'(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds \\
& +\left\langle Cz(1,t) - F(u), \int_0^{+\infty} h(s)\eta^t(s)ds\right\rangle, \qquad (3.7)
\end{aligned}$$

Proof. Differentiating (3.6) with respect to t and using the third equation of 2.1, we find

$$I_1'(t) = -\left\langle u_{tt}(t), \int_0^{+\infty} h(s)\eta^t(s)ds\right\rangle + \left\langle u_t(t), \int_0^{+\infty} h(s)\eta_s^t(s)ds\right\rangle - h_0\|u_t\|^2.$$

Integrating by parts with respect to s the second term in the right hand side of the previous equality and using the fact that $\lim_{s\to+\infty} h(s) = 0,\ \eta^t(0) = 0$, we obtain

$$I_1'(t) = -\left\langle u_{tt}(t), \int_0^{+\infty} h(s)\eta^t(s)ds\right\rangle - \left\langle u_t(t), \int_0^{+\infty} h'(s)\eta^t(s)ds\right\rangle - h_0\|u_t\|^2.$$

On the other hand, by the first equation of (2.1), we have

$$\begin{aligned}
& \left\langle u_{tt}(t), \int_0^{+\infty} h(s)\eta^t(s)ds\right\rangle + \left\langle Au(t), \int_0^{+\infty} h(s)\eta^t(s)ds\right\rangle \\
- \quad & h_0\left\langle Bu(t), \int_0^{+\infty} h(s)\eta^t(s)ds\right\rangle + \left\langle \int_0^{+\infty} h(s)B\eta^t(s)ds, \int_0^{+\infty} h(s)\eta^t(s)ds\right\rangle \\
+ \quad & \left\langle Cz(1,t) - F(u), \int_0^{+\infty} h(s)\eta^t(s)ds\right\rangle = 0,
\end{aligned}$$

using the definitions of $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$, we get

$$\begin{aligned}
I_1'(t) \quad = \quad & -h_0\|u_t\|^2 + \left\langle Cz(1,t) - F(u), \int_0^{+\infty} h(s)\eta^t(s)ds\right\rangle \\
& -\left\langle u_t(t), \int_0^{+\infty} h'(s)\eta^t(s)ds\right\rangle + \left\langle A^{\frac{1}{2}}u(t), \int_0^{+\infty} h(s)A^{\frac{1}{2}}\eta^t(s)ds\right\rangle \\
& \left\|\int_0^{+\infty} h(s)B^{\frac{1}{2}}\eta^t(s)ds\right\|^2 - h_0\left\langle B^{\frac{1}{2}}u(t), \int_0^{+\infty} h(s)B^{\frac{1}{2}}\eta^t(s)ds\right\rangle. \qquad (3.8)
\end{aligned}$$

Let estimate the last three terms in the right hand by using Cauchy-Schwarz and Young's inequalities and the definition of A_n. Then, using (2.2), (3.1) and (2.3), we get

$$-\left\langle u_t(t), \int_0^{+\infty} h'(s)\eta^t(s)ds\right\rangle \leq \varepsilon_1\|u_t\|^2 - \frac{h(0)}{4b\varepsilon_1}\int_0^{+\infty} h'(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds\Big\rangle,$$

$$\begin{aligned}
&\left\langle A^{\frac{1}{2}}u(t), \int_0^{+\infty} h(s)A^{\frac{1}{2}}\eta^t(s)ds\right\rangle \\
= &\left\langle A^{\frac{1}{2}}u(t), \int_{A_n} h(s)A^{\frac{1}{2}}\eta^t(s)ds\right\rangle + \left\langle A^{\frac{1}{2}}u(t), \int_{A_n^c} h(s)A^{\frac{1}{2}}\eta^t(s)ds\right\rangle. \\
\leq &\ \varepsilon_2 \left\|A^{\frac{1}{2}}u\right\|^2 + \frac{dh_0}{4\varepsilon_2}\int_{A_n} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds + \frac{\sqrt{dh_n}}{2}\left\|A^{\frac{1}{2}}u\right\|^2 \\
&+\frac{\sqrt{dh_n}}{2}\int_{A_n^c} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds \\
\leq &\ \varepsilon_2 \left\|A^{\frac{1}{2}}u\right\|^2 - \frac{dnh_0}{4\varepsilon_2}\int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds + \frac{\sqrt{dh_n}}{2}\left\|A^{\frac{1}{2}}u\right\|^2 \\
&+\frac{\sqrt{dh_n}}{2}\int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds,
\end{aligned}$$

$$\begin{aligned}
&\left\|\int_0^{+\infty} h(s)B^{\frac{1}{2}}\eta^t(s)ds\right\|^2 \\
= &\left\|\int_{A_n} h(s)B^{\frac{1}{2}}\eta^t(s)ds + \int_{A_n^c} h(s)B^{\frac{1}{2}}\eta^t(s)ds\right\|^2 \\
\leq &\ 2\left\|\int_{A_n} h(s)B^{\frac{1}{2}}\eta^t(s)ds\right\|^2 + 2\left\|\int_{A_n^c} h(s)B^{\frac{1}{2}}\eta^t(s)ds\right\|^2 \\
\leq &\ 2h_0\int_{A_n} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds + 2h_n\int_{A_n^c} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds \\
\leq &\ -2nh_0\int_0^{+\infty} h'(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds + 2h_n\int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds.
\end{aligned}$$

And for the last one, we have

$$\begin{aligned}
&-h_0\left\langle B^{\frac{1}{2}}u(t), \int_0^{+\infty} h(s)B^{\frac{1}{2}}\eta^t(s)ds\right\rangle \\
= &-h_0^2\left\|B^{\frac{1}{2}}u\right\|^2 + h_0\left\langle B^{\frac{1}{2}}u(t), \int_0^{+\infty} h(s)B^{\frac{1}{2}}u(t-s)ds\right\rangle \\
= &-\frac{h_0^2}{2}\left\|B^{\frac{1}{2}}u\right\|^2 + \frac{h_0}{2}\int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}u(t-s)\right\|^2 ds \\
&-\frac{h_0}{2}\int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds. \qquad (3.9)
\end{aligned}$$

Inserting these four inequalities in (3.8), we get (3.7). □

Lemma 3.3. *Let U be solution of* (2.1). *Then the functional*

$$I_2(t) = \langle u_t(t), u(t)\rangle, \qquad (3.10)$$

satisfies,

$$\begin{aligned}
I_2'(t) = &\ \|u_t\|^2 - \left\|A^{\frac{1}{2}}u\right\|^2 + \frac{h_0}{2}\left\|B^{\frac{1}{2}}u\right\|^2 + \frac{1}{2}\int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}u(t-s)\right\|^2 ds \\
&-\frac{1}{2}\int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds - \left\langle Cz(1,t) + F(u), u\right\rangle. \qquad (3.11)
\end{aligned}$$

Proof. Differentiating (3.10) with respect to t, we find

$$I_2'(t) = \|u_t\|^2 + \langle u_{tt}(t), u(t)\rangle.$$

On the other hand, multiplying the first equation of (2.1) by $u(t)$, we have

$$\langle u_{tt}(t), u(t)\rangle + \langle (A - h_0 B)u(t), u(t)\rangle + \left\langle \int_0^{+\infty} h(s)B\eta^t(s)ds, u(t)\right\rangle$$
$$+\langle Cz(1,t), u(t)\rangle = 0,$$

By the definitions of $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$, we have

$$\langle u_{tt}(t), u(t)\rangle + \left\|A^{\frac{1}{2}}u\right\|^2 - h_0\left\|B^{\frac{1}{2}}u\right\|^2 + \left\langle \int_0^{+\infty} h(s)B\eta^t(s)ds, u(t)\right\rangle$$
$$+\langle Cz(1,t), u(t)\rangle = 0.$$

Consequently,

$$I_2'(t) = \|u_t\|^2 - \left\|A^{\frac{1}{2}}u\right\|^2 + h_0\left\|B^{\frac{1}{2}}u\right\|^2 - \left\langle \int_0^{+\infty} h(s)B\eta^t(s)ds, u(t)\right\rangle - \langle Cz(1,t), u(t)\rangle,$$

By using the inequality (3.9), we get (3.11). □

Similarly to [15], we introduce the following functional.

Lemma 3.4. *Let U be solution of (2.1). Then the functional*

$$I_3(t) = \tau e^{2\tau}\int_0^1 e^{-2\tau\rho}\|z(\rho,t)\|^2 ds, \tag{3.12}$$

satisfies,

$$I_3'(t) \le -2\tau\int_0^1 \|z(\rho,t)\|^2 ds + e^{2\tau}\|u_t\|^2 - \|z(1,t)\|^2. \tag{3.13}$$

Proof. By using the second equation of (2.1), we get

$$\begin{aligned} I_3'(t) &= 2\tau e^{2\tau}\int_0^1 e^{-2\tau\rho}\langle z_t(\rho,t), z(\rho,t)\rangle d\rho \\ &= -2e^{2\tau}\int_0^1 e^{-2\tau\rho}\langle z_\rho(\rho,t), z(\rho,t)\rangle d\rho \\ &= -2e^{2\tau}\int_0^1 e^{-2\tau\rho}\frac{\partial}{\partial\rho}\|z(\rho,t)\|^2 d\rho. \end{aligned}$$

Then, by integrating by parts and $z(0,t) = u_t(t)$, we get

$$I_3'(t) = -2\tau e^{2\tau}\int_0^1 e^{-2\tau\rho}\|z(\rho,t)\|^2 ds + e^{2\tau}\|u_t\|^2 - \|z(1,t)\|^2,$$

which is (3.13) by using the fact that $e^{-2\tau\rho} \ge e^{-2\tau}$, for any $\rho \in]0,1[$. □

Now, we consider two functionals J_1 and J_2 and we give their derivatives in the following lemma.

Lemma 3.5. *Let*

$$J_1(t) = \int_0^t \left(\int_t^{+\infty} h(\pi - s)d\pi\right)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds, \quad \forall t \in \mathbb{R}_+, \tag{3.14}$$

and

$$J_2(t)=\int_0^t\left(\int_t^{+\infty}h(\pi-s)d\pi\right)\left\|A^{\frac{1}{2}}\eta^t(s)\right\|^2ds,\quad \forall t\in\mathbb{R}_+. \tag{3.15}$$

Then, for any $\lambda_1\in]0,1[$,

$$\begin{aligned}J_1'(t)\ &\leq\ h_0\left\|B^{\frac{1}{2}}u\right\|^2-(1-\lambda_1)\xi(t)J_1(t)-\lambda_1\int_0^t h(s)\left\|B^{\frac{1}{2}}u(t-s)\right\|^2ds\\ &\quad+\lambda_1\int_t^{+\infty}h(s)\left\|B^{\frac{1}{2}}u_0(s-t)\right\|^2ds,\qquad \forall t\in\mathbb{R}_+,\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}J_2'(t)\ &\leq\ h_0\left\|A^{\frac{1}{2}}u\right\|^2-(1-\lambda_1)\xi(t)J_2(t)-\frac{\lambda_1}{a}\int_0^t h(s)\left\|B^{\frac{1}{2}}u(t-s)\right\|^2ds\\ &\quad+d\lambda_1\int_t^{+\infty}h(s)\left\|B^{\frac{1}{2}}u_0(s-t)\right\|^2ds,\qquad \forall t\in\mathbb{R}_+.\end{aligned} \tag{3.17}$$

Proof. The functional J_1 is well-defined. Indeed, by using the fact that $\eta\in L_h^2(\mathbb{R}_+,D(B^{\frac{1}{2}}))$ and (3.3), we have

$$J_1(t)\leq\frac{1}{\xi(t)}\int_0^t h(t-s)\left\|B^{\frac{1}{2}}u(s)\right\|^2ds\leq\frac{1}{\xi(t)}\int_0^t h(s)\left\|B^{\frac{1}{2}}u(t-s)\right\|^2ds<+\infty.$$

By (3.1), we conclude that J_2 also is well defined.

Then, differentiating J_1 with respect to t and using the definition of u_0 and (3.3), we obtain

$$\begin{aligned}J_1'(t)\ &=\ \left(\int_t^{+\infty}h(\pi-s)d\pi\right)\left\|B^{\frac{1}{2}}u(t)\right\|^2-\int_0^t h(t-s)\left\|B^{\frac{1}{2}}u(s)\right\|^2ds\\ &=\ h_0\left\|B^{\frac{1}{2}}u\right\|^2-(1-\lambda_1)\int_0^t h(t-s)\left\|B^{\frac{1}{2}}u(s)\right\|^2ds\\ &\quad-\lambda_1\int_{-\infty}^t h(t-s)\left\|B^{\frac{1}{2}}u(s)\right\|^2ds+\lambda_1\int_{-\infty}^0 h(t-s)\left\|B^{\frac{1}{2}}u(s)\right\|^2ds\\ &\leq\ h_0\left\|B^{\frac{1}{2}}u\right\|^2-(1-\lambda_1)\xi(t)J_1(t)-\lambda_1\int_0^{+\infty}h(s)\left\|B^{\frac{1}{2}}u(t-s)\right\|^2ds\\ &\quad+a\lambda_1\int_t^{+\infty}h(s)\left\|B^{\frac{1}{2}}u_0(s-t)\right\|^2ds,\end{aligned}$$

which is exactly (3.16). A similar argument yields the relation (3.17). □

In this case, the Lyapunov functional L we will work with is

$$L(t)=E(t)+\epsilon\big(N_1I_1(t)+N_2I_2(t)+I_3(t)\big)+M_1J_1(t)+aM_1J_2(t), \tag{3.18}$$

where $\epsilon,N_1,N_2,M_1>0$ are positive constants to be chosen later.

Now we are in position to state and prove the decay result of solution of problem (2.1).

Theorem 3.6. *Assume that (A1)-(A7) hold. For any initial datum* $U_0\in\mathcal{H}$. *Assume that* h *satisfies*

$$\int_0^{+\infty}h(s)ds<\frac{\gamma^2}{b}, \tag{3.19}$$

and there exists a positive constant δ_0 independent of μ such that, if

$$|\mu| < \delta_0, \tag{3.20}$$

then, for any $U_0 \in \mathcal{H}$, there exist positive constants δ_1 and δ_2 such that

$$E(t) \leq \delta_2 e^{-\delta_1 t}\left(1 + \int_0^t e^{\delta_1 s}\int_s^{+\infty} h(\pi)\left\|B^{\frac{1}{2}}u_0(\pi - s)\right\|^2 d\pi ds\right), \ \ \forall t \in \mathbb{R}_+, \tag{3.21}$$

if $\lim_{t\to+\infty}\xi(t) > 0$, *and*

$$E(t) \leq \delta_2 e^{-\delta_1 \hat{\xi}(t)}\left(1 + \int_0^t e^{\delta_1 \hat{\xi}(s)}\int_s^{+\infty} h(\pi)\left\|B^{\frac{1}{2}}u_0(\pi - s)\right\|^2 d\pi ds\right), \ \ \forall t \in \mathbb{R}_+, \tag{3.22}$$

if $\lim_{t\to+\infty}\xi(t) = 0$, *where*

$$\hat{\xi}(s) = \int_0^s \xi(\pi)d\pi, \quad \forall t \in \mathbb{R}_+. \tag{3.23}$$

Proof. In order to proof the decay estimates, we start by the derivative of the function L. On the other hand, by using (A6) and (2.2), we have

$$\begin{aligned}
-\left\langle F(u), \int_0^{+\infty} h(s)\eta^t(s)ds\right\rangle \ &\leq \ \frac{1}{b}\|F(u)\|^2 + \frac{b}{4}\left\|\int_0^{+\infty} h(s)\eta^t(s)ds\right\|^2 \\
&\leq \ \frac{\gamma^2}{b}\left\|A^{\frac{1}{2}}u\right\|^2 + \frac{h_0}{4}\int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds,
\end{aligned}$$

Combining (3.5), (3.7), (3.11), (3.13), (3.16) and (3.17), we obtain

$$\begin{aligned}
L'(t) \ \leq \ & -\epsilon\Big[(C_1 - \frac{|\mu|}{\epsilon})\|u_t\|^2 + C_2\left\|A^{\frac{1}{2}}u\right\|^2 + C_3 h_0\left\|B^{\frac{1}{2}}u\right\|^2 - 2\tau\int_0^1 \|z(\rho,t)\|^2 d\rho \\
& + \int_0^{+\infty} h(s)\left(C_4\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 + C_5\left\|B^{\frac{1}{2}}u(t-s)\right\|^2\right)ds - 2N_2\psi(u)\Big] \\
& + \frac{\sqrt{dh_n}}{2}\epsilon N_1\left\|A^{\frac{1}{2}}u\right\|^2 + \left(2h_n + \frac{\sqrt{dh_n}}{2}\right)\epsilon N_1\int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds \\
& + \left(\frac{1}{2} - \epsilon C_6\right)\int_0^{+\infty} h'(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds - C_7\xi(t)(J_1(t) + J_2(t)) \\
& + C_8\int_t^{+\infty} h(s)\left\|B^{\frac{1}{2}}u_0(s-t)\right\|^2 ds - \epsilon\|z(1,t)\|^2 \\
& + \epsilon\left\langle Cz(1,t), N_1\int_0^{+\infty} h(s)\eta^t(s)ds - N_2 u\right\rangle,
\end{aligned} \tag{3.24}$$

where

$$\begin{aligned}
& C_1 = (h_0 - \varepsilon_1)N_1 - N_2 - e^{2\tau}, && C_2 = N_2 - (\varepsilon_2 + \frac{\gamma_2}{b})N_1 - \frac{ah_0}{\epsilon}M_1, \\
& C_3 = \frac{h_0}{2}N_1 - \frac{N_2}{2} - \frac{M_1}{\epsilon}, && C_4 = \frac{h_0}{4}N_1 + \frac{N_2}{2}, \\
& C_5 = \frac{2\lambda_1}{\epsilon}M_1 - \frac{h_0}{2}N_1 - \frac{N_2}{2}, && C_6 = \left(2nh_0 + \frac{dnh_0}{4\varepsilon_2} + \frac{h(0)}{4b\varepsilon_1}\right)N_1, \\
& C_7 = (1 - \lambda_1)M_1\min\{1, a\}, && C_8 = M_1\lambda_1(1 + ad).
\end{aligned} \tag{3.25}$$

At this point, we choose the different constants to obtain some results. First, we select $N_2 = (1 + ah_0)e^{2\tau}$ and we choose M_1, N_1 such that

$$\frac{\epsilon N_2}{2(1 + ah_0)} < M_1 < \frac{e^{2\tau}}{2\epsilon}.$$

$$\max\left\{\frac{b}{bh_0 - 2\gamma^2}\left(2(1 + ah_0)\frac{M_1}{\epsilon} - N_2\right), \frac{1}{h_0}(N_2 + e^{2\tau})\right\} < N_1 < \frac{1}{h_0}\left(N_2 + \frac{2M_1}{\epsilon}\right).$$

Note that M_1 exists as a result of the selection of N_2 for certain value of ϵ to be choose later and the choice of M_1 and N_2 guarantees the existence of N_1. Now, let pick ε_1, ε_2 and λ_1 such that

$$0 < \varepsilon_1 < h_0 - \frac{N_2 + e^{2\tau}}{N_1},$$

$$\varepsilon_2 = \frac{h_0}{2} - \frac{\gamma_2}{b} + \frac{1}{2N_1}\left(N_2 - 2(1 + ah_0)\frac{M_1}{\epsilon}\right),$$

and

$$\frac{\epsilon}{4M_1}(N_2 + h_0 N_1) \leq \lambda_1 < 1,$$

ε_2 and λ_1 exist by the previous selection of N_1 and N_2. Consequently, it result that $C_1 > 0$, $C_2 = -C_3$, $C_3 < 0$ and $C_5 \geq 0$. Moreover, it's clear that $C_4 > 0$, so, we have

$$\begin{aligned}
&-\epsilon\Bigg[C_1\|u_t\|^2 + C_2\left(\left\|A^{\frac{1}{2}}u\right\|^2 - h_0\left\|B^{\frac{1}{2}}u\right\|^2\right) - 2N_2\psi(u) \\
&+\int_0^{+\infty} h(s)\left(C_4\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 + C_5\left\|B^{\frac{1}{2}}u(t-s)\right\|^2\right)ds\Bigg] \\
\leq\ & -\epsilon C_9\left(\|u_t\|^2 + \left\|A^{\frac{1}{2}}u\right\|^2 - h_0\left\|B^{\frac{1}{2}}u\right\|^2 - 2\psi(u) + \int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds\right)
\end{aligned}$$

where

$$C_9 = \frac{1}{N_2}\min\{C_1, C_2, C_4\}.$$

Observe that C_9 is positive and independent on μ. Next, using Cauchy-Schwarz's and Young's inequalities for estimate the last term in the right hand in (3.24). Then, by (2.2) and (2.4), we get

$$\begin{aligned}
&\epsilon\left\langle Cz(1,t), N_1\int_0^{+\infty} h(s)\eta^t(s)ds - N_2 u\right\rangle \\
\leq\ & \epsilon\|z(1,t)\|^2 + \epsilon|\mu|C_{10}\left(\left\|A^{\frac{1}{2}}u\right\|^2 + \int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds\right),
\end{aligned}$$

where

$$C_{10} = \frac{1}{2b}\max\{aN_2^2, h_0 N_1^2\}.$$

Inserting the above inequality and (3.26) in (3.24), we obtain

$$\begin{aligned} L'(t) \quad \leq \quad & -\epsilon C_{11}E(t) + \left(\frac{4h_n + \sqrt{dh_n}}{2}\right)\epsilon N_1 E(t) \\ & + \left(\frac{1}{2} - \epsilon C_6\right) \int_0^{+\infty} h'(s) \left\| B^{\frac{1}{2}} \eta^t(s) \right\|^2 ds - C_7 \xi(t)(J_1(t) + J_2(t)) \\ & + C_8 \int_t^{+\infty} h(s) \left\| B^{\frac{1}{2}} u_0(s-t) \right\|^2 ds, \end{aligned} \tag{3.26}$$

where

$$C_{11} = 2 \min \left\{ C_9 - \frac{|\mu|}{\epsilon}, \frac{2}{|\mu|}, C_9 - \epsilon|\mu|C_{10} \right\}.$$

Finally, we assume that $|\mu|$ satisfies (3.20) under the following choice of δ_0

$$\delta_0 = \min \left\{ \frac{C_9}{C_6}, \frac{C_9\sqrt{2}}{\sqrt{C_{10}}} \right\}. \tag{3.27}$$

Then, we can choose n big enough and we fix ϵ such that

$$\frac{|\mu|}{2C_9} < \epsilon \leq \frac{1}{2C_6} < \frac{1}{M}, \tag{3.28}$$

where

$$M = N_1 \max\left\{1, \frac{h_0}{b}\right\} + N_2 \max\left\{1, \frac{a}{b}\right\} + \frac{2e^{2\tau}}{|\mu|}.$$

which imply that E is equivalent to $E + \epsilon(N_1 I_1 + N_2 I_2 + I_3)$. Indeed, by using Cauchy-Schwarz's and Young's inequalities, we have

$$|I_1(t)| \quad \leq \quad \frac{1}{2}\left(\|u_t\|^2 + \frac{h_0}{b} \int_0^{+\infty} h(s) \left\| B^{\frac{1}{2}} \eta^t(s) \right\|^2 ds \right) \tag{3.29}$$

$$\leq \quad \max\left\{1, \frac{h_0}{b}\right\} E(t) \tag{3.30}$$

and

$$|I_2(t)| \quad \leq \quad \frac{1}{2}\left(\|u_t\|^2 + \frac{a}{b} \left\| A^{\frac{1}{2}} u \right\|^2 \right) \leq \max\left\{1, \frac{a}{b}\right\} E(t). \tag{3.31}$$

From (3.12), it follows

$$|I_3(t)| = \tau e^{2\tau} \int_0^1 e^{-2\tau\rho} \|z(\rho,t)\|^2 ds \leq \tau e^{2\tau} \int_0^1 e^{-2\tau\rho} \|z(\rho,t)\|^2 ds \leq \frac{2e^{2\tau}}{|\mu|} E(t) \tag{3.32}$$

Combining (3.4), (3.29), (3.31) and (3.32) and by using (3.28), we have

$$E \sim E + \epsilon(N_1 I_1 + N_2 I_2 + I_3).$$

Moreover, the third term in the right hand of (3.26) is non-positive. Note that δ_0 is a positive constant independent of μ. Under the condition (3.20), we conclude that C_{11} is a positive constant and by using the fact that $\lim_{n\to+\infty} h_n = 0$, we get

$$C_{12} = \epsilon C_{11} + \left(\frac{4h_n + \sqrt{dh_n}}{2}\right)\epsilon N_1 > 0.$$

Consequently, we obtain, for all $t \in \mathbb{R}_+$,

$$\begin{aligned} L'(t) &\leq -C_{12}E(t) - C_7\xi(t)\Big(J_1(t) + J_2(t)\Big) \\ &\quad +C_8 \int_t^{+\infty} h(s) \left\|B^{\frac{1}{2}}u_0(s-t)\right\|^2 ds. \end{aligned} \tag{3.33}$$

Let distinguish two cases corresponding to the limit of ξ at infinity.
► If $\lim_{t\to+\infty} \xi(t) > 0$, there exist $t_0 \geq 0$ and $\xi_0 > 0$ such that $\xi(t) \geq \xi_0$, for all $t \geq t_0$. Therefore, using (3.18), we find

$$L'(t) \leq -\delta_1 L(t) + C_8 \int_t^{+\infty} h(s) \left\|B^{\frac{1}{2}}u_0(s-t)\right\|^2 ds, \quad \forall t \in \mathbb{R}_+, \tag{3.34}$$

where

$$\delta_1 = \min\left\{\frac{C_{12}}{1+\epsilon M}, \frac{C_7\xi_0}{M_1}, \frac{C_7\xi_0}{aM_1}\right\}.$$

Then, integrating the differential inequality (3.34) over $[t_0, t]$, we obtain

$$L(t) \leq e^{-\delta_1 t}\left(e^{\delta_1 t_0}L(t_0) + C_7\int_0^t e^{\delta_1 s}\int_s^{+\infty} h(\pi)\left\|B^{\frac{1}{2}}u_0(\pi - s)\right\|^2 d\pi ds\right),\ \forall t \in \mathbb{R}_+.$$

So, using (3.18) and (3.34), we get, for all $t \geq t_0$,

$$\begin{aligned} E(t) &\leq \frac{1}{1-\epsilon M}L(t) \\ &\leq \frac{1}{1-\epsilon M}\max\left\{C_7, e^{\delta_1 t_0}L(t_0)\right\}\times \\ &\quad \times\left(1 + \int_0^t e^{\delta_1 s}\int_s^{+\infty} h(\pi)\left\|B^{\frac{1}{2}}u_0(\pi - s)\right\|^2 d\pi ds\right). \end{aligned} \tag{3.35}$$

For $t \in [0, t_0]$, we have

$$E(t) \leq \frac{1}{1-\epsilon M}L(t)e^{\delta_1 t}e^{-\delta_1 t} \leq \frac{1}{1-\epsilon M}\max_{s\in[0,t_0]} L(s)e^{\delta_1 t_0}e^{-\delta_1 t}. \tag{3.36}$$

Inequalities (3.35) and (3.36) gives (3.21) with

$$\delta_2 = \frac{1}{1-\epsilon M}\left\{C_7, e^{\delta_1 t_0}\max_{s\in[0,t_0]} L(s)\right\}.$$

► If $\lim_{t\to+\infty} \xi(t) = 0$, there exist $t_0 \geq 0$ such that $\xi(t) \leq C_{12}$, for all $t \geq t_0$. Therefore, using (3.18), we obtain, for

$$\delta_1 = \min\left\{\frac{1}{1+\epsilon M}, \frac{C_7}{M_1}, \frac{C_7}{aM_1}\right\},$$

$$L'(t) \leq -\delta_1\xi(t)L(t) + C_8 \int_t^{+\infty} h(s) \left\|B^{\frac{1}{2}}u_0(s-t)\right\|^2 ds, \quad \forall t \in \mathbb{R}_+, \tag{3.37}$$

By integrating the above differential inequality over $[t_0, t]$, we get, for all $t \in \mathbb{R}_+$,

$$L(t) \leq e^{-\delta_1\hat{\xi}(t)}\left(e^{\delta_1\hat{\xi}(t_0)}L(t_0) + C_7\int_0^t e^{\delta_1\hat{\xi}(s)}\int_s^{+\infty} h(\pi)\left\|B^{\frac{1}{2}}u_0(\pi - s)\right\|^2 d\pi ds\right).$$

Then, using (3.18) and (3.37), we get, for all $t \geq t_0$,

$$\begin{aligned} E(t) \quad &\leq \quad \frac{1}{1-\epsilon M} \max\left\{C_7, e^{\delta_1 \hat{\xi}(t_0)} L(t_0)\right\} \times \\ &\times\left(1 + \int_0^t e^{\delta_1 s} \int_s^{+\infty} h(\pi) \left\|B^{\frac{1}{2}} u_0(\pi - s)\right\|^2 d\pi ds\right). \end{aligned} \tag{3.38}$$

For $t \in [0, t_0]$, we have

$$E(t) \leq \frac{1}{1-\epsilon M} L(t) e^{\delta_1 \hat{\xi}(t)} e^{-\delta_1 \hat{\xi}(t)} \leq \frac{1}{1-\epsilon M} \max_{s\in[0,t_0]} \left(L(s) e^{\delta_1 \xi(\hat{s})}\right) e^{-\delta_1 \hat{\xi}(t)}. \tag{3.39}$$

Inequalities (3.38) and (3.39) gives (3.21) with

$$\delta_2 = \frac{1}{1-\epsilon M}\left\{C_7, \max_{s\in[0,t_0]} \left(L(s) e^{\delta_1 \xi(\hat{s})}\right)\right\}.$$

Thus the proof of Theorem 3.6 is completed. □

4. Applications

We can seek our results in some problems. In this section, we consider only three illustrative problems. In the whole section, Ω is a bounded and regular domain of $\mathbb{R}^n$, with $n \geq 1$.

1-: Abstract linear problem

$$\begin{cases} u_{tt}(t) + Au(t) - \int_0^{+\infty} h(s) Au(t-s) ds + C u_t(t-\tau) = 0, & t \in (0, +\infty), \\ u_t(t-\tau) = f_0(t-\tau), & t \in (0, \tau), \\ u(-t) = u_0(t), \quad u_t(0) = u_1, & t \geq 0, \end{cases} \tag{4.1}$$

where the operators A and C are a self-adjoint linear positive operators satisfy the assumptions (A1) and (A3), respectively. The memory kernel h satisfying (A2) and (A7).

2-: Let us consider the semilinear problem

$$\begin{cases} u_{tt}(t) + Au(t) + \int_0^{+\infty} h(s) \Delta u(t-s) ds + b(x) u_t(t-\tau) & \\ \qquad\qquad = F(u(t)), & t \in (0, +\infty), \\ u(x,t) = 0, & x \in \partial\Omega, \\ u(x,-t) = u_0(x,t), \quad u_t(x,0) = u_1(x), & x \in \Omega,\ t \geq 0, \\ u_t(t-\tau) = f_0(t-\tau) & t \in (0, \tau), \end{cases} \tag{4.2}$$

with initial data $(u_0, u_1, f_0) \in [H^2(\Omega) \in \cap H_0^1(\Omega)] \times H_0^1(\Omega) \times H^1(0, \tau; L^2(\Omega))$. The constant $\beta > 0$ satisfies a suitable restriction to be specified below. The memory kernel h satisfying (A2) and (A7) and $b \in L^\infty(\Omega)$ is a function such that

$$b(x) \geq 0 \quad \text{a. e. in} \quad \Omega.$$

The source term F be globally Lipschitz continuous functional such that $F(0) = 0$ and satisfies (3.2).Our results hold with $H = L^2(\Omega)$ and the operators A, B are given by

$$A : D(A) \longrightarrow H \ : u \mapsto -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u}{\partial x_j}\right), \quad B : D(B) \longrightarrow H \ : u \mapsto -\Delta u,$$

where $D(A) = D(B)) = H^2(\Omega) \cap H_0^1(\Omega)$. $a_{ij} \in C^1(\overline{\Omega})$, is symmetric and

$$\exists a_0 > 0, \quad \sum_{i,j=1}^{n} a_{ij}(x)\zeta_j\zeta_i \geq a_0|\zeta|^2, \quad x \in \overline{\Omega},\ \zeta = (\zeta_1, \cdots, \zeta_n) \in \mathbb{R}^n.$$

The operators A and B are a linear, self-adjoint and positive operators in H such that $D(A^{\frac{1}{2}}) = H_0^1(\Omega)$ with $\left\|A^{\frac{1}{2}}u\right\| = (a(u,u))^{1/2}$ and $\left\|B^{\frac{1}{2}}u\right\| = \|\nabla u\|_2$, where

$$a(u,u) = \sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx.$$

Moreover, by using Poincare's inequality and the Sobolev's embedding theorem, we get (A1) and (A5). Then, the assumption (A3) holds with $Cu(x,t) = b(x)u(x,t)$.

3-: Coupled systems

$$\begin{cases} w_{tt}(t) - \alpha\Delta w(t) + \displaystyle\int_0^{+\infty} h(s) div(a_1(x)\nabla w(t-s))ds & \\ \qquad +\mu w_t(t-\tau) + dv(t) = f_1(w(t)), & t \in (0,+\infty), \\ v_{tt}(t) - \beta\Delta v(t) + \displaystyle\int_0^{+\infty} h(s) div(a_2(x)\nabla v(t-s))ds & \\ \qquad +\mu v_t(t-\tau) + dw(t) = f_2(v(t)), & t \in (0,+\infty), \\ w(x,t) = v(x,t) = 0, & x \in \partial\Omega, \\ w(x,-t) = w_0(x,t), \quad v(x,-t) = v_0(x,t), & x \in \Omega,\ t \geq 0, \\ w_t(x,0) = w_1(x), \quad v_t(x,0) = v_1(x), & x \in \Omega,\ t \geq 0, \\ w_t(t-\tau) = f_0(t-\tau), \quad v_t(t-\tau) = f_0(t-\tau), & t \in (0,\tau), \end{cases} \tag{4.3}$$

where α and β are positive constants, $a_1,\ a_2 \in C^1(\Omega),\ a_1(x),\ a_2(x) > 0$ with The memory kernel h satisfying (A2) and (A7). The above system is equivalent to (1.1) where $u = (w,v)$, $f_0 = (l_0, m_0)$ and $H = (L^2(\Omega))^2$ with

$$\langle (w_1,v_1),(w_2,v_2)\rangle = \int_{\Omega} w_1w_2 + v_1v_2 dx.$$

We take $D(A) = D(B)) = (H^2(\Omega) \cap H_0^1(\Omega))^2$ and the operators A, B are given by

$$Au = -(\alpha\Delta w, \beta\Delta v) + d(v,w),$$
$$Bu = -(div(a_1(x)\nabla w), div(a_2(x)\nabla w)).$$

The function $F_2(u(t)) = (f_1(w(t)), f_2(v(t)))$ satisfies (A6).

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