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COMMON FIXED POINTS OF GERAGHTY-SUZUKI TYPE CONTRACTION MAPS IN b-METRIC SPACES

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Abstract. In this paper, we prove the existence and uniqueness of common fixed points for two pairs of selfmaps satisfying a Geraghty-Suzuki type contraction condition in which one pair is compatible, b-continous and the another one is weakly compatible in complete b-metric spaces. Further, we prove the same with different hypotheses on two pairs of selfmaps which satisfy b-(E.A)-property. We draw some corollaries from our results and provide examples in support of our results.

1. Introduction

The development of fixed point theory is based on the generalization of contraction conditions in one direction or/and generalization of ambient spaces of the operator under consideration on the other. Banach contraction principle plays an important role in solving nonlinear equations, and it is one of the most useful results in fixed point theory. In the direction of generalization of contraction conditions, in 1973, Geraghty [17] proved a fixed point theorem, generalizing Banach contraction principle. In 1975, Dass and Gupta [14] extended contraction map to contraction map with rational expression and proved the existence of fixed points in complete metric spaces. In 2008, Suzuki [30] proved two fixed point theorems, one of which is a new type of generalization of the Banach contraction principle and does characterize the metric completeness.

The main idea of b-metric was initiated from the works of Bourbaki [10] and Bakhtin [6]. The concept of b-metric space or metric type space was introduced by Czerwik [12] as a generalization of metric space. Afterwards, many authors studied fixed point theorems for single-valued and multi-valued mappings in b-metric spaces, we refer [2, 3, 8, 9, 13, 22, 28, 29].

In 2002, Aamari and Moutawakil [1] introduced the notion of property (E.A). Different authors applied this concept to prove the existence of common fixed points, we refer [4, 5, 25, 26, 27].

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We denote $\mathbb{N}$, the set of all natural numbers and $\mathbb{R}^+ = [0, \infty)$.

Definition 1.1. [12] *Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric if the following conditions are satisfied: for any $x, y, z \in X$*

(b_1) $0 \leq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,
(b_2) $d(x, y) = d(y, x)$,
(b_3) there exists $s \geq 1$ such that $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space with coefficient s.

Every metric space is a b-metric space with $s = 1$. In general, every b-metric space is not a metric space.

Definition 1.2. [9] *Let (X, d) be a b-metric space and $\{x_n\}$ a sequence in X.*

(i) A sequence $\{x_n\}$ in X is called b-convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim\limits_{n\to\infty} x_n = x$.
(ii) A sequence $\{x_n\}$ in X is called b-Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.
(iii) A b-metric space (X, d) is said to be a complete b-metric space if every b-Cauchy sequence in X is b-convergent.
(iv) A set $B \subset X$ is said to be b-closed if for any sequence $\{x_n\}$ in B such that $\{x_n\}$ is b-convergent to $z \in X$ then $z \in B$.

In general, a b-metric is not necessarily continuous.

Example 1.1. [19] *Let $X = \mathbb{N} \cup \{\infty\}$. We define a mapping $d : X \times X \to \mathbb{R}^+$ as follows:*

$$d(m, n) = \begin{cases} 0 & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}| & \text{if one of } m, n \text{ is even and the other is even or } \infty, \\ 5 & \text{if one of } m, n \text{ is odd and the other is odd or } \infty, \\ 2 & \text{otherwise.} \end{cases}$$

Then (X, d) is a b-metric space with coefficient $s = \frac{5}{2}$.

Definition 1.3. [9] *Let (X, d_X) and (Y, d_Y) be two b-metric spaces. A function $f : X \to Y$ is a b-continuous at a point $x \in X$, if it is b-sequentially continuous at x. i.e., whenever $\{x_n\}$ is b-convergent to x, fx_n is b-convergent to fx.*

Definition 1.4. [20] *A pair (A, B) of selfmaps on a metric space (X, d) is said to be compatible if $\lim\limits_{n\to\infty} d(BAx_n, ABx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Bx_n = z$ for some $z \in X$.*

Definition 1.5. [1] *A pair (A, B) of selfmaps on a metric space (X, d) is said to be satisfy (E.A)-property if there exists a sequence $\{x_n\}$ in X such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Bx_n = z$ for some $z \in X$.*

Definition 1.6. [25] *A pair (A, B) of selfmaps on a b-metric space (X, d) is said to be satisfy b-(E.A)-property if there exists a sequence $\{x_n\}$ in X such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Bx_n = z$ for some $z \in X$.*

Definition 1.7. [21] *A pair (A, B) of selfmaps on a set X is said to be weakly compatible if $ABx = BAx$ whenever $Ax = Bx$ for any $x \in X$.*

In 1973, Geraghty [17] introduced a class of functions
$\mathfrak{S} = \{\beta : [0, \infty) \to [0, 1) / \lim\limits_{n\to\infty} \beta(t_n) = 1 \implies \lim\limits_{n\to\infty} t_n = 0\}$.

Theorem 1.1. [17] *Let (X,d) be a complete metric space. Let $T : X \to X$ be a selfmap satisfying the following: there exists $\beta \in \mathfrak{S}$ such that*
$d(Tx,Ty) \leq \beta(d(x,y))d(x,y)$ for all $x,y \in X$. Then T has a unique fixed point.

We denote $\mathfrak{B} = \{\alpha : [0,\infty) \to [0,\frac{1}{s})/ \lim\limits_{n\to\infty} \alpha(t_n) = \frac{1}{s} \implies \lim\limits_{n\to\infty} t_n = 0\}$.

In 2011, Dukic, Kadelburg and Radenović [15] extended Theorem 1.9 to the case of b-metric spaces as follows.

Theorem 1.2. [15] *Let (X,d) ba a complete b-metric space with coefficient $s \geq 1$ and let $T : X \to X$ be a selfmap of X. Suppose that there exists $\alpha \in \mathfrak{B}$ such that $d(Tx,Ty) \leq \alpha(d(x,y))d(x,y)$ for all $x,y \in X$. Then T has a unique fixed point in X.*

The following lemmas are useful in proving our main results.

Lemma 1.3. [18] *Let (X,d) be a b-metric space with coefficient $s \geq 1$. Suppose that $\{x_n\}$ is a sequence in X such that $d(x_n,x_{n+1}) \leq kd(x_{n-1},x_n)$ for all $n \in \mathbb{N}$, where $k \in [0,1)$ is a constant. Then $\{x_n\}$ is a b-Cauchy sequence in X.*

Lemma 1.4. [2] *Let (X,d) be a b-metric space with coefficient $s \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x and y respectively, then we have*
$$\frac{1}{s^2}d(x,y) \leq \liminf_{n\to\infty} d(x_n,y_n) \leq \limsup_{n\to\infty} d(x_n,y_n) \leq s^2 d(x,y).$$
In particular, if $x = y$, then we have $\lim\limits_{n\to\infty} d(x_n,y_n) = 0$. Moreover for each $z \in X$ we have
$$\frac{1}{s}d(x,z) \leq \liminf_{n\to\infty} d(x_n,z) \leq \limsup_{n\to\infty} d(x_n,z) \leq sd(x,z).$$

In 2015, Latif, Parvaneh, Salimi and Al-Mazrooei [23] proved the existence and uniqueness of fixed points of a single selfmap satisfying Suzuki type contraction condition in b-metric spaces as follows.

Theorem 1.5. [23] *Let (X,d) be a complete b-metric space (with parameter $s > 1$) and let $f : X \to X, \alpha : X \times X \to [0,\infty)$ satisfying*
(a) $\alpha(x,y) \geq 1 \implies \alpha(fx,fy) \geq 1$,
(b) $\alpha(x,z) \geq 1, \alpha(z,y) \geq 1 \implies \alpha(x,y) \geq 1, x,y,z \in X$. Suppose that $\beta \in \mathfrak{B}$ such that $\frac{1}{2s}d(x,fx) \leq d(x,y) \implies s\alpha(x,y)d(fx,fy) \leq \beta(M(x,y))M(x,y)$ for all $x,y \in X$, where
$M(x,y) = \max\{d(x,y), \frac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{1+s[d(x,y)+d(fx,fy)]}, \frac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{1+d(x,fy)+d(y,fx)}\}$.
Also, suppose that the following assertions hold:
(i) there exists $x_0 \in X$such that $\alpha(x_0,fx_0) \geq 1$;
(ii) for any sequence $\{x_n\}$ in X with $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$ such that $x_n \to x$ as $n \to \infty$, we have $\alpha(x_n,x) \geq 1$ for all $n \in \mathbb{N}\cup\{0\}$.
Then, f has a fixed point.

The set $\{x_0, fx_0, f^2x_0, f^3x_0, \ldots\}$ is called an orbit of f at the point x_0 and is denoted by $O_f(x_0)$ [7].

Definition 1.8. [11] *A b-metric space X is said to be f-orbitally complete if every Cauchy sequence in $O_f(x_0)$ converges in X, where f is a selfmapping on X and $x_0 \in X$.*

Definition 1.9. [24] *Let X be any nonempty set and $\alpha : X \times X \to \mathbb{R}$. A selfmap $f : X \to X$ is said to have a property (H), if for any $x,y \in X$ with $x \neq y$, there exists $z \in X$ such that $\alpha(x,z) \geq 1, \alpha(y,z) \geq 1$ and $\alpha(z,fz) \geq 1$.*

Definition 1.10. [24] *Let (X,d) be a b-metric space with parameter $s \geq 1$ and $\alpha : X \times X \to \mathbb{R}$. A selfmap $f : X \to X$ is called a generalized α-Suzuki-Geraghty contraction if there exists a $\beta \in \mathfrak{B}$ such that for any $x, y \in X$,*
$\frac{1}{2s}d(x,fx) \leq sd(x,y) \implies d(fx,fy) \leq \beta(M(x,y))M(x,y),$
where
$M(x,y) = \max\{d(x,y), d(x,fx), d(y,fy), d(f^2x,fx), d(f^2x,y), \frac{d(f^2x,fy)}{s}, \frac{d(f^2x,x)}{2s}, \frac{d(x,fy)+d(y,fx)}{2s}, \frac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{1+s[d(x,y)+d(fx,fy)]}, \frac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{1+d(x,fy)+d(y,fx)}\}.$

Theorem 1.6. [24] *Let (X,d) be a complete b-metric space with parameter $s \geq 1, \alpha : X \times X \to \mathbb{R}$ and $f : X \to X$. Assume that X is f-orbitally complete and the following conditions hold:*
(i) there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;
(ii) f is a generalized $\alpha - Suzuki - Geraghty$ contraction and a triangular α-orbital admissible;
(iii) either f is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ such that $x_n \to x$ as $n \to \infty$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.
Then f has a fixed point z in X and $\{f^n x_0\}$ converges to z. Moreover, f has a unique fixed point if condition (i) is replaced with the property (H).

Throughout this paper we denote
$\mathfrak{F} = \{\beta : [0,\infty) \to [0, \frac{1}{s}) / \limsup_{n\to\infty} \beta(t_n) = \frac{1}{s} \implies \lim_{n\to\infty} t_n = 0\}.$
In 2019, Faraji, Savić and Radenović [16] proved the following theorem.

Theorem 1.7. [16] *Let (X,d) be a complete b-metric space with parameter $s \geq 1$. Let $T, S : X \to X$ be selfmaps on X which satisfy: there exists $\beta \in \mathfrak{F}$ such that*
$$sd(Tx, Sy) \leq \beta(M(x,y))M(x,y) \text{ for all } x, y \in X,$$
where $M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Sy)\}$.
If either T or S is continuous, then T and S have a unique common fixed point.

Motivated by Theorem 1.5 and Theorem 1.6, in Section 2 of this paper, we prove the existence and uniqueness of common fixed points for two pairs of selfmaps satisfying a Geraghty-Suzuki type contraction condition in which one pair is compatible, b-continous and the another one is weakly compatible in complete b-metric spaces. Further, we prove the same with different hypotheses on two pairs of selfmaps which satisfy b-(E.A)-property. In Section 3, we draw some corollaries and examples in support of our results.

2. Main Results

Let A, B, S and T be mappings from a b-metric space (X,d) into itself and satisfying
$$A(X) \subseteq T(X) \text{ and } B(X) \subseteq S(X). \tag{2.1}$$
Now by (2.1), for any $x_0 \in X$, there exists $x_1 \in X$ such that $y_0 = Ax_0 = Tx_1$. In the same way for this x_1, we can choose a point $x_2 \in X$ such that $y_1 = Bx_1 = Sx_2$ and so on. In general, we define
$$y_{2n} = Ax_{2n} = Tx_{2n+1} \text{ and } y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \text{ for } n = 0, 1, 2, \ldots. \tag{2.2}$$

Proposition 2.1. *Let (X,d) be a b-metric space wuth coefficient $s \geq 1$. Assume that A, B, S and T are selfmappings of X which satisfy the following condition:*

there exists $\beta \in \mathfrak{F}$ *such that*

$$\frac{1}{2s}\min\{d(Sx,Ax),d(Ty,By)\} \leq \max\{d(Sx,Ty),d(Ax,By)\} \implies s^4 d(Ax,By) \leq \beta(M(x,y))M(x,y) \tag{2.3}$$

where

$$M(x,y) = \max\{d(Sx,Ty),d(Sx,Ax),d(Ty,By),\tfrac{d(Sx,By)}{2s},\tfrac{d(Ty,Ax)}{2s}, \tfrac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)},\tfrac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\},$$

for all $x, y \in X$. *Then we have the following:*

- (i) *If* $A(X) \subseteq T(X)$ *and the pair* (B,T) *is weakly compatible and if* z *is a common fixed point of* A *and* S *then* z *is a common fixed point of* A, B, S *and* T *and it is unique.*
- (ii) *If* $B(X) \subseteq S(X)$ *and the pair* (A,S) *is weakly compatible and if* z *is a common fixed point of* B *and* T *then* z *is a common fixed point of* A, B, S *and* T *and it is unique.*

Proof. First, we assume that (i) holds. Let z be a common fixed point of A and S. Then $Az = Sz = z$. Since $A(X) \subseteq T(X)$, there exists $u \in X$ such that $Tu = z$. Therefore $Az = Sz = Tu = z$.
We now prove that $Az = Bu$. Suppose that $Az \neq Bu$.
Since $\frac{1}{2s}\min\{d(Sz,Az),d(Tu,Bu)\} \leq \max\{d(Sz,Tu),d(Az,Bu)\}$.
From the inequality (2.3), we have

$$s^4 d(Az,Bu) \leq \beta(M(z,u))M(z,u) \tag{2.4}$$

where

$$\begin{aligned} M(z,u) &= \max\{d(Sz,Tu),d(Sz,Az),d(Tu,Bu),\tfrac{d(Sz,Bu)}{2s},\tfrac{d(Tu,Az)}{2s}, \tfrac{d(Sz,Az)d(Tu,Bu)}{1+d(Sz,Tu)+d(Az,Bu)},\tfrac{d(Sz,Bu)d(Tu,Az)}{1+s^4[d(Sz,Tu)+d(Az,Bu)]}\} \\ &= \max\{0,0,d(Az,Bu),\tfrac{d(Az,Bu)}{2s},0,0,0\} = d(Az,Bu). \end{aligned}$$

From the inequality (2.4), we have
$s^4 d(Az,Bu) \leq \beta(d(z,u))d(z,u) \leq \frac{d(Az,Bu)}{s}$ so that $(s^5-1)d(Az,Bu) \leq 0$.
Since $(s^5-1) \geq 0$, it follows that $d(Az,Bu) = 0$.
Hence $Az = Bu$. Therefore $Az = Bu = Sz = Tu = z$.
Since the pair (B,T) is weakly compatible and $Bu = Tu$, we have $BTu = TBu$. i.e., $Bz = Tz$.
Now we show that $Bz = z$.
If $Bz \neq z$, then we have
$\frac{1}{2s}\min\{d(Sz,Az),d(Tz,Bz)\} \leq \max\{d(Sz,Tz),d(Az,Bz)\}$
From the inequality (2.3), we have

$$s^4 d(z,Bz) = s^4 d(Az,Bz) \leq \beta(M(z,z))M(z,z) \tag{2.5}$$

where

$$\begin{aligned} M(z,z) &= \max\{d(Sz,Tz),d(Sz,Az),d(Tz,Bz),\tfrac{d(Sz,Bz)}{2s},\tfrac{d(Tz,Az)}{2s}, \tfrac{d(Sz,Az)d(Tz,Bz)}{1+d(Sz,Tz)+d(Az,Bz)},\tfrac{d(Sz,Bz)d(Tz,Az)}{1+s^4[d(Sz,Tz)+d(Az,Bz)]}\} \\ &= \max\{d(z,Bz),0,0,\tfrac{d(z,Bz)}{2s},\tfrac{d(z,Bz)}{2s},0,\tfrac{[d(z,Bz)]^2}{1+2s^4[d(z,Bz)]}\} = d(z,Bz). \end{aligned}$$

From the inequality (2.5), we have
$s^4 d(z,Bz) \leq \beta(M(z,z))M(z,z) = \beta(d(z,Bz))d(z,Bz) \leq \frac{d(z,Bz)}{s}$ so that $(s^5-1)d(z,Bz) \leq 0$.
Since $(s^5-1) \geq 0$, it follows that $d(z,Bz) = 0$.

Hence $Bz = z$. Therefore $Az = Bz = Sz = Tz = z$.
Therefore z is a common fixed point of A, B, S and T.

In a similar way, under the assumption (ii), the conclusion of the proposition follows.

Uniqueness follows from the inequality (2.3). $\square$

Remark. *Selfmaps A, B, S and T of a b-metric space X that satisfy* (2.3) *is said to be Geraghty-Suzuki type contraction maps on X.*

Proposition 2.2. *Let A, B, S and T be selfmaps of a b-metric space (X, d) and satisfy* (2.1) *and Geraghty-Suzuki type contraction maps. Then for any $x_0 \in X$, the sequence $\{y_n\}$ defined by* (2.2) *is b-Cauchy in X.*

Proof. Let $x_0 \in X$ and let $\{y_n\}$ be defined by (2.2). Assume that $y_n = y_{n+1}$ for some n.
Case (i): n even.
We write $n = 2m$ for some $m \in \mathbb{N}$. Suppose that $d(y_{n+1}, y_{n+2}) > 0$. Since

$$\frac{1}{2s}\min\{d(Sx_{2m+2}, Ax_{2m+2}), d(Tx_{2m+1}, Bx_{2m+1})\} \leq \max\{d(Sx_{2m+2}, Tx_{2m+1}), \\ d(Ax_{2m+2}, Bx_{2m+1})\}$$

From the inequality (2.3), we have

$$\begin{aligned} s^4 d(y_{n+1}, y_{n+2}) &= s^4 d(y_{2m+1}, y_{2m+2}) \\ &= s^4 d(y_{2m+2}, y_{2m+1}) \\ &= s^4 d(Ax_{2m+2}, Bx_{2m+1}) \\ &\leq \beta(M(x_{2m+2}, x_{2m+1}))M(x_{2m+2}, x_{2m+1}) \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} M(x_{2m+2}, x_{2m+1}) = \max\{&d(Sx_{2m+2}, Tx_{2m+1}), d(Sx_{2m+2}, Ax_{2m+2}), \\ &d(Tx_{2m+1}, Bx_{2m+1}), \frac{d(Sx_{2m+2}, Bx_{2m+1})}{2s}, \frac{d(Tx_{2m+1}, Ax_{2m+2})}{2s}, \\ &\frac{d(Sx_{2m+2}, Ax_{2m+2})d(Tx_{2m+1}, Bx_{2m+1})}{1+d(Sx_{2m+2}, Tx_{2m+1})+d(Ax_{2m+2}, Bx_{2m+1})}, \\ &\frac{d(Sx_{2m+2}, Bx_{2m+1})d(Tx_{2m+1}, Ax_{2m+2})}{1+s^4[d(Sx_{2m+2}, Tx_{2m+1})+d(Ax_{2m+2}, Bx_{2m+1})]}\} \\ = \max\{&0, d(y_{n+1}, y_{n+2}), 0, 0, \frac{d(y_n, y_{n+2})}{2s}, 0, 0\} = d(y_{n+1}, y_{n+2}). \end{aligned}$$

From the inequality (2.6), we have

$$\begin{aligned} s^4 d(y_{n+1}, y_{n+2}) &\leq \beta(M(x_{2m+2}, x_{2m+1}))M(x_{2m+1}, x_{2m+1}) \\ &\leq \beta(d(y_{n+1}, y_{n+2}))d(y_{n+1}, y_{n+2}) \leq \frac{d(y_{n+1}, y_{n+2})}{s} \end{aligned}$$

which implies that $(s^5 - 1)d(y_{n+1}, y_{n+2}) \leq 0$.
Since $(s^5 - 1) \geq 0$, we have $d(y_{n+1}, y_{n+2}) \leq 0$.
Therefore $y_{n+2} = y_{n+1} = y_n$.
In general, we have $y_{n+k} = y_n$ for $k = 0, 1, 2, \ldots$.
Case (ii): n odd.
We write $n = 2m + 1$ for some $m \in \mathbb{N}$.
Since

$$\frac{1}{2s}\min\{d(Sx_{2m+2}, Ax_{2m+2}), d(Tx_{2m+3}, Bx_{2m+3})\} \leq \max\{d(Sx_{2m+2}, Tx_{2m+3}), \\ d(Ax_{2m+2}, Bx_{2m+3})\},$$

from the inequality (2.3), we have

$$\begin{aligned} s^4 d(y_{n+1}, y_{n+2}) = s^4 d(y_{2m+2}, y_{2m+3}) &= d(Ax_{2m+2}, Bx_{2m+3}) \\ &\leq \beta(M(x_{2m+2}, x_{2m+3}))M(x_{2m+2}, x_{2m+3}) \end{aligned} \tag{2.7}$$

where

$$M(x_{2m+2}, x_{2m+3}) = \max\{d(Sx_{2m+2}, Tx_{2m+3}), d(Sx_{2m+2}, Ax_{2m+2}),$$

$$d(Tx_{2m+3},Bx_{2m+3}),\frac{d(Sx_{2m+2},Bx_{2m+3})}{2s},\frac{d(Tx_{2m+3},Ax_{2m+2})}{2s},$$
$$\frac{d(Sx_{2m+2},Ax_{2m+2})d(Tx_{2m+3},Bx_{2m+3})}{1+d(Sx_{2m+2},Tx_{2m+3})+d(Ax_{2m+2},Bx_{2m+3})},$$
$$\frac{d(Sx_{2m+2},Bx_{2m+3})d(Tx_{2m+3},Ax_{2m+2})}{1+s^4[d(Sx_{2m+2},Tx_{2m+3})+d(Ax_{2m+2},Bx_{2m+3})]}\}$$
$$=\max\{0,0,d(y_{n+1},y_{n+2}),\frac{d(y_n,y_{n+2})}{2s},0,0,0\}=d(y_{n+1},y_{n+2}).$$

From the inequality (2.7), we have

$$s^4d(y_{n+1},y_{n+2})\leq\beta(M(x_{2m+2},x_{2m+3}))M(x_{2m+2},x_{2m+3})$$
$$\leq\beta(d(y_{n+1},y_{n+2}))d(y_{n+1},y_{n+2})\leq\frac{d(y_{n+1},y_{n+2})}{s}$$

which implies that $(s^5-1)d(y_{n+1},y_{n+2})\leq 0$.
Since $(s^5-1)\geq 0$, we have $d(y_{n+1},y_{n+2})\leq 0$.
Therefore $y_{n+2}=y_{n+1}=y_n$.
In general, we have $y_{n+k}=y_n$ for $k=0,1,2,\ldots$.
From Case (i) and Case (ii), we have $y_{n+k}=y_n$ for all $k=0,1,2,\ldots$.
Hence $\{y_{n+k}\}$ is a constant sequence and hence $\{y_n\}$ is Cauchy.
Now we assume that $y_{n-1}\neq y_n$ for all $n\in\mathbb{N}$.
If n is odd, then $n=2m+1$ for some $m\in\mathbb{N}$.
Since

$$\frac{1}{2s}\min\{d(Sx_{2m+2},Ax_{2m+2}),d(Tx_{2m+1},Bx_{2m+1})\}\leq\max\{d(Sx_{2m+2},Tx_{2m+1}),$$
$$d(Ax_{2m+2},Bx_{2m+1})\}.$$

From the inequality (2.3), we have

$$s^4d(y_n,y_{n+1})=s^4d(y_{2m+1},y_{2m+2})=s^4d(y_{2m+2},y_{2m+1})$$
$$=s^4d(Ax_{2m+2},Bx_{2m+1})\leq\beta(M(x_{2m+2},x_{2m+1}))M(x_{2m+2},x_{2m+1}) \tag{2.8}$$

where

$$M(x_{2m+2},x_{2m+1})=\max\{d(Sx_{2m+2},Tx_{2m+1}),d(Sx_{2m+2},Ax_{2m+2}),$$
$$d(Tx_{2m+1},Bx_{2m+1}),\frac{d(Sx_{2m+2},Bx_{2m+1})}{2s},\frac{d(Tx_{2m+1},Ax_{2m+2})}{2s},$$
$$\frac{d(Sx_{2m+2},Ax_{2m+2})d(Tx_{2m+1},Bx_{2m+1})}{1+d(Sx_{2m+2},Tx_{2m+1})+d(Ax_{2m+2},Bx_{2m+1})},$$
$$\frac{d(Sx_{2m+2},Bx_{2m+1})d(Tx_{2m+1},Ax_{2m+2})}{1+s^4[d(Sx_{2m+2},Tx_{2m+1})+d(Ax_{2m+2},Bx_{2m+1})]}\}$$
$$\leq\max\{d(y_{n-1},y_n),d(y_n,y_{n+1}),d(y_{n-1},y_n),0,\frac{d(y_n,y_n)+d(y_n,y_{n+1})}{2},$$
$$\frac{d(y_n,y_{n+1})d(y_{n-1},y_n)}{1+d(y_{n-1},y_n)+d(y_n,y_{n+1})},0\}$$
$$\leq\max\{d(y_{n-1},y_n),d(y_n,y_{n+1})\}.$$

Suppose $M(x_{2m+2},x_{2m+1})=d(y_n,y_{n+1})$.
Then from the inequality (2.8), we have

$$s^4d(y_n,y_{n+1})\leq\beta(M(x_{2m+2},x_{2m+1}))M(x_{2m+2},x_{2m+1})$$
$$\leq\beta(d(y_n,y_{n+1}))d(y_n,y_{n+1})\leq\frac{d(y_n,y_{n+1})}{s}$$

which implies that $(s^5-1)d(y_n,y_{n+1})\leq 0$.
Since $(s^5-1)\geq 0$, we have $d(y_n,y_{n+1})\leq 0$.
Therefore $M(x_{2m+2},x_{2m+1})=d(y_{n-1},y_n)$.
From the inequality (2.8), we have

$$s^4d(y_n,y_{n+1})\leq\beta(M(x_{2m+2},x_{2m+1}))M(x_{2m+2},x_{2m+1})$$
$$\leq\beta(d(y_{n-1},y_n))d(y_{n-1},y_n)\leq\frac{d(y_{n-1},y_n)}{s}. \tag{2.9}$$

Also, it is easy to see that (2.9) is valid when n is even.
Hence we have $d(y_n,y_{n+1})\leq\frac{1}{s^5}d(y_{n-1},y_n)$ for all $n\in\mathbb{N}$.
From Lemma 1.3, we have the sequence $\{y_n\}$ is a b-Cauchy sequence in X. □

The following is the main result of this paper.

Theorem 2.3. *Let A, B, S and T be selfmaps on a complete b-metric space (X, d) and satisfy (2.1) and Geraghty-Suzuki type contractive maps. If either*

(i) *the pair (A, S) compatible, A (or) S is b-continuous and the pair (B, T) is weakly compatible*

or

(ii) *the pair (B, T) compatible, B (or) T is b-continuous and the pair (A, S) is weakly compatible*

then A, B, S and T have a unique common fixed point in X.

Proof. By Proposition 2.2, the sequence $\{y_n\}$ is b-Cauchy in X.
Since X is b-complete, there exists $z \in X$ such that $\lim_{n\to\infty} y_n = z$. Thus

$$\begin{cases} \lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z \text{ and} \\ \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z. \end{cases} \tag{2.10}$$

Assume that (i) holds.
Since S is b-continuous, it follows that $\lim_{n\to\infty} SSx_{2n+2} = Sz$, $\lim_{n\to\infty} SAx_{2n} = Sz$.
By the b-triangle inequality, we have $d(ASx_{2n}, Sz) \leq s[d(ASx_{2n}, SAx_{2n}) + d(SAx_{2n}, Sz)]$.
Since the pair (A, S) is compatible, $\lim_{n\to\infty} d(ASx_{2n}, SAx_{2n}) = 0$.
Taking limit superior as $n \to \infty$, we have
$\limsup_{n\to\infty} d(ASx_{2n}, Sz) \leq s[\limsup_{n\to\infty} d(ASx_{2n}, SAx_{2n}) + \limsup_{n\to\infty} d(SAx_{2n}, Sz)] = 0$.
Therefore $\lim_{n\to\infty} ASx_{2n} = Sz$.
We now prove that $Sz = z$.
Suppose that $Sz \neq z$. Since

$$\frac{1}{2s}\min\{d(SSx_{2m+2}, ASx_{2m+2}), d(Tx_{2m+1}, Bx_{2m+1})\} \leq \max\{d(SSx_{2m+2}, Tx_{2m+1}), d(ASx_{2m+2}, Bx_{2m+1})\}$$

From the inequality (2.3), we have

$$s^4 d(ASx_{2n+2}, Bx_{2n+1}) \leq \beta(M(Sx_{2n+2}, x_{2n+1}))M(Sx_{2n+2}, x_{2n+1}) \tag{2.11}$$

where

$$\begin{aligned} M(Sx_{2n+2}, x_{2n+1}) = \max\{ & d(SSx_{2n+2}, Tx_{2n+1}), d(SSx_{2n+2}, ASx_{2n+s}), \\ & d(Tx_{2n+1}, Bx_{2n+1}), \tfrac{d(SSx_{2n+2}, Bx_{2n+1})}{2s}, \tfrac{d(Tx_{2n+1}, ASx_{2n+2})}{2s}, \\ & \tfrac{d(SSx_{2n+2}, ASx_{2n+2})d(Tx_{2n+1}, Bx_{2n+1})}{1+d(SSx_{2n+2}, Tx_{2n+1})+d(ASx_{2n+2}, Bx_{2n+1})}, \\ & \tfrac{d(SSx_{2n+2}, Bx_{2n+1})d(Tx_{2n+1}, ASx_{2n+2})}{1+s^4[d(SSx_{2n+2}, Tx_{2n+1})+d(ASx_{2n+2}, Bx_{2n+1})]}\}. \end{aligned}$$

By taking limit superior as $n \to \infty$ on $M(Sx_{2n+2}, x_{2n+1})$ and using Lemma 1.4, we obtain

$$\begin{aligned} \limsup_{n\to\infty} M(Sx_{2n+2}, x_{2n+1}) &\leq \max\{s^2 d(Sz, z), 0, 0, \tfrac{s^2 d(Sz,z)}{2s}, \tfrac{s^2 d(Sz,z)}{2s}, 0, \tfrac{s^4[d(Sz,z)]^2}{1+2s^4 d(Sz,z)}\} \\ &= s^2 d(Sz, z). \end{aligned}$$

Therefore

$$\frac{1}{s^2} d(Sz, z) \leq \liminf_{n\to\infty} M(Sx_{2n+2}, x_{2n+1}) \leq \limsup_{n\to\infty} M(Sx_{2n+2}, x_{2n+1}) \leq s^2 d(Sz, z). \tag{2.12}$$

Taking limit superior as $n \to \infty$ in the inequality (2.11) and using Lemma 1.4, we get

$$\begin{aligned} s^4 \tfrac{1}{s^2} d(Sz, z) &\leq s^4 \limsup_{n\to\infty} d(ASx_{2n+2}, Bx_{2n+1}) \\ &= \limsup_{n\to\infty} s^4 d(ASx_{2n+2}, Bx_{2n+1}) \end{aligned}$$

$$\le \limsup_{n\to\infty}[\beta(M(Sx_{2n+2},x_{2n+1}))M(Sx_{2n+2},x_{2n+1})]$$
$$= \limsup_{n\to\infty}\beta(M(Sx_{2n+2},x_{2n+1}))\limsup_{n\to\infty}M(Sx_{2n+2},x_{2n+1})$$
$$\le \limsup_{n\to\infty}\beta(M(Sx_{2n+2},x_{2n+1}))s^2d(Sz,z).$$

Therefore
$\frac{1}{s} \le 1 \le \limsup_{n\to\infty}\beta(M(Sx_{2n+2},x_{2n+1})) \le \frac{1}{s}$ which implies that
$\limsup_{n\to\infty}\beta(M(Sx_{2n+2},x_{2n+1})) = \frac{1}{s}$.
Since $\beta \in \mathfrak{F}$, it follows that $\lim_{n\to\infty}M(Sx_{2n+2},x_{2n+1}) = 0$.
Therefore from the inequality (2.12), we have
$\frac{1}{s^2}d(Sz,z) \le \lim_{n\to\infty}M(Sx_{2n+2},x_{2n+1}) = 0$ which implies that $d(Sz,z) \le 0$.
Therefore $Sz = z$.
We now show that $Az = z$. Suppose that $Az \neq z$.
Since
$\frac{1}{2s}\min\{d(Sz,Az),d(Tx_{2m+1},Bx_{2m+1})\} \le \max\{d(Sz,Tx_{2m+1}),d(Az,Bx_{2m+1})\}$
From the inequality (2.3), we have

$$s^4d(Az,Bx_{2n+1}) \le \beta(M(z,x_{2n+1}))M(z,x_{2n+1}) \tag{2.13}$$

where
$$M(z,x_{2n+1}) = \max\{d(Sz,Tx_{2n+1}),d(Sz,Az),d(Tx_{2n+1},Bx_{2n+1}), \frac{d(Sz,Bx_{2n+1})}{2s},\frac{d(Tx_{2n+1},Az)}{2s},\frac{d(Sz,Az)d(Tx_{2n+1},Bx_{2n+1})}{1+d(Sz,Tx_{2n+1})+d(Az,Bx_{2n+1})}, \frac{d(Sz,Bx_{2n+1})d(Tx_{2n+1},Az)}{1+s^4[d(Sz,Tx_{2n+1})+d(Az,Bx_{2n+1})]}\}.$$

By taking limit superior as $n\to\infty$ on $M(z,x_{2n+1})$ and using Lemma 1.4, we obtain
$$\limsup_{n\to\infty}M(z,x_{2n+1}) \le \max\{s^2d(Az,z),0,0,\frac{s^2d(Az,z)}{2s},\frac{s^2d(Az,z)}{2s},0,\frac{s^4[d(Az,z)]^2}{1+2s^4d(Az,z)}\}$$
$$= s^2d(Az,z).$$

Therefore
$$\frac{1}{s^2}d(Az,z) \le \liminf_{n\to\infty}M(z,x_{2n+1}) \le \limsup_{n\to\infty}M(z,x_{2n+1}) \le s^2d(Az,z). \tag{2.14}$$

Taking limit superior as $n\to\infty$ in the inequality (2.13) and using Lemma 1.4, we get
$$s^4\frac{1}{s^2}d(Az,z) \le s^4\limsup_{n\to\infty}d(Az,Bx_{2n+1})$$
$$= \limsup_{n\to\infty}s^4d(Az,Bx_{2n+1})$$
$$\le \limsup_{n\to\infty}[\beta(M(z,x_{2n+1}))M(z,x_{2n+1})]$$
$$= \limsup_{n\to\infty}\beta(M(z,x_{2n+1}))\limsup_{n\to\infty}M(z,x_{2n+1})$$
$$\le \limsup_{n\to\infty}\beta(M(z,x_{2n+1}))s^2d(Az,z).$$

Hence
$\frac{1}{s} \le 1 \le \limsup_{n\to\infty}\beta(M(z,x_{2n+1})) \le \frac{1}{s}$ which implies that
$\limsup_{n\to\infty}\beta(M(z,x_{2n+1})) = \frac{1}{s}$.
Since $\beta \in \mathfrak{F}$, it follows that $\lim_{n\to\infty}M(z,x_{2n+1}) = 0$.
Therefore from the inequality (2.14), we have
$\frac{1}{s^2}d(Az,z) \le \lim_{n\to\infty}M(z,x_{2n+1}) = 0$ which implies that $d(Az,z) \le 0$.
Therefore $Az = Sz = z$. Hence z is a common fixed point of A and S.

Now by Proposition 2.1, we have z is a unique common fixed point of A, B, S and T.
Assume that A is b-continuous, it follows that
$\lim_{n\to\infty} AAx_{2n} = Az$, $\lim_{n\to\infty} ASx_{2n+2} = Az$.
By the b-triangle inequality, we have
$d(SAx_{2n}, Az) \leq s[d(SAx_{2n}, ASx_{2n}) + d(ASx_{2n}, Az)]$.
Since the pair (A, S) is compatible, $\lim_{n\to\infty} d(ASx_{2n}, SAx_{2n}) = 0$.
Taking limit superior as $n \to \infty$, we have
$\limsup_{n\to\infty} d(SAx_{2n}, Az) \leq s[\limsup_{n\to\infty} d(SAx_{2n}, ASx_{2n}) + \limsup_{n\to\infty} d(ASx_{2n}, Az)] = 0$.
Therefore $\lim_{n\to\infty} SAx_{2n} = Az$.
Now we prove that $Az = z$. Suppose that $Az \neq z$.
Since
$\frac{1}{2s}\min\{d(SAx_{2n}, AAx_{2n}), d(Tx_{2n+1}, Bx_{2n+1})\} \leq \max\{d(SAx_{2n}, Tx_{2n+1}), d(AAx_{2n}, Bx_{2n+1})\}$
From the inequality (2.3), we have

$$s^4 d(ASx_{2n+2}, Bx_{2n+1}) \leq \beta(M(Sx_{2n+2}, x_{2n+1}))M(Sx_{2n+2}, x_{2n+1}) \tag{2.15}$$

where
$$M(Ax_{2n}, x_{2n+1}) = \max\{d(SAx_{2n}, Tx_{2n+1}), d(SAx_{2n}, AAx_{2n}), d(Tx_{2n+1}, Bx_{2n+1}), \frac{d(SAx_{2n}, Bx_{2n+1})}{2s}, \frac{d(Tx_{2n+1}, AAx_{2n})}{2s}, \frac{d(SAx_{2n}, AAx_{2n})d(Tx_{2n+1}, Bx_{2n+1})}{1+d(SAx_{2n}, Tx_{2n+1})+d(AAx_{2n}, Bx_{2n+1})}, \frac{d(SAx_{2n}, Bx_{2n+1})d(Tx_{2n+1}, AAx_{2n})}{1+s^4[d(SAx_{2n}, Tx_{2n+1})+d(AAx_{2n}, Bx_{2n+1})]}\}.$$
By taking limit superior as $n \to \infty$ on $M(Ax_{2n}, x_{2n+1})$ and using Lemma 1.4, we obtain
$$\begin{aligned}\limsup_{n\to\infty} M(Ax_{2n}, x_{2n+1}) &\leq \max\{s^2 d(Az, z), 0, 0, \tfrac{s^2 d(Az,z)}{2s}, \tfrac{s^2 d(Az,z)}{2s}, 0, \tfrac{s^4[d(Az,z)]^2}{1+2s^2 d(Az,z)}\} \\ &= s^2 d(Az, z).\end{aligned}$$
Therefore
$$\frac{1}{s^2} d(Az, z) \leq \liminf_{n\to\infty} M(Ax_{2n}, x_{2n+1}) \leq \limsup_{n\to\infty} M(Ax_{2n}, x_{2n+1}) \leq s^2 d(Az, z). \tag{2.16}$$
Taking limit superior as $n \to \infty$ in the inequality (2.15) and using Lemma 1.4, we get
$$\begin{aligned} s^4 \tfrac{1}{s^2} d(Az, z) &\leq s^4 \limsup_{n\to\infty} d(AAx_{2n}, Bx_{2n+1}) \\ &= \limsup_{n\to\infty} s^4 d(AAx_{2n}, Bx_{2n+1}) \\ &\leq \limsup_{n\to\infty} [\beta(M(Ax_{2n}, x_{2n+1}))M(Ax_{2n}, x_{2n+1}) \\ &= \limsup_{n\to\infty} \beta(M(Ax_{2n}, x_{2n+1})) \limsup_{n\to\infty} M(Ax_{2n}, x_{2n+1}) \\ &\leq \limsup_{n\to\infty} \beta(M(Ax_{2n}, x_{2n+1}))s^2 d(Az, z). \end{aligned}$$
Thus
$\frac{1}{s} \leq 1 \leq \limsup_{n\to\infty} \beta(M(Ax_{2n}, x_{2n+1})) \leq \frac{1}{s}$ which implies that
$\limsup_{n\to\infty} \beta(M(Ax_{2n}, x_{2n+1})) = \frac{1}{s}$.
Since $\beta \in \mathfrak{F}$, it follows that $\lim_{n\to\infty} M(Ax_{2n}, x_{2n+1}) = 0$.
Therefore from the inequality (2.16), we have
$\frac{1}{s^2} d(Az, z) \leq \lim_{n\to\infty} M(Ax_{2n}, x_{2n+1}) = 0$ which implies that $d(Az, z) \leq 0$.
Therefore $Az = z$.

Since $A(X) \subseteq T(X)$, there exists $u \in X$ such that $z = Tu$.
We now show that $Bu = z$. Suppose that $Bu \neq z$.
Since
$\frac{1}{2s}\min\{d(Sx_{2n}, Ax_{2n}), d(Tu, Bu)\} \leq \max\{d(Sx_{2n}, Tu), d(Ax_{2n}, Bu)\}$
From the inequality (2.3), we have

$$s^4 d(Ax_{2n}, Bu) \leq \beta(M(x_{2n}, u))M(x_{2n}, u) \tag{2.17}$$

where
$M(x_{2n}, u) = \max\{d(Sx_{2n}, Tu), d(Sx_{2n}, Ax_{2n}), d(Tu, Bu), \frac{d(Sx_{2n},Bu)}{2s}, \frac{d(Tu,Ax_{2n})}{2s},$
$\frac{d(Sx_{2n},Ax_{2n})d(Tu,Bu)}{1+d(Sx_{2n},Tu)+d(Ax_{2n},Bu)}, \frac{d(Sx_{2n},Bu)d(Tu,Ax_{2n})}{1+s^4[d(Sx_{2n},Tu)+d(Ax_{2n},Bu)]}\}.$
By taking limit superior as $n \to \infty$ on $M(x_{2n}, u)$ and using Lemma 1.4, we obtain
$\limsup_{n\to\infty} M(x_{2n}, u) \leq \max\{s^2 d(z, Bu), 0, 0, \frac{s^2 d(z,Bu)}{2s}, \frac{s^2 d(z,Bu)}{2s}, 0, \frac{s^4[d(z,Bu)]^2}{1+2s^2 d(z,Bu)}\}$
$= s^2 d(Az, z).$
Therefore

$$\frac{1}{s^2} d(z, Bu) \leq \liminf_{n\to\infty} M(x_{2n}, u) \leq \limsup_{n\to\infty} M(x_{2n}, u) \leq s^2 d(z, Bu). \tag{2.18}$$

Taking limit superior as $n \to \infty$ in the inequality (2.17) and using Lemma 1.4, we get

$$\begin{aligned}
s^4 \tfrac{1}{s^2} d(z, Bu) &\leq s^4 \limsup_{n\to\infty} d(Ax_{2n}, Bu) \\
&= \limsup_{n\to\infty} s^4 d(Ax_{2n}, Bu) \\
&\leq \limsup_{n\to\infty} [\beta(M(x_{2n}, u))M(x_{2n}, u) \\
&= \limsup_{n\to\infty} \beta(M(x_{2n}, u)) \limsup_{n\to\infty} M(x_{2n}, u) \\
&\leq \limsup_{n\to\infty} \beta(M(x_{2n}, u)) s^2 d(z, Bu).
\end{aligned}$$

Therefore
$\frac{1}{s} \leq 1 \leq \limsup_{n\to\infty} \beta(M(x_{2n}, u)) \leq \frac{1}{s}$ which implies that $\limsup_{n\to\infty} \beta(M(x_{2n}, u)) = \frac{1}{s}$.
Since $\beta \in \mathfrak{F}$, it follows that $\lim_{n\to\infty} M(x_{2n}, u) = 0$.
Therefore from the inequality (2.18), we have
$\frac{1}{s^2} d(z, Bu) \leq \lim_{n\to\infty} M(x_{2n}, u) = 0.$ implies that $d(z, Bu) \leq 0$.
Therefore $Bu = Tu = z$. Since the pair (B, T) is weakly compatible and $Bu = Tu$, we have
$BTu = TBu$. i.e., $Bz = Tz$.
We now show that $Bz = z$. Suppose that $Bz \neq z$.
Since
$\frac{1}{2s}\min\{d(Sx_{2n}, Ax_{2n}), d(Tz, Bz)\} \leq \max\{d(Sx_{2n}, Tz), d(Ax_{2n}, Bz)\}$
From the inequality (2.3), we have

$$s^4 d(Ax_{2n}, Bz) \leq \beta(M(x_{2n}, z))M(x_{2n}, z) \tag{2.19}$$

where
$M(x_{2n}, z) = \max\{d(Sx_{2n}, Tz), d(Sx_{2n}, Ax_{2n}), d(Tz, Bz), \frac{d(Sx_{2n},Bz)}{2s}, \frac{d(Tz,Ax_{2n})}{2s},$
$\frac{d(Sx_{2n},Ax_{2n})d(Tz,Bz)}{1+d(Sx_{2n},Tz)+d(Ax_{2n},Bz)}, \frac{d(Sx_{2n},Bz)d(Tz,Ax_{2n})}{1+s^4 d(Sx_{2n},Tz)+d(Ax_{2n},Bz)}\}.$
By taking limit superior as $n \to \infty$ on $M(x_{2n}, z)$ and using Lemma 1.4, we obtain
$\limsup_{n\to\infty} M(x_{2n}, z) \leq \max\{s^2 d(z, Bz), 0, 0, \frac{s^2 d(z,Bz)}{2s}, \frac{s^2 d(z,Bz)}{2s}, 0, \frac{s^6[d(z,Bz)]^2}{1+2s^2 d(z,Bz)}\}$

$= s^2 d(Az, z).$

Therefore

$$\frac{1}{s^2} d(z, Bz) \leq \liminf_{n\to\infty} M(x_{2n}, z) \leq \limsup_{n\to\infty} M(x_{2n}, z) \leq s^2 d(z, Bz). \tag{2.20}$$

Taking limit superior as $n \to \infty$ in the inequality (2.19) and using Lemma 1.4, we get

$$\begin{aligned} s^4 \tfrac{1}{s^2} d(z, Bz) &\leq s^4 \limsup_{n\to\infty} d(Ax_{2n}, Bz) \\ &= \limsup_{n\to\infty} s^4 d(Ax_{2n}, Bz) \\ &\leq \limsup_{n\to\infty} [\beta(M(x_{2n}, z)) M(x_{2n}, z) \\ &= \limsup_{n\to\infty} \beta(M(x_{2n}, z)) \limsup_{n\to\infty} M(x_{2n}, z) \\ &\leq \limsup_{n\to\infty} \beta(M(x_{2n}, z)) s^2 d(z, Bz). \end{aligned}$$

Therefore

$\frac{1}{s} \leq 1 \leq \limsup_{n\to\infty} \beta(M(x_{2n}, z)) \leq \frac{1}{s}$ which implies that $\limsup_{n\to\infty} \beta(M(x_{2n}, z)) = \frac{1}{s}$.

Since $\beta \in \mathfrak{F}$, it follows that $\lim_{n\to\infty} M(x_{2n}, z) = 0$.

Therefore from the inequality (2.20), we have

$\frac{1}{s^2} d(z, Bz) \leq \lim_{n\to\infty} M(x_{2n}, z) = 0$. implies that $d(z, Bz) \leq 0$.

Hence $Bz = z$.

Therefore $Bz = Tz = z$.

Hence z is a common fixed point of A and S.

Now by Proposition 2.1, we have z is a unique common fixed point of A, B, S and T.

In a similar way, under the assumption (ii), the conclusion of the theorem holds. □

Theorem 2.4. *Let (X, d) be a b-metric space with coefficient $s \geq 1$. Let $A, B, S, T : X \to X$ be selfmaps of X and satisfy* (2.1) *and Geraghty-Suzuki type contractive maps. Suppose that one of the pairs (A, S) and (B, T) satisfies the b-(E.A)-property and that one of the subspace $A(X), B(X), S(X)$ and $T(X)$ is b-closed in X. Then the pairs (A, S) and (B, T) have a point of coincidence in X. Moreover, if the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.*

Proof. We first assume that the pair (A, S) satisfies the b-(E.A)-property. So there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = q \tag{2.21}$$

for some $q \in X$.

Since $A(X) \subseteq T(X)$, there exists a sequence $\{y_n\}$ in X such that $Ax_n = Ty_n$, and hence

$$\lim_{n\to\infty} Ty_n = q. \tag{2.22}$$

Now we show that $\lim_{n\to\infty} By_n = q$.

Since $\frac{1}{2s} \min\{d(Sx_n, Ax_n), d(Ty_n, By_n)\} \leq \max\{d(Sx_n, Ty_n), d(Ax_n, By_n)\}$.

From the inequality (2.3), we have

$$s^4 d(Ax_n, By_n) \leq \beta(M(x_n, y_n)) M(x_n, y_n) \tag{2.23}$$

where
$M(x_n,y_n)=\max\{d(Sx_n,Ty_n),d(Sx_n,Ax_n),d(Ty_n,By_n),\frac{d(Sx_n,By_n)}{2s},\frac{d(Ty_n,Ax_n)}{2s},$
$\frac{d(Sx_n,Ax_n)d(Ty_n,By_n)}{1+d(Sx_n,Ty_n)+d(Ax_n,By_n)},\frac{d(Sx_n,By_n)d(Ty_n,Ax_n)}{1+s^4[d(Sx_n,Ty_n)+d(Ax_n,By_n)]}\}.$
By taking limit superior as $n\to\infty$ on $M(x_n,y_n)$, and using (2.21) and (2.22), we obtain

$$\left\{\begin{aligned}\limsup_{n\to\infty}M(x_n,y_n)&=\max\{0,0,\limsup_{n\to\infty}d(Ax_n,By_n),\frac{\limsup_{n\to\infty}d(Ax_n,By_n)}{2s},0,0,0\}\\&=\limsup_{n\to\infty}d(Ax_n,By_n).\end{aligned}\right. \tag{2.24}$$

On taking limit superior as $n\to\infty$ in (2.23), and using (2.24), we get
$$\begin{aligned}s^4\limsup_{n\to\infty}d(Ax_n,By_n)&=\limsup_{n\to\infty}[\beta(M(x_n,y_n))M(x_n,y_n)]\\&=\limsup_{n\to\infty}\beta(M(x_n,y_n))\limsup_{n\to\infty}M(x_n,y_n)\\&=\limsup_{n\to\infty}\beta(M(x_n,y_n))\limsup_{n\to\infty}d(Ax_n,By_n).\end{aligned}$$
Therefore
$\frac{1}{s}\le 1\le\limsup_{n\to\infty}\beta(M(x_n,y_n))\le\frac{1}{s^5}\le\frac{1}{s}$ which implies that
$\limsup_{n\to\infty}\beta(M(x_n,y_n))=\frac{1}{s}$.
Since $\beta\in\mathfrak{F}$, we have $\lim_{n\to\infty}M(x_n,y_n)=0$. i.e., $\limsup_{n\to\infty}d(Ax_n,By_n)=0$.
Therefore
$$\lim_{n\to\infty}d(Ax_n,By_n)=0. \tag{2.25}$$
By the b-triangular inequality, we have
$$d(q,By_n)\le s[d(q,Ax_n)+d(Ax_n,By_n)]. \tag{2.26}$$
On taking limits as $n\to\infty$ in (2.26), and using (2.21) and (2.25), we get
$\lim_{n\to\infty}d(q,By_n)\le s[\lim_{n\to\infty}d(q,Ax_n)+\lim_{n\to\infty}d(Ax_n,By_n)]=0.$
Therefore $\lim_{n\to\infty}d(q,By_n)=0$.
Case (i): Assume that $T(X)$ is a b-closed subset of X.
In this case $q\in T(X)$, we can choose $r\in X$ such that $Tr=q$.
We now prove that $Br=q$. Suppose that $d(Br,q)>0$.
Since $\frac{1}{2s}\min\{d(Sx_n,Ax_n),d(Tr,Br)\}\le\max\{d(Sx_n,Tr),d(Ax_n,Br)\}$
From the inequality (2.3), we have
$$s^4d(Ax_n,Br)\le\beta(M(x_n,r))M(x_n,r) \tag{2.27}$$
where
$M(x_n,r)=\max\{d(Sx_n,Tr),d(Sx_n,Ax_n),d(Tr,Br),\frac{d(Sx_n,Br)}{2s},\frac{d(Tr,Ax_n)}{2s},$
$\frac{d(Sx_n,Ax_n)d(Tr,Br)}{1+d(Sx_n,Tr)+d(Ax_n,Br)},\frac{d(Sx_n,Br)d(Tr,Ax_n)}{1+s^4[d(Sx_n,Tr)+d(Ax_n,Br)]}\}.$
By taking limit superior as $n\to\infty$ on $M(x_n,r)$, and using (2.21), (2.22) and Lemma 1.4, we obtain
$$\limsup_{n\to\infty}M(x_n,r)\le\max\{0,0,d(q,Br),\frac{d(q,Br)}{2},0,0,0\}=d(q,Br). \tag{2.28}$$
We have
$$\begin{aligned}d(Br,q)&\le s[d(Br,Sx_n)+d(Sx_n,q)]\\&=2s^2[\tfrac{d(Br,Sx_n)}{2s}]+sd(Sx_n,q)\le 2s^2M(x_n,r)+sd(Sx_n,q).\end{aligned}$$
On taking limit inferior as $n\to\infty$, we get

Therefore $\frac{1}{2s^2}d(Br,q) \leq \liminf\limits_{n\to\infty} M(x_n,r)$.
Taking limit superior as $n\to\infty$ in (2.27) and using (2.28) and Lemma 1.4, we have

$$\begin{aligned}
s^4(\tfrac{1}{s}d(q,Br)) &\leq s^4 \limsup_{n\to\infty} d(Ax_n,Br)\\
&= \limsup_{n\to\infty}[\beta(M(x_n,r))M(x_n,r)]\\
&= \limsup_{n\to\infty}\beta(M(x_n,r)) \limsup_{n\to\infty} M(x_n,r)\\
&\leq \limsup_{n\to\infty}\beta(M(x_n,r))d(q,Br).
\end{aligned}$$

Therefore
$\frac{1}{s} \leq \limsup\limits_{n\to\infty}\beta(M(x_n,r)) \leq \frac{1}{s}$ which implies that $\limsup\limits_{n\to\infty}\beta(M(x_n,r)) = \frac{1}{s}$.
Since $\beta \in \mathfrak{F}$, we have $\lim\limits_{n\to\infty} M(x_n,r) = 0$.
Therefore $\frac{1}{2s^2}d(Br,q) \leq \lim\limits_{n\to\infty} M(x_n,r) = 0$.
Thus $Br = q$. Hence $Br = Tr = q$, so that q is a coincidence point of B and T.
Since $B(X) \subseteq S(X)$, we have $q \in S(X)$, there exists $z \in X$ such that $Sz = q = Br$.
Now we show that $Az = q$. Suppose $Az \neq q$.
Since
$\frac{1}{2s}\min\{d(Sz,Az), d(Tr,Br)\} \leq \max\{d(Sz,Tr), d(Az,Br)\}$
From the inequality (2.3), we have

$$s^4 d(Az,q) = s^4 d(Az,Br) \leq \beta(M(z,r))M(z,r) \tag{2.29}$$

where

$$\begin{aligned}
M(z,r) = \max\{&d(Sz,Tr), d(Sz,Az), d(Tr,Br), \tfrac{d(Sz,Br)}{2s}, \tfrac{d(Tr,Az)}{2s},\\
&\tfrac{d(Sz,Az)d(Tr,Br)}{1+d(Sz,Tr)+d(Az,Br)}, \tfrac{d(Sz,Br)d(Tr,Az)}{1+s^4[d(Sz,Tr)+d(Az,Br)]}\}\\
= \max\{&0, d(q,Az), 0, 0, \tfrac{d(q,Az)}{2s}, 0, 0\} = d(q,Az).
\end{aligned}$$

From the inequality (2.29), we have
$s^4 d(Az,q) \leq \beta(d(Az,q)d(Az,q)) < d(Az,q)$,
a contradiction.
Therefore $Az = Sz = q$ so that z is a coincidence point of A and S.
Since the pairs (A,S) and (B,T) are weakly compatible, we have $Aq = Sq$ and $Bq = Tq$.
Therefore q is also a coincidence point of the pairs (A,S) and (B,T).
We now show that q is a common fixed point of A, B, S and T.
Suppose $Aq \neq q$.
Since $\frac{1}{2s}\min\{d(Sq,Aq), d(Tr,Br)\} \leq \max\{d(Sq,Tr), d(Aq,Br)\}$,
from the inequality (2.3), we have

$$s^4 d(Aq,q) = s^4 d(Aq,Br) \leq \beta(M(q,r))M(q,r) \tag{2.30}$$

where

$$\begin{aligned}
M(q,r) = \max\{&d(Sq,Tr), d(Sq,Aq), d(Tr,Br), \tfrac{d(Sq,Br)}{2s}, \tfrac{d(Tr,Aq)}{2s},\\
&\tfrac{d(Sq,Aq)d(Tr,Br)}{1+d(Sq,Tr)+d(Aq,Br)}, \tfrac{d(Sq,Br)d(Tr,Aq)}{1+s^4[d(Sq,Tr)+d(Aq,Br)]}\}\\
= \max\{&d(Aq,q), 0, 0, \tfrac{d(Aq,q)}{2s}, \tfrac{d(Aq,q)}{2s}, 0, 0\} = d(Aq,q).
\end{aligned}$$

Now, from the inequality (2.30), we have
$s^4 d(Aq,q) \leq \beta(d(Aq,q)d(Aq,q)) < d(Aq,q)$,
a contradiction.
Therefore $Aq = Sq = q$ so that q is a common fixed point of A and S.

By Proposition 2.1, we get that q is a unique common fixed point of A, B, S and T.
Case (ii): Suppose $A(X)$ is b-closed.
In this case, we have $q \in A(X)$ and since $A(X) \subseteq T(X)$, we choose $r \in X$ such that $q = Tr$.
The proof follows as in Case (i).
Case (iii): Suppose $S(X)$ is b-closed.
We follow the argument similar as Case (i) and we get conclusion.
Case (iv): Suppose $B(X)$ is b-closed. As in Case (ii), we get the conclusion.

For the case of (B, T) satisfies the b-(E.A)-property, we follow the argument similar to the case (A, S) satisfies the b-(E.A)-property. □

3. Corollaries and examples

In this section we draw some corollaries from our main results and provide examples in support of our results.

If we take $A = B = f$ and $S = T = g$ in Theorem 2.3 and Theorem 2.4, we get Corollary 3.1 and Corollary 3.2, respectively.

Corollary 3.1. *Let (X, d) be a b-metric space and f and g be selfmaps of X. Assume that there exists $\beta \in \mathfrak{F}$ such that*

$$\begin{aligned} \tfrac{1}{2s}\min\{d(fx,gx),d(fy,gy)\} &\leq \max\{d(gx,gy),d(fx,fy)\} \\ &\implies s^4 d(fx,fy) \leq \beta(M(x,y))M(x,y) \end{aligned} \tag{3.1}$$

where
$$M(x,y) = \max\{d(gx,gy), d(gx,fx), d(gy,fy), \tfrac{d(gx,fy)}{2s}, \tfrac{d(gy,fx)}{2s}, \tfrac{d(gx,fx)d(gy,fy)}{1+d(gx,gy)+d(fx,fy)}, \tfrac{d(gx,fy)d(gy,fx)}{1+s^4[d(gx,gy)+d(fx,fy)]}\},$$
for all $x, y \in X$. If $f(X) \subseteq g(X)$, the pair (f, g) is compatible and f or g is b-continuous then f and g have a unique common fixed point in X.

Corollary 3.2. *Let (X, d) be a b-metric space with coefficient $s \geq 1$. Let $f, g : X \to X$ be selfmaps of X and satisfy $f(X) \subseteq g(X)$ and the inequality (3.1). Suppose that the pair (f, g) satisfies the b-(E.A)-property and that one of the subspace $f(X)$ and $g(X)$ is b-closed in X. Then the pairs (f, g) have a point of coincidence in X. Moreover, if the pair (f, g) is weakly compatible, then f and g have a unique common fixed point in X.*

The following is an example in support of Theorem 2.3.

Example 3.1. Let $X = \mathbb{R}^+$ and let $d : X \times X \to \mathbb{R}^+$ defined by
$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 4 & \text{if } x, y \in [0,1], \\ 5 + \frac{1}{x+y} & \text{if } x, y \in (1, \infty), \\ \frac{27}{10} & \text{otherwise.} \end{cases}$$
The b-metric conditions (b_1) and (b_2) are trivially hold for this example.
Let us now check (b_3).
For this purpose we consider the following nontrivial case.
Let $y \in [0, 1]$ and $x, z \in (1, \infty)$.
Then $d(x, z) = 5 + \frac{1}{x+z}, d(x, y) = \frac{27}{10}, d(y, z) = \frac{27}{10}$.
We have
$2 \leq x + z \implies \frac{1}{x+z} \leq \frac{1}{2}$ so that $5 + \frac{1}{x+z} \leq 5 + \frac{1}{2} < \frac{489}{480}(\frac{27}{5})$.
Therefore $d(x, z) = 5 + \frac{1}{x+z} < \frac{489}{480}(\frac{27}{10} + \frac{27}{10}) = s[d(x, y) + d(y, z)]$ where $s = \frac{489}{480}$

so that (b_3) holds.
Thus d is a b-metric with $s = \frac{489}{480}$.
Clearly this d is complete so that (X, d) is a complete b-metric space.

Here we observe that when $x = \frac{101}{100}, z = \frac{102}{100} \in (1, \infty)$ and $y \in [0, 1)$, we have $d(x, z) = \frac{1115}{203} \nleq \frac{27}{5} = d(x, y) + d(y, z)$ so that d is not a metric.
We define $A, B, S, T : X \to X$ by

$$A(x) = 1 \text{ if } x \in [0, \infty),\ B(x) = \begin{cases} x^2 + 2 & \text{if } x \in [0, 1) \\ \frac{x^2+1}{2} & \text{if } x \in [1, \infty), \end{cases}$$

$$S(x) = \begin{cases} x + 2 & \text{if } x \in [0, 1) \\ \frac{x+1}{2} & \text{if } x \in [1, \infty), \end{cases} \text{ and } T(x) = \begin{cases} 3x^2 + 4 & \text{if } x \in [0, 1) \\ \frac{x(x+2)}{3} & \text{if } x \in [1, \infty). \end{cases}$$

Clearly $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$.
Here A is b-continuous.
We choose a sequence $\{x_n\}$ with $\{x_n\} = 1 + \frac{1}{2n}, n \geq 1$, we have
$ASx_n = A(\frac{1+\frac{1}{2n}+1}{2}) = 1$ and $SAx_n = S1 = 1$.
Therefore $\lim\limits_{n\to\infty} d(ASx_n, SAx_n) = 0$ so that the pair (A, S) is compatible and clearly the pair (B, T) is weakly compatible.
We define $\beta : [0, \infty) \to [0, \frac{1}{s})$ by $\beta(t) = \frac{480}{489} e^{\frac{-t}{100}}$. Then we have $\beta \in \mathfrak{F}$.
Case (i): $x, y \in [0, 1)$.
$d(Ax, By) = \frac{27}{10}, d(Sx, Ty) = 5 + \frac{1}{x+y}, d(Ty, By) = 5 + \frac{1}{x+y}, d(Sx, Ax) = \frac{27}{10}$,
$d(Ax, Ty) = \frac{27}{10}, d(Sx, By) = 5 + \frac{1}{x+y}$ and

$$\begin{aligned} M(x, y) &= \max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \tfrac{d(Sx,By)}{2s}, \tfrac{d(Ty,Ax)}{2s}, \\ &\qquad \tfrac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)}, \tfrac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\} \\ &= \max\{5 + \tfrac{1}{x+y}, \tfrac{27}{10}, 5 + \tfrac{1}{x+y}, (\tfrac{240}{489})(5 + \tfrac{1}{x+y}), (\tfrac{240}{489})(\tfrac{27}{10}), \tfrac{(\frac{27}{10})(5+\frac{1}{x+y})}{1+5+\frac{1}{x+y}+\frac{27}{10}}, \\ &\qquad \tfrac{(5+\frac{1}{x+y})(\frac{27}{10})}{1+(\frac{489}{480})^4(5+\frac{1}{x+y}+\frac{27}{10})}\} \\ &= 5 + \tfrac{1}{x+y}. \end{aligned}$$

Since

$$\begin{aligned} \tfrac{1}{2s} \min\{d(Sx, Ax), d(Ty, By)\} &= \tfrac{240}{489} \min\{\tfrac{27}{10}, 5 + \tfrac{1}{x+y}\} \\ &= (\tfrac{240}{489})(\tfrac{27}{10}) \\ &\leq \max\{5 + \tfrac{1}{x+y}, \tfrac{27}{10}\} \\ &= \max\{d(Sx, Ty), d(Ax, By)\}. \end{aligned}$$

Now we consider

$$s^4 d(Ax, By) = (\tfrac{489}{480})^4(\tfrac{27}{10}) \leq \tfrac{480}{489} e^{\frac{-(5+\frac{1}{x+y})}{100}} 5 + \tfrac{1}{x+y} = \beta(M(x, y))M(x, y).$$

Case (ii): $x, y \in (1, \infty)$.
$d(Ax, By) = \frac{27}{10}, d(Sx, Ty) = 5 + \frac{1}{x+y}, d(Ty, By) = 5 + \frac{1}{x+y}, d(Sx, Ax) = \frac{27}{10}$,
$d(Ax, Ty) = \frac{27}{10}, d(Sx, By) = 5 + \frac{1}{x+y}$ and

$$\begin{aligned} M(x, y) &= \max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \tfrac{d(Sx,By)}{2s}, \tfrac{d(Ty,Ax)}{2s}, \\ &\qquad \tfrac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)}, \tfrac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\} \\ &= \max\{5 + \tfrac{1}{x+y}, \tfrac{27}{10}, 5 + \tfrac{1}{x+y}, (\tfrac{240}{489})(5 + \tfrac{1}{x+y}), (\tfrac{240}{489})(\tfrac{27}{10}), \tfrac{(\frac{27}{10})(5+\frac{1}{x+y})}{1+5+\frac{1}{x+y}+\frac{27}{10}}, \\ &\qquad \tfrac{(5+\frac{1}{x+y})(\frac{27}{10})}{1+(\frac{489}{480})^4(5+\frac{1}{x+y}+\frac{27}{10})}\} \\ &= 5 + \tfrac{1}{x+y}. \end{aligned}$$

Since

$$\begin{aligned}\frac{1}{2s}\min\{d(Sx,Ax),d(Ty,By)\} &= \tfrac{240}{489}\min\{\tfrac{27}{10},5+\tfrac{1}{x+y}\}\\ &= (\tfrac{240}{489})(\tfrac{27}{10})\\ &\leq \max\{5+\tfrac{1}{x+y},\tfrac{27}{10}\}\\ &= \max\{d(Sx,Ty),d(Ax,By)\}.\end{aligned}$$

Now we consider

$s^4d(Ax,By)=(\frac{489}{480})^4(\frac{27}{10})\leq \frac{480}{489}e^{\frac{-(5+\frac{1}{x+y})}{100}}5+\frac{1}{x+y}=\beta(M(x,y))M(x,y)$.

Case (iii): $x\in[0,1), y\in(1,\infty)$.

$d(Ax,By)=\frac{27}{10}, d(Sx,Ty)=5+\frac{1}{x+y}, d(Ty,By)=5+\frac{1}{x+y}, d(Sx,Ax)=\frac{27}{10}$,
$d(Ax,Ty)=\frac{27}{10}, d(Sx,By)=5+\frac{1}{x+y}$ and

$$\begin{aligned}M(x,y) &= \max\{d(Sx,Ty),d(Sx,Ax),d(Ty,By),\tfrac{d(Sx,By)}{2s},\tfrac{d(Ty,Ax)}{2s},\\ &\qquad \tfrac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)},\tfrac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\}\\ &= \max\{5+\tfrac{1}{x+y},\tfrac{27}{10},5+\tfrac{1}{x+y},(\tfrac{240}{489})(5+\tfrac{1}{x+y}),(\tfrac{240}{489})(\tfrac{27}{10}),\tfrac{(\frac{27}{10})(5+\frac{1}{x+y})}{1+5+\frac{1}{x+y}+\frac{27}{10}},\\ &\qquad \tfrac{(5+\frac{1}{x+y})(\frac{27}{10})}{1+(\frac{489}{480})^4(5+\frac{1}{x+y}+\frac{27}{10})}\}\\ &= 5+\tfrac{1}{x+y}.\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{2s}\min\{d(Sx,Ax),d(Ty,By)\} &= \tfrac{240}{489}\min\{\tfrac{27}{10},5+\tfrac{1}{x+y}\}\\ &= (\tfrac{240}{489})(\tfrac{27}{10})\\ &\leq \max\{5+\tfrac{1}{x+y},\tfrac{27}{10}\}\\ &= \max\{d(Sx,Ty),d(Ax,By)\}.\end{aligned}$$

Now we consider

$s^4d(Ax,By)=(\frac{489}{480})^4(\frac{27}{10})\leq \frac{480}{489}e^{\frac{-(5+\frac{1}{x+y})}{100}}(5+\frac{1}{x+y})=\beta(M(x,y))M(x,y)$.

Case (iv): $x\in(1,\infty), y\in[0,1)$.

$d(Ax,By)=\frac{27}{10}, d(Sx,Ty)=5+\frac{1}{x+y}, d(Ty,By)=5+\frac{1}{x+y}, d(Sx,Ax)=\frac{27}{10}$,
$d(Ax,Ty)=\frac{27}{10}, d(Sx,By)=5+\frac{1}{x+y}$ and

$$\begin{aligned}M(x,y) &= \max\{d(Sx,Ty),d(Sx,Ax),d(Ty,By),\tfrac{d(Sx,By)}{2s},\tfrac{d(Ty,Ax)}{2s},\\ &\qquad \tfrac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)},\tfrac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\}\\ &= \max\{5+\tfrac{1}{x+y},\tfrac{27}{10},5+\tfrac{1}{x+y},(\tfrac{240}{489})(5+\tfrac{1}{x+y}),(\tfrac{240}{489})(\tfrac{27}{10}),\tfrac{(\frac{27}{10})(5+\frac{1}{x+y})}{1+5+\frac{1}{x+y}+\frac{27}{10}},\\ &\qquad \tfrac{(5+\frac{1}{x+y})(\frac{27}{10})}{1+(\frac{489}{480})^4(5+\frac{1}{x+y}+\frac{27}{10})}\}\\ &= 5+\tfrac{1}{x+y}.\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{2s}\min\{d(Sx,Ax),d(Ty,By)\} &= \tfrac{240}{489}\min\{\tfrac{27}{10},5+\tfrac{1}{x+y}\}\\ &= (\tfrac{240}{489})(\tfrac{27}{10})\\ &\leq \max\{5+\tfrac{1}{x+y},\tfrac{27}{10}\}\\ &= \max\{d(Sx,Ty),d(Ax,By)\}.\end{aligned}$$

Now we consider

$s^4d(Ax,By)=(\frac{489}{480})^4(\frac{27}{10})\leq \frac{480}{489}e^{\frac{-(5+\frac{1}{x+y})}{100}}(5+\frac{1}{x+y})=\beta(M(x,y))M(x,y)$.

Case (v): $x=1, y\in[0,1)$.

$d(Ax,By)=\frac{27}{10}, d(Sx,Ty)=\frac{27}{10}, d(Ty,By)=5+\frac{1}{x+y}, d(Sx,Ax)=0$,
$d(Ax,Ty)=\frac{27}{10}, d(Sx,By)=\frac{27}{10}$ and

$M(x,y)=\max\{d(Sx,Ty),d(Sx,Ax),d(Ty,By),\frac{d(Sx,By)}{2s},\frac{d(Ty,Ax)}{2s},$

$$\frac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)}, \frac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\}$$
$$= \max\{\tfrac{27}{10}, 0, 5+\tfrac{1}{x+y}, (\tfrac{240}{489})(\tfrac{27}{10}), (\tfrac{240}{489})(\tfrac{27}{10}), 0, \frac{(\frac{27}{10})(\frac{27}{10})}{1+(\frac{489}{480})^4(5+\frac{1}{x+y}+\frac{27}{10})}\}$$
$$= 5+\tfrac{1}{x+y}.$$

Since

$$\begin{aligned}\tfrac{1}{2s}\min\{d(Sx,Ax), d(Ty,By)\} &= \tfrac{240}{489}\min\{0, 5+\tfrac{1}{x+y}\}\\ &= 0 \le \max\{\tfrac{27}{10}, \tfrac{27}{10}\}\\ &\quad = \max\{d(Sx,Ty), d(Ax,By)\}.\end{aligned}$$

Now we consider

$$s^4 d(Ax,By) = (\tfrac{489}{480})^4(\tfrac{27}{10}) \le \tfrac{480}{489}e^{\frac{-(5+\frac{1}{x+y})}{100}}(5+\tfrac{1}{x+y}) = \beta(M(x,y))M(x,y).$$

Case (vi): $x=1, y\in(1,\infty)$.

$d(Ax,By)=\frac{27}{10}, d(Sx,Ty)=\frac{27}{10}, d(Ty,By)=5+\frac{1}{x+y}, d(Sx,Ax)=0,$
$d(Ax,Ty)=\frac{27}{10}, d(Sx,By)=\frac{27}{10}$ and

$$M(x,y) = \max\{d(Sx,Ty), d(Sx,Ax), d(Ty,By), \tfrac{d(Sx,By)}{2s}, \tfrac{d(Ty,Ax)}{2s},$$
$$\frac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)}, \frac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\}$$
$$= \max\{\tfrac{27}{10}, 0, 5+\tfrac{1}{x+y}, (\tfrac{240}{489})(\tfrac{27}{10}), (\tfrac{240}{489})(\tfrac{27}{10}), 0, \frac{(\frac{27}{10})(\frac{27}{10})}{1+(\frac{489}{480})^4(5+\frac{1}{x+y}+\frac{27}{10})}\}$$
$$= 5+\tfrac{1}{x+y}.$$

Since

$$\begin{aligned}\tfrac{1}{2s}\min\{d(Sx,Ax), d(Ty,By)\} &= \tfrac{240}{489}\min\{0, 5+\tfrac{1}{x+y}\}\\ &= 0 \le \max\{\tfrac{27}{10}, \tfrac{27}{10}\}\\ &\quad = \max\{d(Sx,Ty), d(Ax,By)\}.\end{aligned}$$

Now we consider

$$s^4 d(Ax,By) = (\tfrac{489}{480})^4(\tfrac{27}{10}) \le \tfrac{480}{489}e^{\frac{-(5+\frac{1}{x+y})}{100}}(5+\tfrac{1}{x+y}) = \beta(M(x,y))M(x,y).$$

Case (vii): $x\in[0,1), y=1$.

Here $d(Ax,By)=0$. Clearly the inequality (2.3) holds in this case.

Case (viii): $x\in[0,1), y=1$.

Here $d(Ax,By)=0$. In this case the inequality (2.3) holds clearly.

From all the above four cases, A, B, S and T are Geraghty-Suzuki type contraction maps. Therefore A, B, S and T satisfy all the hypotheses of Theorem 2.3 and 1 is the unique common fixed point of A, B, S and T.

The following is an example in support of Theorem 2.4.

Example 3.2. Let $X=[0,1]$ and let $d: X\times X\to\mathbb{R}^+$ defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x=y,\\ \frac{11}{15} & \text{if } x,y\in[0,\frac{2}{3}),\\ \frac{23}{25}+\frac{x+y}{26} & \text{if } x,y\in[\frac{2}{3},1],\\ \frac{121}{250} & \text{otherwise.}\end{cases}$$

The conditions (b_1) and (b_2) are trivially hold.

We now verify condition (b_3) for nontrivial case.

Let $y\in[0,\frac{2}{3})$ and $x,z\in[\frac{2}{3},1]$.

Then $d(x,z)=\frac{23}{25}+\frac{x+z}{26}, d(x,y)=\frac{121}{250}, d(y,z)=\frac{121}{250}$.

We have

$x+z\le 2 \implies \frac{x+z}{26}\le\frac{1}{13}$ so that $\frac{23}{25}+\frac{x+z}{26}\le\frac{23}{25}+\frac{1}{13}<\frac{51}{49}(\frac{121}{125})$.

Therefore $d(x,z)=\frac{23}{25}+\frac{x+z}{26}<\frac{51}{49}(\frac{121}{250}+\frac{121}{250})=s[d(x,y)+d(y,z)]$ where $s=\frac{51}{49}$.

The other cases also trivially hold with $s=\frac{51}{49}$ so that (b_3) holds and d is a b-metric.

Clearly this metric d is complete so that (X,d) is a complete b-metric space.

Here we observe that when $x = \frac{9}{10}, z = 1 \in [\frac{2}{3}, 1]$ and $y \in [0, \frac{2}{3})$, we have $d(x, z) = \frac{1291}{1300} \nleq \frac{121}{125} = d(x, y) + d(y, z)$ so that d is not a metric.
We define $A, B, S, T : X \to X$ by

$$A(x) = \frac{2}{3} \text{ if } x \in [0,1],\ B(x) = \begin{cases} \frac{1}{3} & \text{if } x \in [0, \frac{2}{3}) \\ 1 - \frac{x}{2} & \text{if } x \in [\frac{2}{3}, 1], \end{cases}$$

$$S(x) = \begin{cases} x & \text{if } x \in [0, \frac{2}{3}) \\ \frac{4}{3} - x & \text{if } x \in [\frac{2}{3}, 1], \end{cases} \text{ and } T(x) = \begin{cases} \frac{1}{4} & \text{if } x \in [0, \frac{2}{3}) \\ \frac{4}{3} - x & \text{if } x \in [\frac{2}{3}, 1]. \end{cases}$$

Clearly $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$. $A(X) = \{\frac{2}{3}\}$ is b-closed.
We choose a sequence $\{x_n\}$ with $\{x_n\} = \frac{2}{3} + \frac{1}{2n}, n \geq 2$ with $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Sx_n = \frac{2}{3}$, hence the pair (A, S) satisfies the b-(E.A)-property.
Clearly the pairs (A, S) and (B, T) are weakly compatible.
We define $\beta : [0, \infty) \to [0, \frac{1}{s})$ by $\beta(t) = \frac{49}{51} e^{\frac{-t}{100}}$.
Then we have $\beta \in \mathfrak{F}$.

Case (i): $x, y \in [0, \frac{2}{3})$.
$d(Ax, By) = \frac{121}{250}, d(Sx, Ty) = \frac{11}{15}, d(Ty, By) = \frac{11}{15}, d(Sx, Ax) = \frac{121}{250}$,
$d(Ax, Ty) = \frac{121}{250}, d(Sx, By) = \frac{11}{15}$ and

$$\begin{aligned} M(x, y) &= \max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \tfrac{d(Sx,By)}{2s}, \tfrac{d(Ty,Ax)}{2s}, \\ &\qquad \tfrac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)}, \tfrac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\} \\ &= \max\{\tfrac{11}{15}, \tfrac{121}{250}, \tfrac{11}{15}, (\tfrac{49}{102})(\tfrac{11}{15}), (\tfrac{49}{102})(\tfrac{121}{250}), \tfrac{(\frac{121}{250})(\frac{11}{15})}{1+\frac{11}{15}+\frac{121}{250}}, \tfrac{(\frac{11}{15})(\frac{121}{250})}{1+(\frac{51}{49})^4(\frac{11}{15}+\frac{121}{250})}\} = \tfrac{11}{15}. \end{aligned}$$

Since

$$\begin{aligned} \tfrac{1}{2s}\min\{d(Sx, Ax), d(Ty, By)\} &= \tfrac{49}{102}\min\{\tfrac{121}{250}, \tfrac{11}{15}\} \\ &= (\tfrac{49}{102})(\tfrac{121}{250}) \\ &\leq \max\{\tfrac{11}{15}, \tfrac{121}{250}\} \\ &= \max\{d(Sx, Ty), d(Ax, By)\}. \end{aligned}$$

Now we consider

$$s^4 d(Ax, By) = (\tfrac{51}{49})^4(\tfrac{121}{250}) \leq \tfrac{49}{51} e^{\frac{-(\frac{11}{15})}{100}} \tfrac{11}{15} = \beta(M(x, y))M(x, y).$$

Case (ii): $x, y \in (\frac{2}{3}, 1]$.
$d(Ax, By) = \frac{121}{250}, d(Sx, Ty) = \frac{11}{15}, d(Ty, By) = \frac{11}{15}, d(Sx, Ax) = \frac{121}{250}$,
$d(Ax, Ty) = \frac{121}{250}, d(Sx, By) = \frac{11}{15}$ and

$$\begin{aligned} M(x, y) &= \max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \tfrac{d(Sx,By)}{2s}, \tfrac{d(Ty,Ax)}{2s}, \\ &\qquad \tfrac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)}, \tfrac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\} \\ &= \max\{\tfrac{11}{15}, \tfrac{121}{250}, \tfrac{11}{15}, (\tfrac{49}{102})(\tfrac{11}{15}), (\tfrac{49}{102})(\tfrac{121}{250}), \tfrac{(\frac{121}{250})(\frac{11}{15})}{1+\frac{11}{15}+\frac{121}{250}}, \tfrac{(\frac{11}{15})(\frac{121}{250})}{1+(\frac{51}{49})^4(\frac{11}{15}+\frac{121}{250})}\} = \tfrac{11}{15}. \end{aligned}$$

Since

$$\begin{aligned} \tfrac{1}{2s}\min\{d(Sx, Ax), d(Ty, By)\} &= \tfrac{49}{102}\min\{\tfrac{121}{250}, \tfrac{11}{15}\} \\ &= (\tfrac{49}{102})(\tfrac{121}{250}) \\ &\leq \max\{\tfrac{11}{15}, \tfrac{121}{250}\} \\ &= \max\{d(Sx, Ty), d(Ax, By)\}. \end{aligned}$$

Now we consider $s^4 d(Ax, By) = (\frac{51}{49})^4(\frac{121}{250}) \leq \frac{49}{51} e^{\frac{-(\frac{11}{15})}{100}} \frac{11}{15} = \beta(M(x, y))M(x, y)$.

Case (iii): $x \in [0, \frac{2}{3}), y \in (\frac{2}{3}, 1]$.
$d(Ax, By) = \frac{121}{250}, d(Sx, Ty) = \frac{11}{15}, d(Ty, By) = \frac{11}{15}, d(Sx, Ax) = \frac{121}{250}$,
$d(Ax, Ty) = \frac{121}{250}, d(Sx, By) = \frac{11}{15}$ and

$$\begin{aligned} M(x, y) &= \max\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \tfrac{d(Sx,By)}{2s}, \tfrac{d(Ty,Ax)}{2s}, \\ &\qquad \tfrac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)}, \tfrac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\} \end{aligned}$$

$= \max\{\frac{11}{15}, \frac{121}{250}, \frac{11}{15}, (\frac{49}{102})(\frac{11}{15}), (\frac{49}{102})(\frac{121}{250}), \frac{(\frac{121}{250})(\frac{11}{15})}{1+\frac{11}{15}+\frac{121}{250}}, \frac{(\frac{11}{15})(\frac{121}{250})}{(1+(\frac{51}{49})^4\frac{11}{15}+\frac{121}{250})}\} = \frac{11}{15}.$

Since

$$\begin{aligned}\frac{1}{2s}\min\{d(Sx,Ax), d(Ty,By)\} &= \frac{49}{102}\min\{\frac{121}{250}, \frac{11}{15}\}\\ &= (\frac{49}{102})(\frac{121}{250})\\ &\leq \max\{\frac{11}{15}, \frac{121}{250}\}\\ &= \max\{d(Sx,Ty), d(Ax,By)\}.\end{aligned}$$

Now we consider

$s^4 d(Ax,By) = (\frac{51}{49})^4(\frac{121}{250}) \leq \frac{49}{51}e^{\frac{-(\frac{11}{15})}{100}}\frac{11}{15} = \beta(M(x,y))M(x,y).$

Case (iv): $x \in (\frac{2}{3}, 1], y \in [0, \frac{2}{3})$.

$d(Ax,By) = \frac{121}{250}, d(Sx,Ty) = \frac{11}{15}, d(Ty,By) = \frac{11}{15}, d(Sx,Ax) = \frac{121}{250},$
$d(Ax,Ty) = \frac{121}{250}, d(Sx,By) = \frac{11}{15}$ and

$$\begin{aligned}M(x,y) &= \max\{d(Sx,Ty), d(Sx,Ax), d(Ty,By), \frac{d(Sx,By)}{2s}, \frac{d(Ty,Ax)}{2s},\\ &\qquad \frac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)}, \frac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\}\\ &= \max\{\frac{11}{15}, \frac{121}{250}, \frac{11}{15}, (\frac{49}{102})(\frac{11}{15}), (\frac{49}{102})(\frac{121}{250}), \frac{(\frac{121}{250})(\frac{11}{15})}{1+\frac{11}{15}+\frac{121}{250}}, \frac{(\frac{11}{15})(\frac{121}{250})}{1+(\frac{51}{49})^4(\frac{11}{15}+\frac{121}{250})}\}\\ &= \frac{11}{15}.\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{2s}\min\{d(Sx,Ax), d(Ty,By)\} &= \frac{49}{102}\min\{\frac{121}{250}, \frac{11}{15}\}\\ &= (\frac{49}{102})(\frac{121}{250})\\ &\leq \max\{\frac{11}{15}, \frac{121}{250}\}\\ &= \max\{d(Sx,Ty), d(Ax,By)\}.\end{aligned}$$

Now we consider $s^4 d(Ax,By) = (\frac{51}{49})^4(\frac{121}{250}) \leq \frac{49}{51}e^{\frac{-(\frac{11}{15})}{100}}\frac{11}{15} = \beta(M(x,y))M(x,y).$

Case (v): $x = \frac{2}{3}, y \in [0, \frac{2}{3})$.

$d(Ax,By) = \frac{121}{250}, d(Sx,Ty) = \frac{121}{250}, d(Ty,By) = \frac{11}{15}, d(Sx,Ax) = 0,$
$d(Ax,Ty) = \frac{121}{250}, d(Sx,By) = \frac{121}{250}$ and

$$\begin{aligned}M(x,y) &= \max\{d(Sx,Ty), d(Sx,Ax), d(Ty,By), \frac{d(Sx,By)}{2s}, \frac{d(Ty,Ax)}{2s},\\ &\qquad \frac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)}, \frac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\}\\ &= \max\{\frac{121}{250}, 0, \frac{11}{15}, (\frac{49}{102})(\frac{121}{250}), (\frac{49}{102})(\frac{121}{250}), 0, \frac{(\frac{121}{250})(\frac{121}{250})}{(1+(\frac{51}{49})^4\frac{11}{15}+\frac{121}{250})}\} = \frac{11}{15}.\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{2s}\min\{d(Sx,Ax), d(Ty,By)\} &= \frac{240}{489}\min\{0, \frac{11}{15}\}\\ &= 0 \leq \max\{\frac{121}{250}, \frac{121}{250}\}\\ &\qquad = \max\{d(Sx,Ty), d(Ax,By)\}.\end{aligned}$$

Now we consider

$s^4 d(Ax,By) = (\frac{51}{49})^4(\frac{121}{250}) \leq \frac{49}{51}e^{\frac{-(\frac{11}{15})}{100}}\frac{11}{15} = \beta(M(x,y))M(x,y).$

Case (vi): $x = \frac{2}{3}, y \in (\frac{2}{3}, 1]$.

$d(Ax,By) = \frac{121}{250}, d(Sx,Ty) = \frac{121}{250}, d(Ty,By) = \frac{11}{15}, d(Sx,Ax) = 0,$
$d(Ax,Ty) = \frac{121}{250}, d(Sx,By) = \frac{121}{250}$ and

$$\begin{aligned}M(x,y) &= \max\{d(Sx,Ty), d(Sx,Ax), d(Ty,By), \frac{d(Sx,By)}{2s}, \frac{d(Ty,Ax)}{2s},\\ &\qquad \frac{d(Sx,Ax)d(Ty,By)}{1+d(Sx,Ty)+d(Ax,By)}, \frac{d(Sx,By)d(Ty,Ax)}{1+s^4[d(Sx,Ty)+d(Ax,By)]}\}\\ &= \max\{\frac{121}{250}, 0, \frac{11}{15}, (\frac{49}{102})(\frac{121}{250}), (\frac{49}{102})(\frac{121}{250}), 0, \frac{(\frac{121}{250})(\frac{121}{250})}{1+(\frac{51}{49})^4(\frac{11}{15}+\frac{121}{250})}\} = \frac{11}{15}.\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{2s}\min\{d(Sx,Ax), d(Ty,By)\} &= \frac{240}{489}\min\{0, \frac{11}{15}\}\\ &= 0 \leq \max\{\frac{121}{250}, \frac{121}{250}\}\\ &\qquad = \max\{d(Sx,Ty), d(Ax,By)\}.\end{aligned}$$

Now we consider
$s^4 d(Ax,By) = (\frac{51}{49})^4(\frac{121}{250}) \leq \frac{49}{51}e^{\frac{-(\frac{11}{15})}{100}}\frac{11}{15} = \beta(M(x,y))M(x,y).$
Case (vii): $x \in [0,\frac{2}{3}), y = \frac{2}{3}$.
Here $d(Ax,By) = 0$. Clearly the inequality (2.3) holds in this case.
Case (viii): $x \in [0,\frac{2}{3}), y = \frac{2}{3}$.
Here $d(Ax,By) = 0$. In this case the inequality (2.3) holds clearly.
From all the above four cases, A, B, S and T are Geraghty-Suzuki type contraction maps. Therefore A, B, S and T satisfy all the hypotheses of Theorem 2.4 and $\frac{2}{3}$ is the unique common fixed point of A, B, S and T.

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