Proceedings of International Mathematical Sciences
ISSN:2717-6355, URL:https://dergipark.org.tr/tr/pub/pims
Volume II Issue 1 (2020), Pages 48-59.

EXISTENCE OF SOLUTION FOR A SYSTEM OF COUPLED FRACTIONAL BOUNDARY VALUE PROBLEM

NOUREDDINE BOUTERAA*, HABIB DJOURDEM**, AND SLIMANE BENAICHA***
*LABORATORY OF FUNDAMENTAL AND APPLIED MATHEMATICS OF ORAN (LMFAO), UNIVERSITY OF ORAN1, AHMED BENBELLA. ALGERIA
**LABORATORY OF FUNDAMENTAL AND APPLIED MATHEMATICS OF ORAN (LMFAO), UNIVERSITY OF ORAN1, AHMED BENBELLA. ALGERIA
***LABORATORY OF FUNDAMENTAL AND APPLIED MATHEMATICS OF ORAN (LMFAO), UNIVERSITY OF ORAN1, AHMED BENBELLA. ALGERIA

Abstract. This paper deals with the existence and uniqueness of solutions for a coupled system of fractional differential equations with coupled nonlocal and integral boundary conditions. The existence results are obtained by using Leray-Schauder nonlinear alternative and Banach contraction principle. An illustrative example is presented at the end of the paper to illustrate the validity of our results.

1. Introduction

In this paper, we are interested in the existence of solutions for the nonlinear fractional differential equations

$$\begin{cases} {}^{c}D^{\alpha}u\left(t\right)=f\left(t,u\left(t\right),v\left(t\right)\right), & t\in\left[0,1\right],\ 2<\alpha\leq3, \\ {}^{c}D^{\beta}v\left(t\right)=g\left(t,u\left(t\right),v\left(t\right)\right), & t\in\left[0,1\right],\ 2<\beta\leq3, \end{cases} \tag{1.1}$$

subject to three-point coupled boundary conditions

$$\begin{cases} \lambda u\left(0\right)+\gamma u\left(1\right)=v\left(\eta\right),\ \lambda v\left(0\right)+\gamma v\left(1\right)=u\left(\eta\right), \\ u\left(0\right)=\int_{0}^{\eta}v\left(s\right)ds,\ u\left(0\right)=\int_{0}^{\eta}v\left(s\right)ds, \\ \lambda^{C}D^{P}u\left(0\right)+\gamma^{C}D^{P}u\left(1\right)=v\left(\eta\right),\ 1<p\leq2 \\ \lambda^{C}D^{P}v\left(0\right)+\gamma^{C}D^{P}v\left(1\right)=u\left(\eta\right),\ 1<p\leq2 \end{cases} \tag{1.2}$$

where $\gamma,\ \lambda\in\mathbb{R}^{+},\ f,\ g\in C\left(\left[0,1\right]\times\mathbb{R}^{2},\mathbb{R}\right)$ and ${}^{c}D^{\alpha},\ {}^{c}D^{\beta}$ denote the Caputo fractional derivatives of order α and β respectively.

The concept of fractional calculus has played an important role in improving the work based on integer-order (classical) calculus in several diverse disciplines of science and engineering and the details of its basic notions, results and methods

2020 *Mathematics Subject Classification.* Primary: 34B15 ; Secondaries: 34B10 .
Key words and phrases. Fractional differential system; existence; nonlocal boundary; fixed-point theorem.

Submitted on September 3th, 2019. Published on June 30th, 2020.
Area Editors: Valéria Neves DOMINGOS CAVALCANTİ and Idris DAG.

can be found in the texts ([2, 17]) and papers ([1, 21, 23]). The nonlocal nature of a fractional order differential operator, which take into account hereditary properties of various material and processes, has helped to improve the mathematical modeling of many natural phenomena and physical processes, see for example ([17, 22]). The increasing interest of fractional differential equations and inclusions are motivated by their applications in various fields of science such as physics chemistry, biology, economics, fluid mechanics, control theory, etc, we refer the reader to ([3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 19, 20, 27]) and the references therein.

Coupled systems of fractional-order differential equations constitute an interesting and important field of research in view of their applications in many real world problems such as anomalous diffusion [25], disease models [12], synchronization of chaotic systems [24], etc. For some theoretical works on coupled systems of fractional-order differential equations, we refer the reader to a series of papers ([10, 15, 16, 26, 28, 29]).

The goal of this paper is to establish the existence and uniqueness results for the nonlocal boundary value problem (1.1) – (1.2) by using some well-known tools of fixed point theory such as Banach contraction principle and Leray-Schauder nonlinear alternative. The paper is organized as follows. In Section 2, we recall some preliminary facts that we need in the sequel, for more details; see [17]. Section 3, deals with main results and we give an example to illustrate our results.

2. Preliminaries

In this section, we introduce some definitions and lemmas, see ([17, 18]).

Definition 2.1. *Let $\alpha > 0,\ n-1 < \alpha < n,\ n = [\alpha] + 1$ and $u \in C([0,\infty), \mathbb{R})$. The Caputo derivative of fractional order α for the function u is defined by*

$$^{c}D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s)\, ds,$$

where $\Gamma(\cdot)$ is the Euler Gamma function.

Definition 2.2. *The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $u : (0,\infty) \to \mathbb{R}$ is given by*

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds, \quad t > 0,$$

where $\Gamma(\cdot)$ is the Euler Gamma function, provided that the right side is pointwise defined on $(0,\infty)$.

Lemma 2.1. [18]. *Let $\alpha > 0,\ n-1 < \alpha < n$ and the function $g : [0,T] \to \mathbb{R}$ be continuous for each $T > 0$. Then, the general solution of the fractional differential equation $^{c}D^{\alpha}g(t) = 0$ is given by*

$$g(t) = c_0 + c_1 t + \cdots + c_{n-1}t^{n-1},$$

where $c_0, c_1, ..., c_{n-1}$ are real constants and $n = [\alpha] + 1$.

Also, in [8], authors have been proved that for each $T > 0$ and $u \in C([0,T])$ we have

$$I^{\alpha\, c}D^{\alpha}u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1}t^{n-1},$$

where $c_0, c_1, ..., c_{n-1}$ are real constants and $n = [\alpha] + 1$.

3. Existence results

Let $X = \{u(t) : u(t) \in C([0,1], \mathbb{R})\}$ endowed with the norm $\|u\| = \sup_{t\in[0,1]} |u(t)|$ such that $\|u\| < \infty$. Then $(X, \|.\|)$ is a Banach space and the product space $(X \times X, \|(u,v)\|)$ is also a Banach space equipped with the norm $\|(u,v)\| = \|u\| + \|v\|$.

Throughout the paper, we let

$$M = \frac{\Gamma(3-p)}{|\gamma - \eta^{2-p}|} \neq 0,\ |\lambda + \gamma - 1| \neq 0,\ \left|\gamma - \eta^2\right| \neq 0,\ \ Q = \left|2(1-\eta)(\gamma - \eta) + \eta^2 |\lambda + \gamma - 1|\right| \neq 0,$$

$$A(t) = |\Lambda_1(t)| = |\lambda + \gamma - 1| \left(\eta^2 + 2(1-\eta)t\right),$$

$$B(t) = |\Lambda_2(t)| = \left(\eta^3 |\lambda + \gamma - 1| + 3\left|\gamma - \eta^2\right|(1-\eta)\right)\left(\eta^2 + 2(1-\eta)t\right) - Q\left(\eta^3 + 3(1-\eta)t^2\right),$$

and

$$Q = 2(1-\eta)(\gamma - \eta) + \eta^2(\lambda + \gamma - 1) \neq 0.$$

Lemma 3.1. *Let $y \in C([0,1], \mathbb{R})$. Then the solution of the linear differential system*

$$\begin{cases} {}^cD^{\alpha}u(t) = y(t),\ {}^cD^{\beta}v(t) = h(t),\ t \in [0,1],\ 2 < \alpha,\ \beta \leq 3 \\ \lambda u(0) + \gamma u(1) = v(\eta),\ \lambda v(0) + \gamma v(1) = u(\eta), \\ u(0) = \int_0^{\eta} v(s)\,ds,\ \ v(0) = \int_0^{\eta} u(s)\,ds, \\ \lambda^c D^p u(0) + \gamma^c D^p u(1) =^c D^p v(\eta),\quad 1 < p \leq 2, \\ \lambda^c D^p v(0) + \gamma^c D^p v(1) =^c D^p u(\eta),\quad 1 < p \leq 2, \end{cases} \tag{3.1}$$

is equivalent to the system of integral equations

$$\begin{aligned} u(t) = & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds + \frac{1}{1-\eta}\int_0^{\eta}\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} h(\tau)\,d\tau\right) ds \\ & - \frac{\Lambda_1(t)}{Q(1-\eta)} \int_0^{\eta}\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} h(\tau)\,d\tau\right) ds \\ & - \frac{\Lambda_2(t) M}{6(1-\eta)Q}\left[\int_0^{\eta} \frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)} h(s)\,ds - \gamma \int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\,ds\right] \\ & + \frac{\Lambda_1(t)}{Q(\lambda+\gamma-1)}\left[\int_0^{\eta} \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)} h(s)\,ds - \gamma \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds\right] \end{aligned} \tag{3.2}$$

and

$$v(t)=\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}y(s)\,ds+\frac{1}{1-\eta}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}h(\tau)\,d\tau\right)ds$$
$$-\frac{\Lambda_1(t)}{Q(1-\eta)}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}h(\tau)\,d\tau\right)ds$$
$$-\frac{\Lambda_2(t)\,M}{6(1-\eta)\,Q}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}h(s)\,ds-\gamma\int_0^1 \frac{(1-s)^{\beta-p-1}}{\Gamma(\alpha-p)}y(s)\,ds\right]$$
$$+\frac{\Lambda_1(t)}{Q(\lambda+\gamma-1)}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\,ds-\gamma\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\alpha)}y(s)\,ds\right], \tag{3.3}$$

where

$$\Lambda_1(t)=(\lambda+\gamma-1)\left(\eta^2+2(1-\eta)t\right),$$

and

$$\Lambda_2(t)=\left(\eta^3(\lambda+\gamma-1)+3\left(\gamma-\eta^2\right)(1-\eta)\right)\left(\eta^2+2(1-\eta)t\right)-Q\left(\eta^3+3(1-\eta)t^2\right).$$

Proof. It is well known that the solution of equations ${}^cD^\alpha u(t)=y(t)$, ${}^cD^\beta v(t)=h(t)$ can be written as

$$u(t)=I^\alpha y(t)+c_0+c_1t+c_2t^2, \tag{3.4}$$

$$v(t)=I^\beta h(t)+d_0+d_1t+d_2t^2, \tag{3.5}$$

where $c_0,c_1,c_2\in\mathbb{R}$ and $d_0,d_1,d_2\in\mathbb{R}$ are arbitrary constants.
Then, from (3.4) we have

$$u'(t)=I^{\alpha-1}y(t)+c_1+2c_2t,$$

and

$${}^cD^pu(t)=I^{\alpha-p}y(t)+c_2\frac{2t^{2-p}}{\Gamma(3-p)},\quad 1<p\le 2.$$

By using the three-point boundary conditions, we obtain

$$c_2=\frac{M}{2}\left(I^{\beta-p}y(\eta)-\gamma I^{\alpha-p}y(1)\right),$$

$$c_0=-\frac{2\eta^2(\lambda+\gamma-1)}{2(1-\eta)Q}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}h(\tau)\,d\tau\right)ds+\frac{1}{1-\eta}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}h(\tau)\,d\tau\right)ds$$
$$-\frac{\left(\eta^2\left[\eta^3(\lambda+\gamma-1)+3\left(\gamma-\eta^2\right)(1-\eta)\right]-\eta^3Q\right)M}{2(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)}h(s)\,ds-\gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds\right]$$
$$+\frac{\eta^2}{Q}\left[\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}h(s)\,ds-\gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds\right],$$

and

$$c_1=\frac{-2\left(\lambda+\gamma-1\right)}{Q}\int_0^{\eta}\left(\int_0^{s}\frac{\left(s-\tau\right)^{\beta-1}}{\Gamma\left(\beta\right)}h\left(\tau\right)d\tau\right)ds$$

$$-\frac{\left(\eta^3\left(\lambda+\gamma-1\right)+3\left(\gamma-\eta^2\right)\left(1-\eta\right)\right)M}{3Q}\left[\int_0^{\eta}\frac{\left(\eta-s\right)^{\beta-p-1}}{\Gamma\left(\beta-p\right)}h\left(s\right)ds-\gamma\int_0^{1}\frac{\left(1-s\right)^{\alpha-p-1}}{\Gamma\left(\alpha-p\right)}y\left(s\right)ds\right]$$

$$+\frac{2\left(1-\eta\right)}{Q}\left[\int_0^{\eta}\frac{\left(\eta-s\right)^{\beta-1}}{\Gamma\left(\beta\right)}y\left(s\right)ds-\gamma\int_0^{1}\frac{\left(1-s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}y\left(s\right)ds\right].$$

Substituting the values of constants c_0, c_1 and c_2 in (3.4), we get solution (3.2). Similarly, we obtain solution (3.3). The proof is complete. □

The following relations hold:

$\left|A\left(t\right)\right|\leq\left|\beta+\gamma-1\right|\left(\eta^2+2\left(1-\eta\right)\right)=A_1,$

and

$\left|B\left(t\right)\right|\leq\left|\left(\eta^3\left|\beta+\gamma-1\right|+3\left|\gamma-\eta^2\right|\left(1-\eta\right)\right)\left(\eta^2+2\left(1-\eta\right)\right)-Q\left(\eta^3+3\left(1-\eta\right)\right)\right|=B_1,$

For the sake of brevity, we set

$\Delta_1=\frac{\eta^{\beta+1}}{(1-\eta)\Gamma(\beta+2)}+\frac{A_1\eta^{\beta+1}}{Q(1-\eta)\Gamma(\beta+2)}+\frac{MB_1\eta^{\beta-p}}{(1-\eta)Q\Gamma(\lambda-p+1)}+\frac{A_1\eta^{\beta}}{Q|\beta+\gamma-1|\Gamma(\beta+1)},$

$\Delta_2=\frac{MB_1\gamma}{6(1-\eta)Q\Gamma(\alpha-p+1)}+\frac{A_1\gamma}{Q|\lambda+\gamma-1|\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha+1)},$

$\Delta_3=\frac{\eta^{\alpha+1}}{(1-\eta)\Gamma(\alpha+2)}+\frac{A_1\eta^{\alpha+1}}{Q(1-\eta)\Gamma(\alpha+2)}+\frac{MB_1\eta^{\alpha-p}}{(1-\eta)Q\Gamma(\alpha-p+1)}+\frac{A_1\eta^{\alpha}}{Q|\lambda+\gamma-1|\Gamma(\alpha+1)},$

and

$\Delta_4=\frac{MB_1\gamma}{6(1-\eta)Q\Gamma(\beta-p+1)}+\frac{A_1\gamma}{Q|\lambda+\gamma-1|\Gamma(\beta+1)}+\frac{1}{\Gamma(\beta+1)}.$

In view of Lemma 1.2, we define the operator $T : X\times X\to X\times X$ by

$$T\left(u,v\right)\left(t\right)=\begin{pmatrix}T_1\left(u,v\right)\left(t\right)\\ T_2\left(u,v\right)\left(t\right)\end{pmatrix},$$

where

$$T_1\left(u,v\right)\left(t\right)=\int_0^{t}\frac{\left(t-s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}f\left(s,u\left(s\right),v\left(s\right)\right)ds+\frac{1}{1-\eta}\int_0^{\eta}\left(\int_0^{s}\frac{\left(s-\tau\right)^{\beta-1}}{\Gamma\left(\beta\right)}g\left(\tau,u\left(\tau\right),v\left(\tau\right)\right)d\tau\right)ds$$

$$-\frac{B\left(t\right)M}{6\left(1-\eta\right)Q}\left[\int_0^{\eta}\frac{\left(\eta-s\right)^{\beta-p-1}}{\Gamma\left(\beta-p\right)}g\left(s,u\left(s\right),v\left(s\right)\right)ds-\gamma\int_0^{1}\frac{\left(1-s\right)^{\alpha-p-1}}{\Gamma\left(\alpha-p\right)}f\left(s,u\left(s\right),v\left(s\right)\right)ds\right]$$

$$+\frac{A\left(t\right)}{Q\left|\beta+\gamma-1\right|}\left[\int_0^{\eta}\frac{\left(\eta-s\right)^{\beta-1}}{\Gamma\left(\beta\right)}g\left(s,u\left(s\right),v\left(s\right)\right)ds-\gamma\int_0^{1}\frac{\left(1-s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}f\left(s,u\left(s\right),v\left(s\right)\right)ds.\right]$$

$$-\frac{A\left(t\right)}{Q\left(1-\eta\right)}\int_0^{\eta}\left(\int_0^{s}\frac{\left(s-\tau\right)^{\beta-1}}{\Gamma\left(\beta\right)}g\left(\tau,u\left(\tau\right),v\left(\tau\right)\right)d\tau\right)ds$$

and

$$T_2\left(u,v\right)\left(t\right)=\int_0^{t}\frac{\left(t-s\right)^{\beta-1}}{\Gamma\left(\beta\right)}g\left(s,u\left(s\right),v\left(s\right)\right)ds+\frac{1}{1-\eta}\int_0^{\eta}\left(\int_0^{s}\frac{\left(s-\tau\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}f\left(\tau,u\left(\tau\right),v\left(\tau\right)\right)d\tau\right)ds$$

$$-\frac{B(t)M}{6(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}f(s,u(s),v(s))\,ds-\gamma\int_0^1 \frac{(1-s)^{\beta-p-1}}{\Gamma(\beta-p)}g(s,u(s),v(s))\,ds\right]$$
$$+\frac{A(t)}{Q|\beta+\gamma-1|}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),v(s))\,ds-\gamma\int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}g(s,u(s),v(s))\,ds.\right]$$
$$-\frac{A(t)}{Q(1-\eta)}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(\tau,u(\tau),v(\tau))\,d\tau\right)ds.$$

Observe that the boundary value problem $(1.1)-(1.2)$ has solutions if the operator equation $(u,v)=T(u,v)$ has fixed points.

Now we are in a position to present the first main results of this paper. The existence results is based on Leray-Schauder nonlinear alternative.

Lemma 3.2. [14] *(Leray-Schauder alternative). Let E be a Banach space and $T:E\to E$ be a completely continuous operator (i.e., a map restricted to any bounded set in E is compact). Let*

$$\varepsilon(T)=\{(u,v)\in X\times X:(u,v)=\lambda T(u,v),\ for\ some\ 0<\lambda<1\}.$$

Then either the $\varepsilon(T)$ is unbounded or T has at least one fixed point.

Theorem 3.3. *Assume that $f,g:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are a continuous function and*
(H_1) there exist constants $k_i>0,\ m_i>0,\ i=0,1,2$ such that $\forall u\in\mathbb{R}$, $\forall v\in\mathbb{R}$, we have

$$|f(t,u,v)|\le k_0+k_1|u|+k_2|v|,$$

and

$$|g(t,u,v)|\le m_0+m_1|u|+m_2|v|.$$

If $(\Delta_2+\Delta_3)k_1+(\Delta_1+\Delta_4)m_1<1$ and $(\Delta_2+\Delta_3)k_2+(\Delta_1+\Delta_4)m_3<1$, where $\Delta_i,\ i=1,2,3,4$ are given above. Then the boundary value problem $(1.1)-(1.2)$ has at least one solution on $[0,1]$.

Proof. It is clear that T is a continuous operator where $T:X\times X\to X\times X$ is defined above. Now, we show that T is completely continuous. Let $\Omega\subset X\times X$ be bounded. Then there exist positive constants L_1 and L_2 such that

$$|f(t,u(t),v(t))|\le L_1,\quad |g(t,u(t),v(t))|\le L_2,\quad \forall (u,v)\in\Omega.$$

Then for any $(u,v)\in\Omega$, we have

$$|T_1(u,v)(t)|\le\frac{L_2}{1-\eta}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau\right)ds$$
$$+\frac{|A(t)|L_2}{Q(1-\eta)}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau\right)ds+L_1\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds$$
$$+\frac{M|B(t)|}{6(1-\eta)Q}\left[L_2\int_0^\eta\frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)}ds+\gamma L_1\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds\right]$$

$$+\frac{|A(t)|}{Q|\lambda+\gamma-1|}\left[L_2\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}ds+\gamma L_1\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right]$$

$$\leq L_2\left\{\frac{1}{1-\eta}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau\right)ds+\frac{A_1}{Q(1-\eta)}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\alpha\beta)}d\tau\right)ds\right.$$

$$\left.+\frac{MB_1}{6(1-\eta)Q}\int_0^\eta \frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)}ds+\frac{A_1}{6|\lambda+\gamma-1|}\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}ds\right\}$$

$$+L_1\left\{\frac{M\gamma B_1}{6(1-\eta)Q}\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds+\frac{A_1\gamma}{Q|\lambda+\gamma-1|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right.$$

$$\left.+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right\}$$

$$\leq L_2\Delta_1+L_1\Delta_2.$$

Hence

$$\|T_1(u,v)\|\leq L_2\Delta_1+L_1\Delta_2. \tag{3.6}$$

In the same way, we can obtain that

$$\|T_2(u,v)\|\leq L_1\Delta_3+L_2\Delta_4. \tag{3.7}$$

Thus, it follows from (3.6) and (3.7) that the operator T is uniformly bounded, since $\|T(u,v)\|\leq L_1(\Delta_1+\Delta_3)+L_2(\Delta_2+\Delta_4)$. Now, we show that T is equicontinuous. Let $t_1,\ t_2\in[0,1]$ with $t_1<t_2$. Then we have

$$|T_1(u(t_2),v(t_2))-T_1(u(t_1),v(t_1))|\leq L_1\int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}ds$$

$$+L_1\int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds+\frac{|A(t_2)-A(t_1)|L_2}{Q(1-\eta)}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau\right)ds$$

$$+\frac{(B(t_2)-B(t_1))M}{6(1-\eta)Q}\left[L_2\int_0^\eta \frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)}ds+\gamma L_1\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds\right]$$

$$+\frac{A(t_2)-A(t_1)}{Q|\lambda+\gamma-1|}\left[L_2\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}ds-\gamma L_1\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right].$$

Obviously, the right-hand side of the above inequality tends to zero as $t_2\to t_1$. Similarly, we have

$$|T_2(u(t_2),v(t_2))-T_2(u(t_1),v(t_1))|\leq L_2\int_0^{t_1} \frac{(t_2-s)^{\beta-1}-(t_1-s)^{\beta-1}}{\Gamma(\beta)}ds$$

$$
+L_2 \int_{t_1}^{t_2} \frac{(t_2 - s)^{\beta-1}}{\Gamma(\beta)} ds + \frac{|A(t_2) - A(t_1)| L_1}{Q(1-\eta)} \int_0^{\eta} \left(\int_0^{s} \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} d\tau \right) ds
$$

$$
+ \frac{(B(t_2) - B(t_1)) M}{6(1-\eta) Q} \left[L_1 \int_0^{\eta} \frac{(\eta - s)^{\alpha-p-1}}{\Gamma(\alpha - p)} ds + \gamma L_2 \int_0^{1} \frac{(1-s)^{\beta-p-1}}{\Gamma(\beta - p)} ds \right]
$$

$$
+ \frac{A(t_2) - A(t_1)}{Q|\lambda + \gamma - 1|} \left[L_1 \int_0^{\eta} \frac{(\eta - s)^{\alpha-1}}{\Gamma(\alpha)} ds - \gamma L_2 \int_0^{1} \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} ds \right].
$$

Again, it is seen that the right-hand side of the above inequality tends to zero as $t_2 \to t_1$. Thus, the operator T is equicontinuous.
Therefore, the operator T is completely continuous.
Finally, it will be verified that the set $\varepsilon(T) = \{(u,v) \in X \times X : (u,v) = \lambda T(u,v),\ 0 \le \lambda \le 1\}$ is bounded. Let $(u,v) \in \varepsilon(T)$, with $(u,v) = \lambda T(u,v)$ for any $t \in [0,1]$, we have

$$
u(t) = \lambda T_1(u,v)(t), \quad v(t) = \lambda T_2(u,v)(t).
$$

Then

$$
\begin{aligned}
|u(t)| &\le \Delta_2 (k_0 + k_1 |u| + k_2 |v|) + \Delta_1 (m_0 + m_1 |u| + m_2 |v|), \\
&= \Delta_2 k_0 + \Delta_1 m_0 + (\Delta_2 k_1 + \Delta_1 m_1) |u| + (\Delta_2 k_2 + \Delta_1 m_2) |v|,
\end{aligned}
$$

and

$$
\begin{aligned}
|v(t)| &\le \Delta_3 (k_0 + k_1 |u| + k_2 |v|) + \Delta_4 (m_0 + m_1 |u| + m_2 |v|), \\
&= \Delta_3 k_0 + \Delta_4 m_0 + (\Delta_3 k_1 + \Delta_4 m_1) |u| + (\Delta_3 k_2 + \Delta_4 m_2) |v|.
\end{aligned}
$$

Hence we have

$$
\|u\| = \Delta_2 k_0 + \Delta_1 m_0 + (\Delta_2 k_1 + \Delta_1 m_1) \|u\| + (\Delta_2 k_2 + \Delta_1 m_2) \|v\|,
$$

and

$$
\|v\| = \Delta_3 k_0 + \Delta_4 m_0 + (\Delta_3 k_1 + \Delta_4 m_1) |u| + (\Delta_3 k_2 + \Delta_4 m_2) |v|,
$$

which imply that

$$
\begin{aligned}
\|u\| + \|v\| = (\Delta_2 + \Delta_3) k_0 + (\Delta_1 + \Delta_4) m_0 + [(\Delta_2 + \Delta_3) k_1 + (\Delta_1 + \Delta_4) m_1] \|u\| \\
+ [(\Delta_2 + \Delta_3) k_2 + (\Delta_1 + \Delta_4) m_2] \|v\|.
\end{aligned}
$$

Consequently,

$$
\|(u,v)\| = \frac{(\Delta_2 + \Delta_3) k_0 + (\Delta_1 + \Delta_4) m_0}{\Delta_0},
$$

where

$$
\Delta_0 = \min\{1 - [(\Delta_2 + \Delta_3) k_1 + (\Delta_1 + \Delta_4) m_1], 1 - [(\Delta_2 + \Delta_3) k_2 + (\Delta_1 + \Delta_4) m_2]\},
$$

which proves that $\varepsilon(T)$ is bounded. Thus, by Lemma 3.2, the operator T has at least one fixed point. Hence boundary value problem (1.1) − (1.2) has at least one solution. The proof is complete. □

Now, we are in a position to present the second main results of this paper

Theorem 3.4. *Assume that $f,\ g : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous functions and there exist positive constants L_1 and L_2 such that for all $t \in [0,1]$ and $u_i\ ,v_i \in \mathbb{R}$, $i = 1,\ 2$, we havre*
(1) $|f(t,u_1,u_2) - f(t,v_1,v_2)| \le L_1(|u_1-v_1| + |u_2-v_2|)$,

(2) $|g(t,u_1,u_2) - g(t,v_1,v_2)| \le L_2(|u_1-v_1| + |u_2-v_2|)$.
Then the boundary value problem (1.1)−(1.2) *has a unique solution on* $[0,1]$ *provided*

$$(\Delta_1+\Delta_3)L_1 + (\Delta_2+\Delta_4)L_2 < 1.$$

Proof. Let us set $\sup_{t\in[0,1]}|f(t,0,0)| = N_1 < \infty$ and $\sup_{t\in[0,1]}|g(t,0,0)| = N_2 < \infty$. For $u \in X$, we observe that

$$\begin{aligned}|f(t,u(t),v(t))| &\le |f(t,u(t)) - f(t,0,0)| + |f(t,0,0)|,\\ &\le L_1(|u(t)| + |v(t)|) + N_1,\\ &\le L_1(\|u\| + \|v\|) + N_1,\end{aligned}$$

and

$$|g(t,u(t),v(t))| \le |g(t,u(t)) - g(t,0,0)| + |g(t,0,0)| \le L_2\|u\| + N_2.$$

Then for $u \in X$, we have

$$|T_1(u,v)(t)| \le \frac{1}{1-\eta}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}[L_2\|(u,v)\| + N_2]\,d\tau\right)ds$$

$$+\frac{|A(t)|}{Q(1-\eta)}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}[L_2\|(u,v)\| + N_2]\,d\tau\right)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}[L_1\|(u,v)\| + N_1]\,ds$$

$$+\frac{M|B(t)|}{6(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)}[L_2\|(u,v)\| + N_2]\,ds + \gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}[L_1\|(u,v)\| + N_1]\,ds\right]$$

$$+\frac{|A(t)|}{Q|\lambda+\gamma-1|}\left[\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}[L_2\|(u,v)\| + N_2]\,ds + \gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}[L_1\|(u,v)\| + N_1]\,ds,\right]$$

$$\le (L_2\|(u,v)\| + N_2)\left\{\frac{1}{1-\eta}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau\right)ds + \frac{A_1}{Q(1-\eta)}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau\right)ds\right.$$

$$\left.+\frac{MB_1}{6(1-\eta)Q}\int_0^\eta \frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)}ds + \frac{A_1}{6|\lambda+\gamma-1|}\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}ds\right\}$$

$$+\frac{MB_1}{6(1-\eta)Q}\int_0^\eta \frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)}ds + \frac{A_1}{6|\lambda+\gamma-1|}\int_0^\eta \frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}ds\bigg\}$$

$$+(L_1\|(u,v)\| + N_1)\left\{\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right.$$

$$+\frac{M\gamma B_1}{6(1-\eta)Q}\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds+\frac{A_1\gamma}{Q|\lambda+\gamma-1|}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\Bigg\},$$

$$\leq (L_2 r+N_2)\Delta_1+(L_1 r+N_1)\Delta_2$$

Hence

$$\|T_1(u,v)\|\leq (L_2\Delta_1+L_1\Delta_2)r+N_2\Delta_1+N_1\Delta_2$$

In the same way, we can obtain that

$$\|T_2(u,v)\|\leq (L_1\Delta_3+L_2\Delta_4)r+N_2\Delta_4+N_1\Delta_3.$$

Consequently,

$$\|T(u,v)\|\leq ((\Delta_2+\Delta_3)L_1+(\Delta_1+\Delta_4)L_2)r+N_2(\Delta_1+\Delta_4)+N_1(\Delta_2+\Delta_3)\leq r.$$

Now, for $(u_1,v_1)\,,\,(u_2,v_2)\in X\times X$ and for each $t\in[0,1]$, it follows from assumption (H_3) that

$$|T_1(u_2,v_2)(t)-T_1(u_1,v_1)(t)|\leq L_2(\|u_2-u_1\|+\|v_2-v_1\|)\Bigg\{\frac{1}{1-\eta}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau\right)ds$$

$$+\frac{A_1}{Q(1-\eta)}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)}d\tau\right)ds$$

$$+\frac{MB_1}{6(1-\eta)Q}\int_0^\eta\frac{(\eta-s)^{\beta-p-1}}{\Gamma(\beta-p)}ds+\frac{A_1}{6|\lambda+\gamma-1|}\int_0^\eta\frac{(\eta-s)^{\beta-1}}{\Gamma(\beta)}ds\Bigg\}$$

$$+L_1(\|u_2-u_1\|+\|v_2-v_1\|)\Bigg\{\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds$$

$$+\frac{M\gamma B_1}{6(1-\eta)Q}\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}ds+\frac{A_1\gamma}{Q|\lambda+\gamma-1|}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\Bigg\}$$

$$\leq (L_2\Delta_1+L_1\Delta_2)(\|u_2-u_1\|+\|v_2-v_1\|).$$

Thus

$$\|T_1(u_2,v_2)-T_1(u_1,v_1)\|\leq (L_2\Delta_1+L_1\Delta_2)(\|u_2-u_1\|+\|v_2-v_1\|). \tag{3.8}$$

. Similarly,

$$\|T_2(u_2,v_2)-T_2(u_1,v_1)\|\leq (L_2\Delta_3+L_1\Delta_4)(\|u_2-u_1\|+\|v_2-v_1\|). \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$\|T(u_2,v_2)-T(u_1,v_1)\|\leq (L_2(\Delta_1+\Delta_3)+L_1(\Delta_2+\Delta_4))(\|u_2-u_1\|+\|v_2-v_1\|).$$

Since $L_2(\Delta_1+\Delta_3)+L_1(\Delta_2+\Delta_4)<1$, thus T is a contraction operator. Hence it follows by Banach's contraction principle that the boundary value problem (1.1) – (1.2) has a unique solution on $[0,1]$.

□

We construct an example to illustrate the applicability of the results presented.

Example 3.1. *Consider the following system fractional differential equation*

$$\begin{cases} {}^{c}D^{3}u\left(t\right)=\frac{t}{8}\left(\left(cost\right)sin\left(\frac{|u(t)|+|v(t)|}{2}\right)\right)+\frac{e^{-(u(t)+v(t))^{2}}}{1+t^{2}}, & t\in\left[0,1\right], \\ {}^{c}D^{3}v\left(t\right)=\frac{1}{32}sin\left(2\pi u\left(t\right)\right)+\frac{|v(t)|}{16(1+|v(t)|)}+\frac{1}{2}, & t\in\left[0,1\right], \end{cases}$$

subject to the three-point coupled boundary conditions

$$\begin{cases} \frac{1}{100}u\left(0\right)+\frac{1}{10}u\left(1\right)=u\left(\frac{1}{2}\right), \\ u\left(0\right)=\int_{0}^{0,5}u\left(s\right)ds, \\ \frac{1}{100}{}^{c}D^{\frac{3}{2}}u\left(0\right)+\frac{1}{10}{}^{c}D^{\frac{3}{2}}u\left(1\right)={}^{c}D^{\frac{3}{2}}u\left(\frac{1}{2}\right), \end{cases}$$

where $f\left(t,u,v\right)=\frac{t}{8}\left(\left(cost\right)sin\left(\frac{|u|+|v|}{2}\right)\right)+\frac{e^{-(u+v)^{2}}}{1+t^{2}}$, $t\in\left[0,1\right],\eta=0,5,\lambda=0,01,\gamma=0,1,p=1,5$ *and* $g\left(t,u,v\right)=\frac{1}{32\pi}sin\left(2\pi u\left(t\right)\right)+\frac{|v(t)|}{16(1+|v(t)|)}+\frac{1}{2}$.
It can be easily found that $M=\frac{20}{3}$ *and* $Q=\frac{9}{400}$.
Furthermore, by simple computation, for every $u_i,\ v_i\in\mathbb{R},\ i=1,2,$ *we have*

$$\left|f\left(t,u_{1},u_{2}\right)-f\left(t,v_{1},v_{2}\right)\right|\leq L\left(\left|u_{1}-v_{1}\right|+\left|u_{2}-v_{2}\right|\right),$$

and

$$\left|g\left(t,u_{1},u_{2}\right)-g\left(t,v_{1},v_{2}\right)\right|\leq L\left(\left|u_{1}-v_{1}\right|+\left|u_{2}-v_{2}\right|\right),$$

where $L_1=L_2=L=\frac{1}{16}$. *It can be easily found that* $\Delta_1=\Delta_3\cong 0,799562,\ \Delta_2=\Delta_4\cong 1,182808$.
Finally, since $L_1\left(\Delta_1+\Delta_3\right)+L_2\left(\Delta_2+\Delta_4\right)=2L\left(\Delta_1+\Delta_2\right)\cong 0,247796<1$, *thus all assumptions and conditions of Theorem* 3.4 *are satisfied. Hence, Theorem implies that the three-point boundary value problem* $(1.1)-(1.2)$ *has a unique solution*

Acknowledgments. The authors want to thank the anonymous referees for the throughout reading of the manuscript and several suggestions that help us improve the presentation of the paper.

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Noureddine Bouteraa,
Address of the first author
Email address: bouteraa-27@hotmail.fr

Habib Djourdem,
Address of the second author
Email address: djourdem.habib7@gmail.com

Slimane Benaicha ,
Address of the third author
Email address: slimanebenaicha@yahoo.fr