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STRONG COUPLED FIXED POINTS OF CHATTERJEA TYPE (ψ, φ)-WEAKLY CYCLIC COUPLED MAPPINGS IN S-METRIC SPACES

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Abstract. In this paper, we introduce Chatterjea type (ψ, φ)-weakly cyclic coupled mapping in S-metric spaces and prove the existence and uniqueness of strong coupled fixed point of such mappings. We give an illustrative example to support of our result.

1. Introduction

In 1972, Chatterjea [8] introduced a contraction map which is not necessarily continuous and is known as Chatterjea contraction map or simply Chatterjea map and proved that every Chatterjea map has a unique fixed point in complete metric spaces. For more works on Chatterjea type mappings, we refer [7], [9], [10], [21], [31]. In 1997, Alber and Guerre-Delabriere [2] introduced the concept of weakly contractive mapping as a generalization of contractive map and proved the existence of fixed points for such mappings in Hilbert spaces. Rhoades [33] extended this study to metric space setting. In 2003, Kirk, Srinivasan and Veeramani [23] introduced cyclic contractions in metric spaces and proved the existence and uniqueness of cyclic contractions in complete metric spaces. After this, many authors introduced various types of cyclic contractions and cyclic weakly contractions and proved fixed point results, some of which are in [3], [5], [19], [20], [22], [24], [26], [27], [29], [30], [34]. Meanwhile, in 2006, Gnana Bhaskar and Lakshmikantham [14] introduced and developed coupled fixed point theory for mixed monotone operators. Later, coupled fixed point results were developed by [14], [18], [25], [28], [32], [37]. In 2013, Chandok and Postolache [7] introduced Chatterjea type cyclic weakly contractive maps and obtained fixed point results in complete metric spaces and in 2017, Choudhury, Maity and Konar [10], introduced Chatterjea type coupling and obtained the existence of strong unique coupled fixed points for such maps.

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Inspired by these works, in section 3 of this paper, we introduce Chatterjea type (ψ, φ)-weakly cyclic coupled mapping and prove the existence and uniqueness of strong coupled fixed point of such map in complete S-metric spaces. Also, we present an illustrative example in support of our result.

2. Preliminaries

We use the following propositions in proving our results.

Proposition 2.1. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. Then $\limsup\limits_{n\to\infty} \max\{a_n, b_n\} = \max\{\limsup\limits_{n\to\infty} a_n, \limsup\limits_{n\to\infty} b_n\}$.

Proposition 2.2. (i) Let $\{c_n\}$, $\{d_n\}$, $\{e_n\}$ and $\{f_n\}$ be real sequences then $\max\{c_n + d_n, e_n + f_n\} \leq \max\{c_n, e_n\} + \max\{d_n, f_n\}$.
(ii) Let $\{a_n\}$, $\{b_n\}$ be two real sequences, $\{b_n\}$ be bounded. Then $\liminf\limits_{n\to\infty}(a_n + b_n) \leq \liminf\limits_{n\to\infty} a_n + \limsup\limits_{n\to\infty} b_n$.

Proposition 2.3. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$ and $\{f_n\}$ be nonnegative sequences satisfying $\max\{a_n, b_n\} \leq \max\{c_n + d_n, e_n + f_n\}$ with $\limsup\limits_{n\to\infty} c_n = 0$ and $\limsup\limits_{n\to\infty} e_n = 0$ then $\liminf\limits_{n\to\infty} \max\{a_n, b_n\} \leq \liminf\limits_{n\to\infty} \max\{d_n, f_n\}$.

Definition 2.1. [23] Let X be a nonempty set and $T : X \to X$ be an operator. If X_i, $i = 1, 2, ...m$ are nonempty subsets of X with $X = \bigcup\limits_{i=1}^{m} X_i$ satisfying $T(X_1) \subset X_2$, ..., $T(X_{m-1}) \subset X_m$, $T(X_m) \subset X_1$ is called a cyclic representation of X with respect to T.

Definition 2.2. [14] Let X be a nonempty set. Let $F : X \times X \to X$ be a mapping. An element $(x, y) \in X \times X$ is said to be a *coupled fixed point* of F if $F(x, y) = x$ and $F(y, x) = y$.

Throughout this paper, we denote the set of all reals by $\mathbb{R}$, the set of all natural numbers by $\mathbb{N}$, and
$\Psi = \{\psi : [0, \infty) \to [0, \infty)$ /(i) ψ is continuous (ii) ψ is nondecreasing
(iii) $\psi(t) = 0$ if and only if $t = 0\}$.

Remark. For any $a, b \in [0, \infty)$, we have $\psi(\max\{a, b\}) = \max\{\psi(a), \psi(b)\}$ for any $\psi \in \Psi$.

Definition 2.3. [7] Let (X, d) be a metric space, m be a natural number, A_1, A_2, ..., A_m be nonempty subsets of X and $Y = \bigcup\limits_{i=1}^{m} A_i$. An operator $T : Y \to Y$ is called a *Chatterjea type cyclic weakly contraction* if
$\bigcup\limits_{i=1}^{m} A_i$ is a cyclic representation of Y with respect to T and if there exist $\psi \in \Psi$ and a function $\varphi : [0, \infty)^2 \to [0, \infty)$ with φ is lower semi continuous, $\varphi(t, t) > 0$ for $t \in (0, \infty)$ and $\varphi(0, 0) = 0$ such that
$\psi(d(Tx, Ty)) \leq \psi(\frac{1}{2}[d(x, Ty) + d(y, Tx)]) - \varphi(d(x, Ty), d(y, Tx))$,
for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, ..., m$, where $A_{m+1} = A_1$.

Theorem 2.4. [7] Let (X,d) be a complete metric space, $m \in N$, A_1, A_2,..., A_m be nonempty closed subsets of X and $Y = \bigcup_{i=1}^{m} A_i$. Suppose that T is a Chatterjea type cyclic weakly contraction. Then T has a fixed point $z \in \bigcap_{i=1}^{m} A_i$.

Choudhury, Maity and Konar [10] extended the above notion of cyclic mapping to the case of mappings defined on $X \times X$ in the following definition.

Definition 2.4. [10] Let A and B be two nonempty subsets of X. A mapping $F : X \times X \to X$ is said to be *cyclic* with respect to A and B if $F(A,B) \subset B$ and $F(B,A) \subset A$. Such a function F is also said to be a coupling with respect to A and B.

Definition 2.5. [10] Let X be a nonempty set. Let $F : X \times X \to X$ be a mapping. An element $(x,x) \in X \times X$ is said to be a *strong coupled fixed point* of F if $F(x,x) = x$.

Definition 2.6. [10] Let A and B be two nonempty subsets of a metric space (X,d). A coupling $F : X \times X \to X$ is called a *Chatterjea type coupling* with respect to A and B if F is cyclic with respect to A and B satisfying, the inequality

$$d(F(x,y),F(u,v)) \leq k[d(x,F(u,v)) + d(u,F(x,y))], \tag{2.1}$$

where $x,v \in A$ and $y,u \in B$, for some $k \in (0,\frac{1}{2})$.

Theorem 2.5. [10] Let A and B be two nonempty closed subsets of a complete metric space (X,d). Let $F : X \times X \to X$ be a *Chatterjea type coupling* with respect to A and B. Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.

In 2012, Sedghi, Shobe and Aliouche [35] introduced a new concept on metric spaces, namely S-metric spaces and studied some properties of these spaces. Subsequently, many authors developed coupled fixed point theorems and cyclic contractions on S-metric spaces. Some of them include [1], [12], [15], [16], [17], [25], [31], [37].

Definition 2.7. [35] *Let X be a nonempty set. An S-metric on X is a function $S : X^3 \to [0,\infty)$ that satisfies the following conditions: for each $x,y,z,a \in X$*

(S1) $S(x,y,z) \geq 0$,
(S2) $S(x,y,z) = 0$ *if and only if* $x = y = z$ *and*
(S3) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.

The pair (X,S) is called an S-metric space.

Example 2.1. [35] Let (X,d) be a metric space. Define $S : X^3 \to [0,\infty)$ by $S(x,y,z) = d(x,y) + d(x,z) + d(y,z)$ for all $x,y,z \in X$. Then S is an S-metric on X and S is called the S-metric induced by the metric d.

Example 2.2. [13] Let $X = \mathbb{R}$ and let $S(x,y,z) = |y+z-2x| + |y-z|$ for all $x,y,z \in X$. Then (X,S) is an S-metric space.

Example 2.3. [36] Let $\mathbb{R}$ be the real line. Then $S(x,y,z) = |x-z| + |y-z|$ for all $x,y,z \in \mathbb{R}$ is an S-metric on $\mathbb{R}$. This S-metric is called the usual S-metric.

Lemma 2.6. [35] In an S-metric space, we have $S(x,x,y) = S(y,y,x)$.

Lemma 2.7. [13] Let (X, S) be an S-metric space. Then
$S(x, x, z) \leq 2S(x, x, y) + S(y, y, z)$.

Definition 2.8. [35] Let (X, S) be an S-metric space.

(i) A sequence $\{x_n\} \subseteq X$ is said to converge to a point $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n, x_n, x) < \epsilon$ and we denote it by $\lim_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\} \subseteq X$ is called Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \epsilon$ for all $n, m \geq n_0$.

(iii) An S-metric space (X, S) is said to be complete if each Cauchy sequence in X is convergent.

Lemma 2.8. [35] Let (X, S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x, then x is unique.

Lemma 2.9. [35] Let (X, S) be an S-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y)$.

Lemma 2.10. [6] Let (X, S) be an S-metric space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X, $\{x_n\}$ converges to x in X. Then $\lim_{n\to\infty} S(x_n, x_n, y_n) = \lim_{n\to\infty} S(x, x, y_n)$.

Lemma 2.11. *(*[4], [11]*)* Let (X, S) be an S-metric space and $\{x_n\}$ a sequence in X such that

$$\lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0.$$

If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $m_k > n_k > k$ such that
$S(x_{m_k}, x_{m_k}, x_{n_k}) \geq \epsilon$ with $S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) < \epsilon$.
Also, we have the following:
$(i)\ \lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}) = \epsilon$ $\qquad (ii)\ \lim_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) = \epsilon$
$(iii)\ \lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k-1}) = \epsilon$ $\qquad (iv)\ \lim_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k-1}) = \epsilon$.

We denote

$$\Phi = \{\varphi : [0, \infty)^2 \to [0, \infty) \text{ such that (i) } \varphi \text{ is continuous in each of its variables, and (ii) } \varphi(t_1, t_2) = 0 \text{ if and only if } t_1 = 0 \text{ and } t_2 = 0\}.$$

3. Chatterjea Type (ψ, φ)- Weakly cyclic Coupled Mapping

In the following, we define Chatterjea type (ψ, φ)-weakly cyclic coupled mapping.

Definition 3.1. Let (X, S) be an S-metric space. Let A and B be two nonempty subsets of X. Let $F : X \times X \to X$ be a mapping. If (i) F is cyclic with respect to A and B and (ii) there exist $\psi \in \Psi$, $\varphi \in \Phi$ such that

$$\begin{aligned}\psi(S(F(x,y), F(u,v), F(w,z))) \leq \psi(&\frac{1}{4}[\max\{S(x,x,F(w,z)), S(x,x,F(u,v))\} \\ &+ \max\{S(w,w,F(x,y)), S(u,u,F(x,y))\}]) \\ &-\varphi(\max\{S(x,x,F(w,z)), S(x,x,F(u,v))\}, \\ &\max\{S(w,w,F(x,y)), S(u,u,F(x,y))\}) \end{aligned} \tag{3.1}$$

for any $x, u, z \in A$ and $y, v, w \in B$, then we say that F is a *Chatterjea type* (ψ, φ)-*weakly cyclic coupled mapping with respect to A and B*.

Example 3.1. Let $X = [0, 1]$. We define $S : X^3 \to [0, \infty)$ by

$$S(x, y, z) = \begin{cases} 0 & \textit{if } x = y = z \\ x + y + z & \textit{otherwise.} \end{cases}$$

Then (X, S) is an S-metric space.
Let $A = [0, \frac{1}{2}]$ and $B = [0, 1]$. We define $F : X \times X \to X$ by
$F(x, y) = \frac{xy}{16}$. Then $F(A, B) \subset B$ and $F(B, A) \subset A$ so that F is cyclic with respect to A and B. We define $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \frac{t}{2}$ and $\varphi : [0, \infty)^2 \to [0, \infty)$ by $\varphi(t_1, t_2) = \frac{1}{16}(t_1 + t_2)$. We now verify the inequality (3.1). Let $x, u, z \in A$ and $y, v, w \in B$. We now consider

$$\begin{aligned}
\psi(S(F(x, y), F(u, v), F(w, z))) &= \psi(S(\tfrac{xy}{16}, \tfrac{uv}{16}, \tfrac{wz}{16})) \\
&= \tfrac{1}{2} S(\tfrac{xy}{16}, \tfrac{uv}{16}, \tfrac{wz}{16}) \\
&= \tfrac{xy}{32} + \tfrac{uv}{32} + \tfrac{wz}{32} \\
&\leq \tfrac{1}{32}[x + u + w] \\
&\leq \tfrac{1}{32}[S(x, x, F(w, z)) + S(x, x, F(u, v)) + \\
&\qquad + S(w, w, F(x, y)) + S(u, u, F(x, y))] \\
&\leq \tfrac{2}{32}[\max\{S(x, x, F(w, z)), S(x, x, F(u, v))\} \\
&\qquad + \max\{S(w, w, F(x, y)), S(u, u, F(x, y))\}] \\
&= \tfrac{1}{16}[t_1 + t_2] \\
&= \tfrac{1}{8}[t_1 + t_2] - \tfrac{1}{16}[t_1 + t_2] \\
&= \psi(\tfrac{1}{4}[t_1 + t_2]) - \varphi(t_1, t_2),
\end{aligned}$$

where $t_1 = \max\{S(x, x, F(w, z)), S(x, x, F(u, v))\}$ *and*
$t_2 = \max\{S(w, w, F(x, y)), S(u, u, F(x, y))\}$.

Therefore F is a Chatterjea type (ψ, φ)-weakly cyclic coupled mapping with respect to A and B.

Lemma 3.1. Let (X, S) be an S-metric space. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim\limits_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0$ and $\lim\limits_{n\to\infty} S(y_n, y_n, y_{n+1}) = 0$. If either $\{x_n\}$ or $\{y_n\}$ is not Cauchy, then there exist an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that

$$\max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} \geq \epsilon. \tag{3.2}$$

We choose m_k as the smallest integer with $m_k > n_k$ satisfying (3.2).
i.e., $\max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} \geq \epsilon$ with
$\max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\} < \epsilon$.
Also, the following limits hold.

(i) $\lim\limits_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} = \epsilon$

(ii) $\lim\limits_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k-1}), S(y_{m_k}, y_{m_k}, y_{n_k-1})\} = \epsilon$ and

(iii) $\lim\limits_{k\to\infty} \max\{S(x_{n_k}, x_{n_k}, x_{m_k-1}), S(y_{n_k}, y_{n_k}, y_{m_k-1})\} = \epsilon$.

Proof. (i) We consider

$$\begin{aligned}
S(x_{m_k}, x_{m_k}, x_{n_k}) &\leq 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) \\
&< 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + \epsilon.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
S(y_{m_k}, y_{m_k}, y_{n_k}) &\leq 2S(y_{m_k}, y_{m_k}, y_{m_k-1}) + S(y_{m_k-1}, y_{m_k-1}, y_{n_k}) \\
&< 2S(y_{m_k}, y_{m_k}, y_{m_k-1}) + \epsilon.
\end{aligned}$$

Hence
$\max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} \leq \max\{2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + \epsilon, 2S(y_{m_k}, y_{m_k}, y_{m_k-1}) + \epsilon\}.$

Now, by applying Proposition 2.1, we have

$$\limsup_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} \leq \epsilon. \tag{3.3}$$

We have $\epsilon \leq \max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\}$. Hence
$\epsilon \leq \liminf_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\}$
$\leq \limsup_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} \leq \epsilon$ (from (3.3)).
Hence $\liminf_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} = \epsilon$
$= \limsup_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\}$.
Therefore $\lim_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\}$ exists and
$\lim_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} = \epsilon$.
Hence (i) holds.
(ii) We now consider
$S(x_{m_k}, x_{m_k}, x_{n_k}) = S(x_{n_k}, x_{n_k}, x_{m_k}) \leq 2S(x_{n_k}, x_{n_k}, x_{n_k-1}) + S(x_{m_k}, x_{m_k}, x_{n_k-1})$.
Similarly, we have
$S(y_{m_k}, y_{m_k}, y_{n_k}) = S(y_{n_k}, y_{n_k}, y_{m_k}) \leq 2S(y_{n_k}, y_{n_k}, y_{n_k-1}) + S(y_{m_k}, y_{m_k}, y_{n_k-1})$.
Then
$\max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} \leq \max\{2S(x_{n_k}, x_{n_k}, x_{n_k-1}) + S(x_{m_k}, x_{m_k}, x_{n_k-1}), 2S(y_{n_k}, y_{n_k}, y_{n_k-1}) + S(y_{m_k}, y_{m_k}, y_{n_k-1})\}$.
On taking limit infimum as $k \to \infty$ and using Proposition 2.3, we get
$\liminf_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\}$
$\leq \liminf_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k-1}), S(y_{m_k}, y_{m_k}, y_{n_k-1})\}$.

By using (i), we get $\epsilon \leq \liminf_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k-1}), S(y_{m_k}, y_{m_k}, y_{n_k-1})\}$.
We now consider
$S(x_{m_k}, x_{m_k}, x_{n_k-1}) = S(x_{n_k-1}, x_{n_k-1}, x_{m_k}) \leq 2S(x_{n_k-1}, x_{n_k-1}, x_{n_k}) + S(x_{n_k}, x_{n_k}, x_{m_k})$.
Similarly, we have
$S(y_{m_k}, y_{m_k}, y_{n_k-1}) = S(y_{n_k-1}, y_{n_k-1}, y_{m_k}) \leq 2S(y_{n_k-1}, y_{n_k-1}, y_{n_k}) + S(y_{n_k}, y_{n_k}, y_{m_k})$.
Now,
$\max\{S(x_{m_k}, x_{m_k}, x_{n_k-1}), S(y_{m_k}, y_{m_k}, y_{n_k-1})\}$
$\leq \max\{2S(x_{n_k-1}, x_{n_k-1}, x_{n_k}) + S(x_{n_k}, x_{n_k}, x_{m_k}), 2S(y_{n_k-1}, y_{n_k-1}, y_{n_k}) + S(y_{n_k}, y_{n_k}, y_{m_k})\}$.
On taking limit supremum as $k \to \infty$ and using Proposition 2.1, we get
$\limsup_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k-1}), S(y_{m_k}, y_{m_k}, y_{n_k-1})\}$
$\leq \limsup_{k\to\infty} \max\{S(x_{n_k}, x_{n_k}, x_{m_k}), S(y_{n_k}, y_{n_k}, y_{m_k})\}$
$= \epsilon$ (by (i)).
Therefore we have
$\epsilon \leq \liminf_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k-1}), S(y_{m_k}, y_{m_k}, y_{n_k-1})\}$
$\leq \limsup_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k-1}), S(y_{m_k}, y_{m_k}, y_{n_k-1})\} \leq \epsilon$.

Thus, we have

$$\liminf_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k-1}), S(y_{m_k}, y_{m_k}, y_{n_k-1})\} = \epsilon$$
$$= \limsup_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k-1}), S(y_{m_k}, y_{m_k}, y_{n_k-1})\}.$$

Hence $\lim_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k-1}), S(y_{m_k}, y_{m_k}, y_{n_k-1})\}$ exists and $\lim_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k-1}), S(y_{m_k}, y_{m_k}, y_{n_k-1})\} = \epsilon$. Therefore (ii) holds.

(iii) We consider

$$S(x_{m_k}, x_{m_k}, x_{n_k}) \leq 2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + S(x_{m_k-1}, x_{m_k-1}, x_{n_k})$$

and

$$S(y_{m_k}, y_{m_k}, y_{n_k}) \leq 2S(y_{m_k}, y_{m_k}, y_{m_k-1}) + S(y_{m_k-1}, y_{m_k-1}, y_{n_k}).$$

Now

$$\max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\}$$
$$\leq \max\{2S(x_{m_k}, x_{m_k}, x_{m_k-1}) + S(x_{m_k-1}, x_{m_k-1}, x_{n_k}),$$
$$2S(y_{m_k}, y_{m_k}, y_{m_k-1}) + S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\}.$$

On taking limit infimum as $k \to \infty$ and by Proposition 2.3, we get

$$\epsilon \leq \liminf_{k\to\infty} \max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\}.$$

We have

$$S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) \leq 2S(x_{m_k-1}, x_{m_k-1}, x_{m_k}) + S(x_{m_k}, x_{m_k}, x_{n_k})$$

and

$$S(y_{m_k-1}, y_{m_k-1}, y_{n_k}) \leq 2S(y_{m_k-1}, y_{m_k-1}, y_{m_k}) + S(y_{m_k}, y_{m_k}, y_{n_k}).$$

Then

$$\max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\}$$
$$\leq \max\{2S(x_{m_k-1}, x_{m_k-1}, x_{m_k}) + S(x_{m_k}, x_{m_k}, x_{n_k}),$$
$$2S(y_{m_k-1}, y_{m_k-1}, y_{m_k}) + S(y_{m_k}, y_{m_k}, y_{n_k})\}.$$

On taking limit supremum as $k \to \infty$, we get

$$\limsup_{k\to\infty} \max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\}$$
$$\leq \limsup_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\}$$
$$= \epsilon \text{ (by } (i))$$

so that $\epsilon \leq \liminf_{k\to\infty} \max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\}$

$$\leq \limsup_{k\to\infty} \max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\} \leq \epsilon.$$

Thus $\liminf_{k\to\infty} \max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\} = \epsilon$

$$= \limsup_{k\to\infty} \max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\}.$$

Hence $\lim_{k\to\infty} \max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\}$ exists and $\lim_{k\to\infty} \max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\} = \epsilon$.

This proves (iii). □

Theorem 3.2. Let (X, S) be a complete S-metric space. Let A and B be two nonempty closed subsets of X. Let $F : X \times X \to X$ be a Chatterjea type (ψ, φ)-weakly cyclic coupled mapping with respect to A and B. Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.

Proof. Let $x_0 \in A$ and $y_0 \in B$ be arbitrary. We define the sequences $\{x_n\}$ and $\{y_n\}$ by

$$x_{n+1} = F(y_n, x_n),\ y_{n+1} = F(x_n, y_n),\ n = 0, 1, 2, \dots . \tag{3.4}$$

If $y_n = x_{n+1}$ and $x_n = y_{n+1}$ for some n, then we have

$$\begin{aligned}
\psi(S(x_n, x_n, y_n)) &= \psi(S(y_{n+1}, y_{n+1}, x_{n+1})) \\
&= \psi(S(F(x_n, y_n), F(x_n, y_n), F(y_n, x_n))) \\
&\leq \psi(\tfrac{1}{4}[\max\{S(x_n, x_n, F(y_n, x_n)), S(x_n, x_n, F(x_n, y_n))\} \\
&\qquad + \max\{S(y_n, y_n, F(x_n, y_n)), S(x_n, x_n, F(x_n, y_n))\}]) \\
&\quad - \varphi(\max\{S(x_n, x_n, F(y_n, x_n)), S(x_n, x_n, F(x_n, y_n))\}, \\
&\qquad \max\{S(y_n, y_n, F(x_n, y_n)), S(x_n, x_n, F(x_n, y_n))\}]) \\
&= \psi(\tfrac{1}{4}[\max\{S(x_n, x_n, x_{n+1}), S(x_n, x_n, y_{n+1})\} \\
&\qquad + \max\{S(y_n, y_n, y_{n+1}), S(x_n, x_n, y_{n+1})\}]) \\
&\quad - \varphi(\max\{S(x_n, x_n, x_{n+1}), S(x_n, x_n, y_{n+1})\}, \\
&\qquad \max\{S(y_n, y_n, y_{n+1}), S(x_n, x_n, y_{n+1})\}) \\
&= \psi(\tfrac{1}{4}[\max\{S(x_n, x_n, y_n), S(x_n, x_n, x_n)\} \\
&\qquad + \max\{S(y_n, y_n, x_n), S(x_n, x_n, x_n)\}]) \\
&\quad - \varphi(\max\{S(x_n, x_n, y_n), S(x_n, x_n, x_n)\}, \\
&\qquad \max\{S(y_n, y_n, x_n), S(x_n, x_n, x_n)\}) \\
&= \psi(\tfrac{1}{2}S(x_n, x_n, y_n)) - \varphi(S(x_n, x_n, y_n), S(x_n, x_n, y_n)) \\
&\qquad \text{(by using Lemma 2.6)} \\
&\leq \psi(S(x_n, x_n, y_n)) - \varphi(S(x_n, x_n, y_n), S(x_n, x_n, y_n))
\end{aligned}$$

which implies that $\varphi(S(x_n, x_n, y_n), S(x_n, x_n, y_n)) = 0$ and hence $S(x_n, x_n, y_n) = 0$. Thus $x_n = y_n$ so that $A \cap B \neq \emptyset$ and (x_n, x_n) is a strong coupled fixed point of F and we are through.

If either $y_n \neq x_{n+1}$ or $x_n \neq y_{n+1}$ for all n, then we have the following. If $x_n = y_{n+1}$ and $y_n \neq x_{n+1}$ for all n, then we have

$$\begin{aligned}
\psi(S(y_{n+1}, y_{n+1}, x_{n+2})) &= \psi(S(F(x_n, y_n), F(x_n, y_n), F(y_{n+1}, x_{n+1}))) \\
&\leq \psi(\tfrac{1}{4}[\max\{S(x_n, x_n, F(y_{n+1}, x_{n+1})), \\
&\qquad S(x_n, x_n, F(x_n, y_n))\} \\
&\qquad + \max\{S(y_{n+1}, y_{n+1}, F(x_n, y_n)), \\
&\qquad S(x_n, x_n, F(x_n, y_n))\}]) \\
&\quad - \varphi(\max\{S(x_n, x_n, F(y_{n+1}, x_{n+1})), \\
&\qquad S(x_n, x_n, F(x_n, y_n))\}, \\
&\qquad \max\{S(y_{n+1}, y_{n+1}, F(x_n, y_n)), \\
&\qquad S(x_n, x_n, F(x_n, y_n))\}) \\
&= \psi(\tfrac{1}{4}[\max\{S(x_n, x_n, x_{n+2}), S(x_n, x_n, y_{n+1})\} \\
&\qquad + \max\{S(y_{n+1}, y_{n+1}, y_{n+1}), S(x_n, x_n, y_{n+1})\}]) \\
&\quad - \varphi(\max\{S(x_n, x_n, x_{n+2}), S(x_n, x_n, y_{n+1})\}, \\
&\qquad \max\{S(y_{n+1}, y_{n+1}, y_{n+1}), S(x_n, x_n, y_{n+1})\}) \\
&\leq \psi(\tfrac{1}{4}[\max\{2S(x_n, x_n, y_{n+1}) + S(y_{n+1}, y_{n+1}, x_{n+2}), \\
&\qquad S(x_n, x_n, y_{n+1})\} + S(x_n, x_n, y_{n+1})]) \\
&\quad - \varphi(\max\{S(x_n, x_n, x_{n+2}), S(x_n, x_n, y_{n+1})\}, \\
&\qquad S(x_n, x_n, y_{n+1})) \\
&= \psi(\tfrac{1}{4}[2S(x_n, x_n, y_{n+1}) + S(y_{n+1}, y_{n+1}, x_{n+2}) \\
&\qquad + S(x_n, x_n, y_{n+1})]) \\
&\quad -\varphi(\max\{S(x_n, x_n, x_{n+2}), S(x_n, x_n, y_{n+1})\}, S(x_n, x_n, y_{n+1})) \\
&= \psi(\tfrac{1}{4}S(y_{n+1}, y_{n+1}, x_{n+2})) - \varphi(S(x_n, x_n, x_{n+2}), 0) \\
&\leq \psi(S(y_{n+1}, y_{n+1}, x_{n+2})) - \varphi(S(x_n, x_n, x_{n+2}), 0)
\end{aligned}$$

which implies that $\varphi(S(x_n, x_n, x_{n+2}), 0) = 0$. Therefore $S(x_n, x_n, x_{n+2}) = 0$. Thus $x_n = x_{n+2}$. That is $y_{n+1} = x_{n+2}$ which is a contradiction. Hence this case does not arise.

Similar as above, the case $y_n = x_{n+1}$ and $x_n \neq y_{n+1}$ for all n does not arise.

Hence, we assume that $y_n \neq x_{n+1}$ and $x_n \neq y_{n+1}$ for all n. Now by using (3.1), we have

$$\begin{aligned}
\psi(S(x_1,x_1,y_2)) &= \psi(S(y_2,y_2,x_1)) \\
&= \psi(S(F(x_1,y_1),F(x_1,y_1),F(y_0,x_0))) \\
&\leq \psi(\tfrac{1}{4}[\max\{S(x_1,x_1,F(y_0,x_0)),S(x_1,x_1,F(x_1,y_1))\} \\
&\qquad + \max\{S(y_0,y_0,F(x_1,y_1)),S(x_1,x_1,F(x_1,y_1))\}]) \\
&\quad - \varphi(\max\{S(x_1,x_1,F(y_0,x_0)),S(x_1,x_1,F(x_1,y_1))\}, \\
&\qquad \max\{S(y_0,y_0,F(x_1,y_1)),S(x_1,x_1,F(x_1,y_1))\}) \\
&= \psi(\tfrac{1}{4}[\max\{S(x_1,x_1,x_1),S(x_1,x_1,y_2)\} \\
&\qquad + \max\{S(y_0,y_0,y_2),S(x_1,x_1,y_2)\}]) \\
&\quad - \varphi(\max\{S(x_1,x_1,x_1),S(x_1,x_1,y_2)\}, \\
&\qquad \max\{S(y_0,y_0,y_2),S(x_1,x_1,y_2)\}) \\
&= \psi(\tfrac{1}{4}[S(x_1,x_1,y_2)+\max\{S(y_0,y_0,y_2),S(x_1,x_1,y_2)\}]) \\
&\quad - \varphi(S(x_1,x_1,y_2),\max\{S(y_0,y_0,y_2),S(x_1,x_1,y_2)\}) \\
&\leq \psi(\tfrac{1}{4}[S(x_1,x_1,y_2)+\max\{2S(y_0,y_0,x_1)+S(x_1,x_1,y_2), \\
&\qquad\quad S(x_1,x_1,y_2)\}]) \\
&\quad - \varphi(S(x_1,x_1,y_2),\max\{S(y_0,y_0,y_2),S(x_1,x_1,y_2)\}) \\
&= \psi(\tfrac{1}{4}[2S(x_1,x_1,y_2)+2S(y_0,y_0,x_1)]) \\
&\quad - \varphi(S(x_1,x_1,y_2),\max\{S(y_0,y_0,y_2),S(x_1,x_1,y_2)\}) \\
&< \psi(\tfrac{1}{4}[2S(x_1,x_1,y_2)+2S(y_0,y_0,x_1)]).
\end{aligned}$$

Since ψ is monotonically increasing, it follows that
$S(x_1,x_1,y_2) \leq \frac{1}{2}S(x_1,x_1,y_2)+\frac{1}{2}S(y_0,y_0,x_1)$ so that

$$S(x_1,x_1,y_2) \leq S(y_0,y_0,x_1). \tag{3.5}$$

Similarly, we have

$$\begin{aligned}
\psi(S(y_1,y_1,x_2)) &= \psi(S(F(x_0,y_0),F(x_0,y_0),F(y_1,x_1))) \\
&\leq \psi(\tfrac{1}{4}[\max\{S(x_0,x_0,F(y_1,x_1)),S(x_0,x_0,F(x_0,y_0))\} \\
&\qquad + \max\{S(y_1,y_1,F(x_0,y_0)),S(x_0,x_0,F(x_0,y_0))\}]) \\
&\quad - \varphi(\max\{S(x_0,x_0,F(y_1,x_1)),S(x_0,x_0,F(x_0,y_0))\}, \\
&\qquad \max\{S(y_1,y_1,F(x_0,y_0)),S(x_0,x_0,F(x_0,y_0))\}) \\
&= \psi(\tfrac{1}{4}[\max\{S(x_0,x_0,x_2),S(x_0,x_0,y_1)\} \\
&\qquad + \max\{S(y_1,y_1,y_1),S(x_0,x_0,y_1)\}]) \\
&\quad - \varphi(\max\{S(x_0,x_0,x_2),S(x_0,x_0,y_1)\}, \\
&\qquad \max\{S(y_1,y_1,y_1),S(x_0,x_0,y_1)\}) \\
&\leq \psi(\tfrac{1}{4}[\max\{2S(x_0,x_0,y_1)+S(y_1,y_1,x_2),S(x_0,x_0,y_1)\} \\
&\qquad + S(x_0,x_0,y_1)]) \\
&\quad - \varphi(\max\{S(x_0,x_0,x_2),S(x_0,x_0,y_1)\},S(x_0,x_0,y_1)) \\
&= \psi(\tfrac{1}{4}[2S(x_0,x_0,y_1)+S(y_1,y_1,x_2)+S(x_0,x_0,y_1)]) \\
&\quad - \varphi(\max\{S(x_0,x_0,x_2),S(x_0,x_0,y_1)\},S(x_0,x_0,y_1)) \\
&< \psi(\tfrac{3}{4}S(x_0,x_0,y_1)+\tfrac{1}{4}S(y_1,y_1,x_2)),
\end{aligned}$$

and hence it follows that $S(y_1,y_1,x_2) \leq \frac{3}{4}S(x_0,x_0,y_1)+\frac{1}{4}S(y_1,y_1,x_2)$ so that

$$S(y_1,y_1,x_2) \leq S(x_0,x_0,y_1). \tag{3.6}$$

Again, by using (3.1), we have

$$\begin{aligned}
\psi(S(x_2,x_2,y_3)) &= \psi(S(y_3,y_3,x_2)) \\
&= \psi(S(F(x_2,y_2),F(x_2,y_2),F(y_1,x_1))) \\
&\leq \psi(\tfrac{1}{4}[\max\{S(x_2,x_2,F(y_1,x_1)),S(x_2,x_2,F(x_2,y_2))\}
\end{aligned}$$

$$\begin{aligned}
&+\max\{S(y_1,y_1,F(x_2,y_2)),S(x_2,x_2,F(x_2,y_2))\}])\\
&-\varphi(\max\{S(x_2,x_2,F(y_1,x_1)),S(x_2,x_2,F(x_2,y_2))\},\\
&\quad\max\{S(y_1,y_1,F(x_2,y_2)),S(x_2,x_2,F(x_2,y_2))\})\\
&=\psi(\tfrac{1}{4}[\max\{S(x_2,x_2,x_2),S(x_2,x_2,y_3)\}\\
&\quad+\max\{S(y_1,y_1,y_3),S(x_2,x_2,y_3)\}])\\
&\quad-\varphi(\max\{S(x_2,x_2,x_2),S(x_2,x_2,y_3)\},\\
&\quad\max\{S(y_1,y_1,y_3),S(x_2,x_2,y_3)\})\\
&\leq\psi(\tfrac{1}{4}[S(x_2,x_2,y_3)+\max\{2S(y_1,y_1,x_2)+S(x_2,x_2,y_3),\\
&\qquad S(x_2,x_2,y_3)\}])\\
&\quad-\varphi(S(x_2,x_2,y_3),\max\{S(y_1,y_1,y_3),S(x_2,x_2,y_3)\})\\
&=\psi(\tfrac{1}{4}[S(x_2,x_2,y_3)+2S(y_1,y_1,x_2)+S(x_2,x_2,y_3)])\\
&\quad-\varphi(S(x_2,x_2,y_3),\max\{S(y_1,y_1,y_3),S(x_2,x_2,y_3)\})\\
&<\psi(\tfrac{1}{2}S(x_2,x_2,y_3)+\tfrac{1}{2}S(y_1,y_1,x_2)),
\end{aligned}$$

and hence we have $S(x_2,x_2,y_3)\leq\frac{1}{2}S(x_2,x_2,y_3)+\frac{1}{2}S(y_1,y_1,x_2)$
so that
$S(x_2,x_2,y_3)\leq S(y_1,y_1,x_2)$.
Similarly, we have

$$\begin{aligned}
\psi(S(y_2,y_2,x_3))&=\psi(S(F(x_1,y_1),F(x_1,y_1),F(y_2,x_2)))\\
&\leq\psi(\tfrac{1}{4}[\max\{S(x_1,x_1,F(y_2,x_2)),S(x_1,x_1,F(x_1,y_1))\}\\
&\quad+\max\{S(y_2,y_2,F(x_1,y_1)),S(x_1,x_1,F(x_1,y_1))\}])\\
&\quad-\varphi(\max\{S(x_1,x_1,F(y_2,x_2)),S(x_1,x_1,F(x_1,y_1))\},\\
&\quad\max\{S(y_2,y_2,F(x_1,y_1)),S(x_1,x_1,F(x_1,y_1))\})\\
&=\psi(\tfrac{1}{4}[\max\{S(x_1,x_1,x_3),S(x_1,x_1,y_2)\}+S(x_1,x_1,y_2)])\\
&\quad-\varphi(\max\{S(x_1,x_1,x_3),S(x_1,x_1,y_2)\},S(x_1,x_1,y_2))\\
&\leq\psi(\tfrac{1}{4}[\max\{2S(x_1,x_1,y_2)+S(y_2,y_2,x_3),S(x_1,x_1,y_2)\}\\
&\qquad+S(x_1,x_1,y_2)])\\
&\quad-\varphi(\max\{S(x_1,x_1,x_3),S(x_1,x_1,y_2)\},S(x_1,x_1,y_2))\\
&=\psi(\tfrac{1}{4}[2S(x_1,x_1,y_2)+S(y_2,y_2,x_3)+S(x_1,x_1,y_2)])\\
&\quad-\varphi(\max\{S(x_1,x_1,x_3),S(x_1,x_1,y_2)\},S(x_1,x_1,y_2))\\
&<\psi(\tfrac{3}{4}S(x_1,x_1,y_2)+\tfrac{1}{4}S(y_2,y_2,x_3)),\text{ and hence}
\end{aligned}$$

$S(y_2,y_2,x_3)\leq\frac{3}{4}S(x_1,x_1,y_2)+\frac{1}{4}S(y_2,y_2,x_3)$ so that
$S(y_2,y_2,x_3)\leq S(x_1,x_1,y_2)$.
In general, we have

$$\begin{aligned}
\psi(S(x_{2n+1},x_{2n+1},y_{2n+2}))&=\psi(S(y_{2n+2},y_{2n+2},x_{2n+1}))\\
&=\psi(S(F(x_{2n+1},y_{2n+1}),F(x_{2n+1},y_{2n+1}),F(y_{2n},x_{2n})))\\
&\leq\psi(\tfrac{1}{4}[\max\{S(x_{2n+1},x_{2n+1},F(y_{2n},x_{2n})),\\
&\qquad S(x_{2n+1},x_{2n+1},F(x_{2n+1},y_{2n+1}))\}\\
&\quad+\max\{S(y_{2n},y_{2n},F(x_{2n+1},y_{2n+1})),\\
&\qquad S(x_{2n+1},x_{2n+1},F(x_{2n+1},y_{2n+1}))\}])\\
&\quad-\varphi(\max\{S(x_{2n+1},x_{2n+1},F(y_{2n},x_{2n})),\\
&\qquad S(x_{2n+1},x_{2n+1},F(x_{2n+1},y_{2n+1}))\},\\
&\qquad\max\{S(y_{2n},y_{2n},F(x_{2n+1},y_{2n+1})),\\
&\qquad S(x_{2n+1},x_{2n+1},F(x_{2n+1},y_{2n+1}))\})\\
&=\psi(\tfrac{1}{4}[\max\{S(x_{2n+1},x_{2n+1},x_{2n+1}),S(x_{2n+1},x_{2n+1},y_{2n+2})\}\\
&\quad+\max\{S(y_{2n},y_{2n},y_{2n+2}),S(x_{2n+1},x_{2n+1},y_{2n+2})\}])\\
&\quad-\varphi(\max\{S(x_{2n+1},x_{2n+1},x_{2n+1}),S(x_{2n+1},x_{2n+1},y_{2n+2})\},\\
&\qquad\max\{S(y_{2n},y_{2n},y_{2n+2}),S(x_{2n+1},x_{2n+1},y_{2n+2})\})\\
&\leq\psi(\tfrac{1}{4}[S(x_{2n+1},x_{2n+1},y_{2n+2})+\max\{2S(y_{2n},y_{2n},x_{2n+1})
\end{aligned}$$

$$+S(x_{2n+1},x_{2n+1},y_{2n+2}),S(x_{2n+1},x_{2n+1},y_{2n+2})\}])$$
$$-\varphi(S(x_{2n+1},x_{2n+1},y_{2n+2}),\max\{S(y_{2n},y_{2n},y_{2n+2}),$$
$$S(x_{2n+1},x_{2n+1},y_{2n+2})\}).$$

That is

$$\begin{aligned}\psi(S(x_{2n+1},x_{2n+1},y_{2n+2})) &\leq \psi(\frac{1}{4}[2S(x_{2n+1},x_{2n+1},y_{2n+2})+2S(y_{2n},y_{2n},x_{2n+1})]) \\ &\quad -\varphi(S(x_{2n+1},x_{2n+1},y_{2n+2}),\max\{S(y_{2n},y_{2n},y_{2n+2}), \\ &\quad S(x_{2n+1},x_{2n+1},y_{2n+2})\}) \end{aligned} \tag{3.7}$$

$< \psi(\frac{1}{2}S(x_{2n+1},x_{2n+1},y_{2n+2})+\frac{1}{2}S(y_{2n},y_{2n},x_{2n+1}))$ and hence $S(x_{2n+1},x_{2n+1},y_{2n+2}) \leq \frac{1}{2}S(x_{2n+1},x_{2n+1},y_{2n+2})+\frac{1}{2}S(y_{2n},y_{2n},x_{2n+1})$ so that

$$S(x_{2n+1},x_{2n+1},y_{2n+2}) \leq S(y_{2n},y_{2n},x_{2n+1}), \text{ for each } n=1,2,3,\dots . \tag{3.8}$$

Similarly, we have

$$\begin{aligned}&\psi(S(y_{2n+1},y_{2n+1},x_{2n+2})) \\ &= \psi(S(F(x_{2n},y_{2n}),F(x_{2n},y_{2n}),F(y_{2n+1},x_{2n+1}))) \\ &\leq \psi(\tfrac{1}{4}[\max\{S(x_{2n},x_{2n},F(y_{2n+1},x_{2n+1})),S(x_{2n},x_{2n},F(x_{2n},y_{2n}))\} \\ &\quad +\max\{S(y_{2n+1},y_{2n+1},F(x_{2n},y_{2n})),S(x_{2n},x_{2n},F(x_{2n},y_{2n}))\}]) \\ &\quad -\varphi(\max\{S(x_{2n},x_{2n},F(y_{2n+1},x_{2n+1})),S(x_{2n},x_{2n},F(x_{2n},y_{2n}))\}, \\ &\quad \max\{S(y_{2n+1},y_{2n+1},F(x_{2n},y_{2n})),S(x_{2n},x_{2n},F(x_{2n},y_{2n}))\}) \\ &= \psi(\tfrac{1}{4}[\max\{S(x_{2n},x_{2n},x_{2n+2}),S(x_{2n},x_{2n},y_{2n+1})\}+S(x_{2n},x_{2n},y_{2n+1})]) \\ &\quad -\varphi(\max\{S(x_{2n},x_{2n},x_{2n+2}),S(x_{2n},x_{2n},y_{2n+1})\},S(x_{2n},x_{2n},y_{2n+1})) \\ &\leq \psi(\tfrac{1}{4}[\max\{2S(x_{2n},x_{2n},y_{2n+1})+S(y_{2n+1},y_{2n+1},x_{2n+2}), \\ &\quad S(x_{2n},x_{2n},y_{2n+1})\}+S(x_{2n},x_{2n},y_{2n+1})]) \\ &\quad -\varphi(\max\{S(x_{2n},x_{2n},x_{2n+2}),S(x_{2n},x_{2n},y_{2n+1})\},S(x_{2n},x_{2n},y_{2n+1})).\end{aligned}$$

That is

$$\begin{aligned}\psi(S(y_{2n+1},y_{2n+1},x_{2n+2})) &\leq \psi(\frac{3}{4}S(x_{2n},x_{2n},y_{2n+1})+\frac{1}{4}S(y_{2n+1},y_{2n+1},x_{2n+2})) \\ &\quad -\varphi(\max\{S(x_{2n},x_{2n},x_{2n+2}),S(x_{2n},x_{2n},y_{2n+1})\}, \\ &\quad S(x_{2n},x_{2n},y_{2n+1}))\end{aligned} \tag{3.9}$$

$$< \psi(\tfrac{3}{4}S(x_{2n},x_{2n},y_{2n+1})+\tfrac{1}{4}S(y_{2n+1},y_{2n+1},x_{2n+2}))$$

which implies that

$S(y_{2n+1},y_{2n+1},x_{2n+2}) \leq \frac{3}{4}S(x_{2n},x_{2n},y_{2n+1})+\frac{1}{4}S(y_{2n+1},y_{2n+1},x_{2n+2})$ and hence

$$S(y_{2n+1},y_{2n+1},x_{2n+2}) \leq S(x_{2n},x_{2n},y_{2n+1}) \text{ for each } n=1,2,3,\dots . \tag{3.10}$$

Similarly, we get

$$S(x_{2n},x_{2n},y_{2n+1}) \leq S(y_{2n-1},y_{2n-1},x_{2n}) \text{ for each } n=1,2,3,\dots ; \tag{3.11}$$

and

$$S(y_{2n},y_{2n},x_{2n+1}) \leq S(x_{2n-1},x_{2n-1},y_{2n}) \text{ for each } n=1,2,3,\dots . \tag{3.12}$$

From (3.8) and (3.11) it follows that

$$S(x_n,x_n,y_{n+1}) \leq S(y_{n-1},y_{n-1},x_n) \text{ for } n=1,2,3,\dots \tag{3.13}$$

and from (3.10) and (3.12) it follows that

$$S(y_n,y_n,x_{n+1}) \leq S(x_{n-1},x_{n-1},y_n) \text{ for } n=1,2,3,\dots . \tag{3.14}$$

Hence, from (3.13) and (3.14), it follows that $\{S(x_n, x_n, y_{n+1})\}$ is a decreasing sequence and converges to some $r \geq 0$ and $\{S(y_n, y_n, x_{n+1})\}$ is a decreasing sequence and hence converges to some $s \geq 0$.
From (3.13), we have $r \leq s$ and from (3.14), we have $s \leq r$. Therefore $r = s$.
Now on taking the limits as $n \to \infty$ in (3.7), we have
$$\begin{aligned}\psi(r) &\leq \psi(\tfrac{1}{2}r + \tfrac{1}{2}r) - \varphi(r, \max\{\lim_{n\to\infty} S(y_{2n}, y_{2n}, y_{2n+2}), r\})\\ &= \psi(r) - \varphi(r, \max\{\lim_{n\to\infty} S(y_{2n}, y_{2n}, y_{2n+2}), r\})\end{aligned}$$
which implies that $\varphi(r, \max\{\lim_{n\to\infty} S(y_{2n}, y_{2n}, y_{2n+2}), r\}) = 0$ so that $r = 0$.

Therefore
$$\lim_{n\to\infty} S(x_n, x_n, y_{n+1}) = 0 \text{ and } \lim_{n\to\infty} S(y_n, y_n, x_{n+1}) = 0. \tag{3.15}$$

We now consider
$$\begin{aligned}\psi(S(x_{n+1}, x_{n+1}, y_{n+1})) &= \psi(S(y_{n+1}, y_{n+1}, x_{n+1}))\\
&= \psi(S(F(x_n, y_n), F(x_n, y_n), F(y_n, x_n)))\\
&\leq \psi(\tfrac{1}{4}[\max\{S(x_n, x_n, F(y_n, x_n)), S(x_n, x_n, F(x_n, y_n))\}\\
&\qquad + \max\{S(y_n, y_n, F(x_n, y_n)), S(x_n, x_n, F(x_n, y_n))\}])\\
&\quad - \varphi(\max\{S(x_n, x_n, F(y_n, x_n)), S(x_n, x_n, F(x_n, y_n))\},\\
&\qquad \max\{S(y_n, y_n, F(x_n, y_n)), S(x_n, x_n, F(x_n, y_n))\})\\
&= \psi(\tfrac{1}{4}[\max\{S(x_n, x_n, x_{n+1}), S(x_n, x_n, y_{n+1})\}\\
&\qquad + \max\{S(y_n, y_n, y_{n+1}), S(x_n, x_n, y_{n+1})\}])\\
&\quad - \varphi(\max\{S(x_n, x_n, x_{n+1}), S(x_n, x_n, y_{n+1})\},\\
&\qquad \max\{S(y_n, y_n, y_{n+1}), S(x_n, x_n, y_{n+1})\})\\
&\leq \psi(\tfrac{1}{4}[\max\{2S(x_n, x_n, y_{n+1}) + S(x_{n+1}, x_{n+1}, y_{n+1}),\\
&\qquad S(x_n, x_n, y_{n+1})\} + S(y_n, y_n, y_{n+1}) + S(x_n, x_n, y_{n+1})])\\
&\quad - \varphi(\max\{S(x_n, x_n, x_{n+1}), S(x_n, x_n, y_{n+1})\},\\
&\qquad \max\{S(y_n, y_n, y_{n+1}), S(x_n, x_n, y_{n+1})\})\\
&= \psi(\tfrac{1}{4}[2S(x_n, x_n, y_{n+1}) + S(x_{n+1}, x_{n+1}, y_{n+1})\\
&\qquad + 2S(y_n, y_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, y_{n+1})\\
&\qquad\quad + S(x_n, x_n, y_{n+1})\}])\\
&\quad - \varphi(\max\{S(x_n, x_n, x_{n+1}), S(x_n, x_n, y_{n+1})\},\\
&\qquad \max\{S(y_n, y_n, y_{n+1}), S(x_n, x_n, y_{n+1})\})\\
&= \psi(\tfrac{1}{4}[3S(x_n, x_n, y_{n+1}) + 2S(y_n, y_n, x_{n+1})\\
&\qquad + 2S(x_{n+1}, x_{n+1}, y_{n+1})])\\
&\quad - \varphi(\max\{S(x_n, x_n, x_{n+1}), S(x_n, x_n, y_{n+1})\},\\
&\qquad \max\{S(y_n, y_n, y_{n+1}), S(x_n, x_n, y_{n+1})\})\\
&< \psi(\tfrac{1}{4}[3S(x_n, x_n, y_{n+1}) + 2S(y_n, y_n, x_{n+1})\\
&\qquad + 2S(x_{n+1}, x_{n+1}, y_{n+1})]) \text{ and hence}\end{aligned}$$
$S(x_{n+1}, x_{n+1}, y_{n+1}) \leq \frac{3}{4}S(x_n, x_n, y_{n+1}) + \frac{1}{2}S(y_n, y_n, x_{n+1}) + \frac{1}{2}S(x_{n+1}, x_{n+1}, y_{n+1})$
which implies that
$S(x_{n+1}, x_{n+1}, y_{n+1}) \leq S(y_n, y_n, x_{n+1}) + \frac{3}{2}S(x_n, x_n, y_{n+1})$.
On taking limits as $n \to \infty$ and by using (3.15), we get
$$\lim_{n\to\infty} S(x_{n+1}, x_{n+1}, y_{n+1}) = 0. \tag{3.16}$$

We now consider
$$\begin{aligned}S(x_n, x_n, x_{n+1}) + S(y_n, y_n, y_{n+1}) &\leq 2S(x_n, x_n, y_n) + S(y_n, y_n, x_{n+1})\\
&\qquad + 2S(y_n, y_n, x_n) + S(x_n, x_n, y_{n+1})\\
&= 4S(x_n, x_n, y_n) + S(x_n, x_n, y_{n+1}) + S(y_n, y_n, x_{n+1}).\end{aligned}$$

On taking limits as $n \to \infty$, we get
$\lim\limits_{n\to\infty} S(x_n, x_n, x_{n+1}) + S(y_n, y_n, y_{n+1}) = 0.$
We now prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Suppose that either $\{x_n\}$ or $\{y_n\}$ is not Cauchy. Then there exist $\epsilon > 0$ and subsequences $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that

$$\max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} \geq \epsilon. \tag{3.17}$$

We choose m_k as a smallest integer with $m_k > n_k$ satisfying (3.17).
That is $\max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} \geq \epsilon$
with $\max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\} < \epsilon$.
We now prove the following.

$$\lim_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, y_{n_k}), S(y_{m_k}, y_{m_k}, x_{n_k})\} = \epsilon. \tag{3.18}$$

We consider $S(x_{m_k}, x_{m_k}, y_{n_k}) = S(y_{n_k}, y_{n_k}, x_{m_k})$
$\leq 2S(y_{n_k}, y_{n_k}, x_{n_k}) + S(x_{n_k}, x_{n_k}, x_{m_k})$.
Also, we have
$S(y_{m_k}, y_{m_k}, x_{n_k}) \leq 2S(x_{n_k}, x_{n_k}, y_{n_k}) + S(y_{n_k}, y_{n_k}, y_{m_k})$.
Thus we have
$\max\{S(x_{m_k}, x_{m_k}, y_{n_k}), S(y_{m_k}, y_{m_k}, x_{n_k})\} \leq \max\{2S(y_{n_k}, y_{n_k}, x_{n_k}) + S(x_{n_k}, x_{n_k}, x_{m_k}),$
$2S(x_{n_k}, x_{n_k}, y_{n_k}) + S(y_{n_k}, y_{n_k}, y_{m_k})\}$.
On taking limit supremum as $k \to \infty$, and using Proposition 2.1, we get
$\limsup\limits_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, y_{n_k}), S(y_{m_k}, y_{m_k}, x_{n_k})\} \leq \limsup\limits_{k\to\infty} \max\{S(x_{n_k}, x_{n_k}, x_{m_k}),$
$S(y_{n_k}, y_{n_k}, y_{m_k})\}$
$= \epsilon$ (by (i) of Lemma 3.1).

We now consider
$S(x_{m_k}, x_{m_k}, x_{n_k}) = S(x_{n_k}, x_{n_k}, x_{m_k})$
$\leq 2S(x_{n_k}, x_{n_k}, y_{n_k}) + S(y_{n_k}, y_{n_k}, x_{m_k})$
and
$S(y_{m_k}, y_{m_k}, y_{n_k}) = S(y_{n_k}, y_{n_k}, y_{m_k})$
$\leq 2S(y_{n_k}, y_{n_k}, x_{n_k}) + S(x_{n_k}, x_{n_k}, y_{m_k})$ so that
$\max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\} \leq \max\{2S(x_{n_k}, x_{n_k}, y_{n_k}) + S(y_{n_k}, y_{n_k}, x_{m_k}),$
$2S(y_{n_k}, y_{n_k}, x_{n_k}) + S(x_{n_k}, x_{n_k}, y_{m_k})\}$.
On taking limit infimum as $k \to \infty$ and using (3.16),
$\epsilon \leq \liminf\limits_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, x_{n_k}), S(y_{m_k}, y_{m_k}, y_{n_k})\}$
$\leq \liminf\limits_{k\to\infty} \max\{S(y_{n_k}, y_{n_k}, x_{m_k}), S(x_{n_k}, x_{n_k}, y_{m_k})\}$ (by Proposition 2.3).
From the above we have
$\epsilon \leq \liminf\limits_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, y_{n_k}), S(y_{m_k}, y_{m_k}, x_{n_k})\}$
$\leq \limsup\limits_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, y_{n_k}), S(y_{m_k}, y_{m_k}, x_{n_k})\} \leq \epsilon.$
Hence $\liminf\limits_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, y_{n_k}), S(y_{m_k}, y_{m_k}, x_{n_k})\}$
$= \limsup\limits_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, y_{n_k}), S(y_{m_k}, y_{m_k}, x_{n_k})\} = \epsilon.$
Therefore $\lim\limits_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, y_{n_k}), S(y_{m_k}, y_{m_k}, x_{n_k})\}$ exists and
$\lim\limits_{k\to\infty} \max\{S(x_{m_k}, x_{m_k}, y_{n_k}), S(y_{m_k}, y_{m_k}, x_{n_k})\} = \epsilon.$
Hence (3.18) is proved.

We now consider

$$\begin{aligned}
\psi(S(x_{m_k}, x_{m_k}, y_{n_k})) &= \psi(y_{n_k}, y_{n_k}, x_{m_k}) \\
&= \psi(S(F(x_{n_k-1}, y_{n_k-1}), F(x_{n_k-1}, y_{n_k-1}), F(y_{m_k-1}, x_{m_k-1}))) \\
&\leq \psi(\tfrac{1}{4}[\max\{S(x_{n_k-1}, x_{n_k-1}, F(y_{m_k-1}, x_{m_k-1})), \\
&\qquad S(x_{n_k-1}, x_{n_k-1}, F(x_{n_k-1}, y_{n_k-1}))\} \\
&\quad + \max\{S(y_{m_k-1}, y_{m_k-1}, F(x_{n_k-1}, y_{n_k-1})), \\
&\qquad S(x_{n_k-1}, x_{n_k-1}, F(x_{n_k-1}, y_{n_k-1}))\}]) \\
&\quad - \varphi(\max\{S(x_{n_k-1}, x_{n_k-1}, F(y_{m_k-1}, x_{m_k-1})), \\
&\qquad S(x_{n_k-1}, x_{n_k-1}, F(x_{n_k-1}, y_{n_k-1}))\}, \\
&\qquad \max\{S(y_{m_k-1}, y_{m_k-1}, F(x_{n_k-1}, y_{n_k-1})), \\
&\qquad S(x_{n_k-1}, x_{n_k-1}, F(x_{n_k-1}, y_{n_k-1}))\}) \\
&= \psi(\tfrac{1}{4}[\max\{S(x_{n_k-1}, x_{n_k-1}, x_{m_k}), S(x_{n_k-1}, x_{n_k-1}, y_{n_k})\} \\
&\quad + \max\{S(y_{m_k-1}, y_{m_k-1}, y_{n_k}), S(x_{n_k-1}, x_{n_k-1}, y_{n_k})\}]) \\
&\quad - \varphi(\max\{S(x_{n_k-1}, x_{n_k-1}, x_{m_k}), S(x_{n_k-1}, x_{n_k-1}, y_{n_k})\}, \\
&\qquad \max\{S(y_{m_k-1}, y_{m_k-1}, y_{n_k}), S(x_{n_k-1}, x_{n_k-1}, y_{n_k})\}).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\psi(S(y_{m_k}, y_{m_k}, x_{n_k})) &= \psi(S(F(x_{m_k-1}, y_{m_k-1}), F(x_{m_k-1}, y_{m_k-1}), F(y_{n_k-1}, x_{n_k-1}))) \\
&\leq \psi(\tfrac{1}{4}[\max\{S(x_{m_k-1}, x_{m_k-1}, F(y_{n_k-1}, x_{n_k-1})), \\
&\qquad S(x_{m_k-1}, x_{m_k-1}, F(x_{m_k-1}, y_{m_k-1}))\} \\
&\quad + \max\{S(y_{n_k-1}, y_{n_k-1}, F(x_{m_k-1}, y_{m_k-1})), \\
&\qquad S(x_{m_k-1}, x_{m_k-1}, F(x_{m_k-1}, y_{m_k-1}))\}]) \\
&\quad - \varphi(\max\{S(x_{m_k-1}, x_{m_k-1}, F(y_{n_k-1}, x_{n_k-1})), \\
&\qquad S(x_{m_k-1}, x_{m_k-1}, F(x_{m_k-1}, y_{m_k-1}))\}, \\
&\qquad \max\{S(y_{n_k-1}, y_{n_k-1}, F(x_{m_k-1}, y_{m_k-1})), \\
&\qquad S(x_{m_k-1}, x_{m_k-1}, F(x_{m_k-1}, y_{m_k-1}))\}) \\
&= \psi(\tfrac{1}{4}[\max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(x_{m_k-1}, x_{m_k-1}, y_{m_k})\} \\
&\quad + \max\{S(y_{n_k-1}, y_{n_k-1}, y_{m_k}), S(x_{m_k-1}, x_{m_k-1}, y_{m_k})\}]) \\
&\quad - \varphi(\max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(x_{m_k-1}, x_{m_k-1}, y_{m_k})\}, \\
&\qquad \max\{S(y_{n_k-1}, y_{n_k-1}, y_{m_k}), S(x_{m_k-1}, x_{m_k-1}, y_{m_k})\}).
\end{aligned}$$

We now consider

$$\begin{aligned}
&\psi(\max\{S(x_{m_k}, x_{m_k}, y_{n_k}), S(y_{m_k}, y_{m_k}, x_{n_k})\}) \\
&\quad = \max\{\psi(S(x_{m_k}, x_{m_k}, y_{n_k})), \psi(S(y_{m_k}, y_{m_k}, x_{n_k}))\} \\
&\quad \leq \max\{\psi(\tfrac{1}{4}[\max\{S(x_{n_k-1}, x_{n_k-1}, x_{m_k}), S(x_{n_k-1}, x_{n_k-1}, y_{n_k})\} \\
&\qquad + \max\{S(y_{m_k-1}, y_{m_k-1}, y_{n_k}), S(x_{n_k-1}, x_{n_k-1}, y_{n_k})\}]), \\
&\qquad \psi(\tfrac{1}{4}[\max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(x_{m_k-1}, x_{m_k-1}, y_{m_k})\} \\
&\qquad + \max\{S(y_{n_k-1}, y_{n_k-1}, y_{m_k}), S(x_{m_k-1}, x_{m_k-1}, y_{m_k})\}])\} \\
&\qquad - \min\{\varphi(\max\{S(x_{n_k-1}, x_{n_k-1}, x_{m_k}), S(x_{n_k-1}, x_{n_k-1}, y_{n_k})\}, \\
&\qquad \max\{S(y_{m_k-1}, y_{m_k-1}, y_{n_k}), S(x_{n_k-1}, x_{n_k-1}, y_{n_k})\}), \\
&\qquad \varphi(\max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(x_{m_k-1}, x_{m_k-1}, y_{m_k})\}, \\
&\qquad \max\{S(y_{n_k-1}, y_{n_k-1}, y_{m_k}), S(x_{m_k-1}, x_{m_k-1}, y_{m_k})\})\}.
\end{aligned}$$

On letting $k \to \infty$ and by using (3.15), we get

$$\begin{aligned}
\psi(\epsilon) \leq \max\{&\psi(\tfrac{1}{4}[\lim_{k\to\infty} S(x_{n_k-1}, x_{n_k-1}, x_{m_k}) + \lim_{k\to\infty} S(y_{m_k-1}, y_{m_k-1}, y_{n_k})]), \\
&\psi(\tfrac{1}{4}[\lim_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) + \lim_{k\to\infty} S(y_{n_k-1}, y_{n_k-1}, y_{m_k})])\} \\
- \min\{&\varphi(\lim_{k\to\infty} S(x_{n_k-1}, x_{n_k-1}, x_{m_k}), \lim_{k\to\infty} S(y_{m_k-1}, y_{m_k-1}, y_{n_k})),
\end{aligned}$$

$$\varphi(\lim_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), \lim_{k\to\infty} S(y_{n_k-1}, y_{n_k-1}, y_{m_k}))\} \text{ so that}$$

$$\begin{aligned}\psi(\epsilon) \leq \psi(\max\{&\tfrac{1}{4}[\lim_{k\to\infty} \max\{S(x_{n_k-1}, x_{n_k-1}, x_{m_k}), S(y_{n_k-1}, y_{n_k-1}, y_{m_k})\}\\
&+ \lim_{k\to\infty} \max\{S(y_{m_k-1}, y_{m_k-1}, y_{n_k}), S(x_{m_k-1}, x_{m_k-1}, x_{n_k})\}],\\
&\tfrac{1}{4}[\lim_{k\to\infty} \max\{S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), S(y_{m_k-1}, y_{m_k-1}, y_{n_k})\}\\
&+ \lim_{k\to\infty} \max\{S(y_{n_k-1}, y_{n_k-1}, y_{m_k}), S(x_{n_k-1}, x_{n_k-1}, x_{m_k})\}]\})\\
&- \min\{\varphi(\lim_{k\to\infty} S(x_{n_k-1}, x_{n_k-1}, x_{m_k}), \lim_{k\to\infty} S(y_{m_k-1}, y_{m_k-1}, y_{n_k})),\\
&\varphi(\lim_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), \lim_{k\to\infty} S(y_{n_k-1}, y_{n_k-1}, y_{m_k}))\}.\\
&\text{(By using (ii) and (iii) of Lemma 3.1)}\\
= \psi(\tfrac{\epsilon}{2}) - \min\{&\varphi(\lim_{k\to\infty} S(x_{n_k-1}, x_{n_k-1}, x_{m_k}), \lim_{k\to\infty} S(y_{m_k-1}, y_{m_k-1}, y_{n_k})),\\
&\varphi(\lim_{k\to\infty} S(x_{m_k-1}, x_{m_k-1}, x_{n_k}), \lim_{k\to\infty} S(y_{n_k-1}, y_{n_k-1}, y_{m_k}))\}\\
&\text{(all these limits are positive by using Lemma 2.11)}\\
< \psi(\tfrac{\epsilon}{2}), &\text{ a contradiction.}\end{aligned}$$

Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences and hence convergent. Since A and B are closed subsets of X and $\{x_n\} \subset A$, $\{y_n\} \subset B$, there exist $x \in A$ and $y \in B$ such that

$$x_n \to x,\ y_n \to y \text{ as } n \to \infty. \tag{3.19}$$

By using (3.16), we get $\lim_{n\to\infty} S(x_n, x_n, y_n) = 0$. Now, by Lemma 2.9, we have $S(x, x, y) = 0$ and hence $x = y$ so that $A \cap B \neq \emptyset$ and $x \in A \cap B$.

Now, by (3.1) and (3.4), we have

$$\begin{aligned}\psi(S(x_{n+1}, x_{n+1}, F(x, x)) &= \psi(S(F(x, x), F(x, x), F(y_n, x_n)))\\
&\leq \psi(\tfrac{1}{4}[\max\{S(x, x, F(y_n, x_n)), S(x, x, F(x, x))\}\\
&\quad + \max\{S(y_n, y_n, F(x, x)), S(x, x, F(x, x))\}])\\
&\quad - \varphi(\max\{S(x, x, F(y_n, x_n)), S(x, x, F(x, x))\},\\
&\quad \max\{S(y_n, y_n, F(x, x)), S(x, x, F(x, x))\})\\
&= \psi(\tfrac{1}{4}[\max\{S(x, x, x_{n+1}), S(x, x, F(x, x))\}\\
&\quad + \max\{S(y_n, y_n, F(x, x)), S(x, x, F(x, x))\}])\\
&\quad - \varphi(\max\{S(x, x, x_{n+1}), S(x, x, F(x, x))\},\\
&\quad \max\{S(y_n, y_n, F(x, x)), S(x, x, F(x, x))\}).\end{aligned}$$

On taking limits as $n \to \infty$, we get

$$\begin{aligned}\psi(S(x, x, F(x, x))) &\leq \psi(\tfrac{1}{4}[\max\{S(x, x, x), S(x, x, F(x, x))\}\\
&\quad + \max\{S(y, y, F(x, x)), S(x, x, F(x, x))\}])\\
&\quad - \varphi(\max\{S(x, x, x), S(x, x, F(x, x))\},\\
&\quad \max\{S(y, y, F(x, x)), S(x, x, F(x, x))\})\\
&\leq \psi(S(x, x, F(x, x))) - \varphi(S(x, x, F(x, x)), S(x, x, F(x, x))) \text{ and}\end{aligned}$$

hence $\varphi(S(x, x, F(x, x)), S(x, x, F(x, x))) = 0$ so that $S(x, x, F(x, x)) = 0$. Therefore $x = F(x, x)$ is a strong coupled fixed point of F. We now prove the uniqueness of strong coupled fixed point of F. Suppose (x, x) and (y, y) are two strong coupled fixed points of F. We consider

$$\begin{aligned}\psi(S(x, x, y)) &= \psi(S(F(x, x), F(x, x), F(y, y)))\\
&\leq \psi(\tfrac{1}{4}[\max\{S(x, x, F(y, y)), S(x, x, F(x, x))\}\\
&\quad + \max\{S(y, y, F(x, x)), S(x, x, F(x, x))\}])\\
&\quad - \varphi(\max\{S(x, x, F(y, y)), S(x, x, F(x, x))\},\\
&\quad \max\{S(y, y, F(x, x)), S(x, x, F(x, x))\})\\
&= \psi(\tfrac{1}{4}[S(x, x, y) + S(y, y, x)]) - \varphi(S(x, x, y), S(y, y, x))\end{aligned}$$

$$\leq \psi(S(x,x,y)) - \varphi(S(x,x,y), S(y,y,x))$$

so that $\varphi(S(x,x,y), S(y,y,x)) = 0$. Thus $x = y$. □

By choosing $\psi(t) = t$ in Theorem 3.2, then we have the following.

Corollary 3.3. Let (X, S) be a complete S-metric space. Let A and B be two nonempty closed subsets of X. Let $F : X \times X \to X$ be mapping. If F is cyclic with respect to A and B and there exists $\varphi \in \Phi$ such that

$$\begin{aligned} S(F(x,y), F(u,v), F(w,z)) \leq \frac{1}{4}[&\max\{S(x,x,F(w,z)), S(x,x,F(u,v))\} \\ &+ \max\{S(w,w,F(x,y)), S(u,u,F(x,y))\}] \\ &-\varphi(\max\{S(x,x,F(w,z)), S(x,x,F(u,v))\}, \\ &\max\{S(w,w,F(x,y)), S(u,u,F(x,y))\}) \end{aligned}$$

where $x, u, z \in A$ and $y, v, w \in B$. Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.

By choosing $\varphi(t_1, t_2) = (\frac{1}{4} - k)(t_1 + t_2)$ in Corollary 3.3 then we have the following.

Corollary 3.4. Let (X, S) be a complete S-metric space. Let A and B be two nonempty closed subsets of X. Let $F : X \times X \to X$ be mapping. If F is cyclic with respect to A and B and there exists $k \in (0, \frac{1}{4})$ such that

$$\begin{aligned} S(F(x,y), F(u,v), F(w,z)) \leq k[&\max\{S(x,x,F(w,z)), S(x,x,F(u,v))\} \\ &+ \max\{S(w,w,F(x,y)), S(u,u,F(x,y))\}] \end{aligned}$$

where $x, u, z \in A$ and $y, v, w \in B$. Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.

By choosing $w = u$ and $z = v$ in Corollary 3.4, then we have the following.

Corollary 3.5. Let (X, S) be a complete S-metric space. Let A and B be two nonempty closed subsets of X. Let $F : X \times X \to X$ be mapping. If F is cyclic with respect to A and B and there exists $k \in (0, \frac{1}{4})$ such that

$$S(F(x,y), F(u,v), F(u,v)) \leq k[S(x,x,F(u,v)) + S(u,u,F(x,y))$$

where $x, v \in A$ and $y, u \in B$. Then $A \cap B \neq \emptyset$ and F has a unique strong coupled fixed point in $A \cap B$.

The following example is in support of Theorem 3.2.

Example 3.2. Let $X = [0, 1]$. We define $S : X^3 \to [0, \infty)$ by

$$S(x,y,z) = \begin{cases} 0 & \text{if } x = y = z \\ x + y + z & \text{otherwise.} \end{cases}$$

Then (X, S) is a complete S-metric space.
Let $A = [0, \frac{1}{2}]$ and $B = [0, 1]$. We define $F : X \times X \to X$ by

$$F(x,y) = \begin{cases} \frac{x}{8(x+y+1)} & \text{if } x \in A \text{ and } y \in B \\ 0 & \text{otherwise.} \end{cases}$$

Then $F(A, B) = [0, \frac{1}{16}] \subset B$ and $F(B, A) = \{0\} \subset A$ so that F is cyclic with respect to A and B. We define $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \frac{t}{2}$ and $\varphi : [0, \infty)^2 \to [0, \infty)$ by $\varphi(t_1, t_2) = \frac{1}{16}(t_1 + t_2)$. We now verify the inequality (3.1). Let $x, u, z \in A$ and $y, v, w \in B$. We now consider

$$\begin{aligned}\psi(S(F(x,y),F(u,v),F(w,z))) &= \psi(S(\tfrac{x}{8(x+y+1)},\tfrac{u}{8(u+v+1)},0))\\ &= \tfrac{1}{2}S(\tfrac{x}{8(x+y+1)},\tfrac{u}{8(u+v+1)},0)\\ &= \tfrac{x}{16(x+y+1)}+\tfrac{u}{16(u+v+1)}\\ &\leq \tfrac{1}{16}[x+u]\\ &\leq \tfrac{1}{16}[\max\{S(x,x,F(w,z)),S(x,x,F(u,v))\}\\ &\qquad +\max\{S(w,w,F(x,y)),S(u,u,F(x,y))\}]\\ &= \tfrac{1}{8}[t_1+t_2]-\tfrac{1}{16}[t_1+t_2]\\ &= \psi(\tfrac{1}{4}[t_1+t_2])-\varphi(t_1,t_2),\end{aligned}$$

where $t_1=\max\{S(x,x,F(w,z)),S(x,x,F(u,v))\}$ and
$t_2=\max\{S(w,w,F(x,y)),S(u,u,F(x,y))\}$.

Therefore F is a Chatterjea type (ψ,φ)-weakly cyclic coupled mapping with respect to A and B. Hence F satisfies all the hypotheses of Theorem 3.2 and $(0,0)$ is a unique strong coupled fixed point of F.

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