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A GENERALIZATION OF THE LEE WEIGHT TO $\mathbb{Z}_{p^k}$

B. YILDIZ[1], Z. ÖDEMİŞ ÖZGER[2] §

ABSTRACT. We introduce a new extension of the Lee weight to $\mathbb{Z}_{p^k}$ and later to Galois rings $GR(p^k, m)$. The weight we define is a non-homogeneous weight and is different than the one that is generally labeled as "generalized Lee weight". Unlike the case of generalized Lee weight, we define a distance-preserving Gray map from ($\mathbb{Z}_{p^k}$, extended Lee distance) to ($\mathbb{F}_p^{p^{k-1}}$, Hamming distance), thus making our extension practical for coding theory purposes.

Keywords: Lee weight, extended, Gray map.

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1. Introduction

In the early history of coding theory, codes over finite fields were predominantly studied. The most common weight used for such codes was the Hamming weight, which we will denote by w_H here, and is simply defined to be the number of nonzero coordinates. Many encoding and decoding schemes as well as error correction algorithms are based on the Hamming distance.

Codes over rings have been considered since early seventies (viz. [2], [12]), however it was not until the beginning of the nineties that studying codes over rings became practical. In 1994, Hammons et al.([6]) solved a long standing mystery in non-linear binary codes by constructing the Kerdock and Preparata codes as the Gray images of linear codes over $\mathbb{Z}_4$. This work started an intense activity on codes over rings. The rich algebraic structure that rings bring together with some better than optimal nonlinear codes obtained from linear codes over rings have increased the popularity of this topic. What started with the ring $\mathbb{Z}_4$, later was extended to rings like $\mathbb{Z}_{2^k}$, $\mathbb{Z}_{p^k}$, Galois rings, $\mathbb{F}_q + u\mathbb{F}_q$, etc. What all these rings have in common is that they are finite chain rings which allows introducing the concept of a type and a single form of a generating matrix for codes over these rings.

In studying codes over rings, weights other than the Hamming weight started to appear. For example, in [6], the authors used the Lee weight on $\mathbb{Z}_4$, which we will denote by w_L and was defined as

$$w_L(x) := \begin{cases} 0 & \text{if } x = 0, \\ 2 & \text{if } x = 2, \\ 1 & \text{otherwise.} \end{cases}$$

[1] Department of Mathematics, Fatih University, 34500, Istanbul, Turkey, e-mail: byildiz@fatih.edu.tr

[2] Department of Mathematics, Fatih University, 34500, Istanbul, Turkey, e-mail: zodemis@fatih.edu.tr

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Now, what makes this weight a useful weight is that we can define a so-called Gray map

$$\phi_L : \mathbb{Z}_4 \to \mathbb{F}_2^2,$$

with

$$\phi_L(0) = (00),\ \phi_L(1) = (01),\ \phi_L(2) = (11),\ \phi_L(3) = (10),$$

which turns out to be a non-linear isometry from $(\mathbb{Z}_4^n, \text{Lee distance})$ to $(\mathbb{F}_2^{2n}, \text{Hamming distance})$. This means that if C is a linear code over $\mathbb{Z}_4$ of length n, size M and minimum Lee distance d, then $\phi_L(C)$ is a possibly non-linear binary code with parameters $[n, M, d]$.

In extending the Lee weight to other extensions of $\mathbb{Z}_4$, different approaches were followed. A natural extension to $\mathbb{Z}_{2^k}$ was studied briefly by Carlet in [3] but was dismissed in favor of the so-called homogeneous weight. We call the first extension the extended Lee weight for $\mathbb{Z}_{2^k}$ and is defined as:

$$w_L(x) := \begin{cases} x & \text{if } x \le 2^{k-1}, \\ 2^k - x & \text{if } x > 2^{k-1}. \end{cases} \tag{1}$$

One of the advantages of working with such a weight is that we can define a Gray map from $\mathbb{Z}_{2^k}$ to $\mathbb{F}_2^{2^{k-1}}$ for this weight function in a very simple way:

$$\begin{array}{lcl}
0 & \to & (000\cdots000), \\
1 & \to & (100\cdots000), \\
2 & \to & (110\cdots000), \\
 & . & \\
 & . & \\
2^{k-1} & \to & (111\cdots111), \\
2^{k-1}+1 & \to & (011\cdots111), \\
2^{k-1}+2 & \to & (001\cdots111), \\
 & . & \\
 & . & \\
2^k-2 & \to & (000\cdots011), \\
2^k-1 & \to & (000\cdots001).
\end{array}$$

We simply put 1' s in the first x coordinates and 0' s in the other coordinates for all $x \le 2^{k-1}$. If $x > 2^{k-1}$ then the Gray map takes x to $\overline{1} + \phi_L(2^{k-1} - x)$, where ϕ_L is the Gray map for w_L.

The homogeneous weight for integer rings was defined in [4] and was applied to $\mathbb{Z}_{p^k}$ by most coding theorists as follows:

$$w_{\text{hom}}(x) := \begin{cases} 0 & \text{if } x = 0, \\ p^{k-1} & \text{if } 0 \ne x \in p^{k-1}\mathbb{Z}_{p^k}, \\ (p-1)p^{k-2} & \text{otherwise.} \end{cases}$$

Of course a Gray map for the homogeneous weight for finite chain rings exist. But it is relatively harder to construct and it doesn't have as simple a form as does the extended Lee weight. For the algebraic constructions of the Gray map of w_{hom} one can look at [5] and [9], while in [13], a combinatorial construction of the Gray map using finite geometries is given. Researchers found it advantageous to work with the homogeneous weight for several reasons. Firstly, there are some bounds in Number Theory about exponential sums which are found to be related to the homogeneous weight; for these one can look at [8], [11] and [10]. The homogeneous weight is also found to be related to finite geometries as seen in [13]. Moreover, because there are only two nonzero weights, weight enumerators of codes seem to be simpler and they satisfy certain divisibility properties as seen in [14].

The homogeneous weight is certainly an extension of the Lee weight defined on $\mathbb{Z}_4$, because if we put $p = 2$ and $k = 2$ we get the Lee weight on $\mathbb{Z}_4$. However it doesn't seem to be a natural extension and since there are only two non-zero weights, we don't have a rich class of codes that can be obtained from codes over rings.

Now, just as the Lee weight on $\mathbb{Z}_4$ was extended to (1) for $\mathbb{Z}_{2^k}$, several researches defined a generalized Lee weight on $\mathbb{Z}_q$ as follows([1]):

$$w_L(x) = \min\{x, q - x\}. \tag{2}$$

However, the problem with this weight is that no distance preserving Gray map has been given up to now that takes codes over $\mathbb{Z}_q$ to codes over the residue field. So, it is not clear how this weight can be used in practical applications of coding theory.

In this work, we define a new weight, which we call the extended Lee weight, on $\mathbb{Z}_{p^k}$ and we later extend it to Galois rings as well. The weight we introduce is a non-homogeneous weight, however it has quite a simple Gray map attached to it. The Gray map is a non-linear isometry from $(\mathbb{Z}_{p^k}^n, \text{Lee distance})$ to $(\mathbb{Z}_p^{p^{k-1}n}, \text{Hamming distance})$ and thus the Lee weight in our definition is comparable to the homogeneous weight.

In Section 2, we introduce the wight together with a simple Gray map for this weight. We will prove that the Gray map is distance preserving.

In Section 3, we will extend the Lee weight w_L to Galois rings.

We will finish the paper with concluding remarks as well as possible directions for research on this new weight.

2. The Extended Lee Weight over $\mathbb{Z}_{p^k}$

We introduce a new weight on $\mathbb{Z}_{p^k}$ as follows:

$$w_L(x) := \begin{cases} x & \text{if } x \le p^{k-1}, \\ p^{k-1} & \text{if } p^{k-1} \le x \le p^k - p^{k-1}, \\ p^k - x & \text{if } p^k - p^{k-1} < x \le p^k - 1. \end{cases}$$

Note that for $p = 2$ and $k = 2$ this reduces to the Lee weight for $\mathbb{Z}_4$ and for $p = 2$ and any k, this is the weight that was used briefly by Carlet in [3].

As an example, for $q = 9$, the homogeneous weight on $\mathbb{Z}_9$ is as follows:

$$w_{\text{hom}}(x) := \begin{cases} 0 & \text{if } x = 0, \\ 3 & \text{if } x = 3, 6 \\ 2 & \text{otherwise.} \end{cases}$$

The Lee weight however is given by

$$w_L(x) := \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x = 1, 8 \\ 2 & \text{if } x = 2, 7 \\ 3 & \text{if } x = 3, 4, 5, 6 \end{cases}$$

Of course for the weight to be of use in terms of coding theory we need to introduce a Gray map as well. It turns out that we can define a Gray map from $\mathbb{Z}_{p^k}$ to $\mathbb{Z}_p^{p^{k-1}}$ just as was done for the homogeneous weight as follows:

$$
\begin{array}{lll}
0 & \rightarrow & (000\cdots000),\\
1 & \rightarrow & (100\cdots000),\\
2 & \rightarrow & (110\cdots000),\\
 & . & \\
 & . & \\
p^{k-1} & \rightarrow & (111\cdots111),\\
p^{k-1}+1 & \rightarrow & (211\cdots111),\\
p^{k-1}+2 & \rightarrow & (221\cdots111),\\
 & . & \\
 & . & \\
p^{k-1}+p^{k-1}-1 & \rightarrow & (222\cdots221),\\
2p^{k-1} & \rightarrow & (222\cdots222),\\
2p^{k-1}+1 & \rightarrow & (322\cdots222),\\
 & . & \\
 & . & \\
2p^{k-1}+p^{k-1}-1 & \rightarrow & (333\cdots332),\\
3p^{k-1} & \rightarrow & (333\cdots333),\\
 & . & \\
 & . & \\
(p-1)p^{k-1} & \rightarrow & ((p-1)\cdots(p-1)),\\
(p-1)p^{k-1}+1 & \rightarrow & (0(p-1)\cdots(p-1)),\\
 & . & \\
 & . & \\
p^k-2 & \rightarrow & (000\cdots0(p-1)(p-1)),\\
p^k-1 & \rightarrow & (000\cdots00(p-1)).
\end{array}
$$

We simply put 1' s in the first x coordinates and 0' s in the other coordinates for all $x \le p^{k-1}$. If $x > p^{k-1}$ then the Gray map takes x to $\overline{q} + \phi_L(r)$, where ϕ_L is the Gray map for w_L, $\overline{q} = (qqq\cdots qqq)$ and q and r are such that

$$x = qp^{k-1} + r,$$

which can be found by division algorithm. Here, $0 \le x \le p^k - 1$, $0 \le q \le p-1$, $0 \le r \le p^{k-1} - 1$.

We define the Lee distance on $\mathbb{Z}_{p^k}$ as

$$d_L(x,y) := w_L(x-y), \quad x, y \in \mathbb{Z}_{p^k}. \tag{3}$$

Note that this is a metric on $\mathbb{Z}_{p^k}$ and by extending w_L and d_L linearly to $(\mathbb{Z}_{p^k})^n$ in an obvious way, we get a weight and a metric on $(\mathbb{Z}_{p^k})^n$. Note also that the Gray map in this case has a very simple description compared to the Gray map for the homogeneous weight. However, first we need to prove that the map defined above is indeed a distance preserving Gray map:

Theorem 2.1. *$\phi_L : (\mathbb{Z}_{p^k}, d_L) \longrightarrow (\mathbb{F}_p^{p^{k-1}}, d_H)$ is a distance preserving (not necessarily linear) map, where d_L and d_H denote the Lee and the Hamming distances respectively.*

Proof. We will show that $\forall x_1, x_2 \in \mathbb{Z}_{p^k}$ $d_L(x_1,x_2) = d_H(\phi_L(x_1), \phi_L(x_2))$. Without loss of generality assume that $x_1 < x_2$, and let $x_1 = q_1p^{k-1} + r_1$, $x_2 = q_2p^{k-1} + r_2$, where $0 \le q_1, q_2 \le p-1$, $0 \le r_1, r_2 \le p^{k-1} - 1$. We will consider the problem in three cases.

Case 1: Assume that $q_2 = q_1 = q$. Since $x_1 < x_2$, $r_2 > r_1$. Then,

$$\begin{aligned} d_L(x_1, x_2) &= d_L(qp^{k-1} + r_1, qp^{k-1} + r_2) \\ &= w_L(qp^{k-1} + r_2 - (qp^{k-1} + r_1)) \\ &= w_L(r_2 - r_1) = r_2 - r_1, \end{aligned}$$

and

$$\begin{aligned} d_H(\phi_L(x_1), \phi_L(x_2)) &= d_H(\overline{q} + \phi_L(r_1), \overline{q} + \phi_L(r_2)) \\ &= w_H(\phi_L(r_1) - \phi_L(r_2)) \\ &= r_2 - r_1. \end{aligned}$$

Case 2: Assume that $q_2 \geq q_1$ and $1 \leq q_2 - q_1 \leq p - 2$. Let x_{2i}, x_{1i} denotes the i^{th} coordinates of $\phi_L(x_2)$ and $\phi_L(x_1)$ respectively. Here we have two subcases.

(i) Let $r_2 \geq r_1$, then $r_2 - r_1 \geq 0$.

$$\begin{aligned} d_L(x_1, x_2) &= w_L(x_2 - x_1) \\ &= w_L((q_2 - q_1)p^{k-1} + r_2 - r_1) \\ &= p^{k-1}. \end{aligned}$$

$$d_H(\phi_L(x_1), \phi_L(x_2)) = d_H(\overline{q_1} + \phi_L(r_1), \overline{q_2} + \phi_L(r_2)) = p^{k-1},$$

since when $1 \leq i \leq r_1$ $x_{2i} = q_2 + 1$ and $x_{1i} = q_1 + 1$, when $r_1 + 1 \leq i \leq r_2$ $x_{2i} = q_2 + 1$ and $x_{1i} = q_1$, when $r_2 < i \leq p^{k-1}$ $x_{2i} = q_2$ and $x_{1i} = q_1$. So $\phi_L(x_1)$ and $\phi_L(x_2)$ have p^{k-1} different coordinates.

(ii) Let $r_1 \geq r_2$, then $r_2 - r_1 \leq 0$. Again we have two subcases for this case. First assume that $q_2 = q_1 + 1$. Then,

$$\begin{aligned} d_L(x_1, x_2) &= w_L(x_2 - x_1) \\ &= w_L(q_2 p^{k-1} + r_2 - (q_1 p^{k-1} + r_1)) \\ &= w_L(q_2 p^{k-1} + r_2 - q_1 p^{k-1} - r_1)) \\ &= w_L((q_2 - q_1 - 1)p^{k-1} + p^{k-1} + r_2 - r_1)) \\ &= w_L(p^{k-1} + r_2 - r_1)) \\ &= p^{k-1} + r_2 - r_1, \end{aligned}$$

since $r_2 - r_1 < 0$. Hence

$$\begin{aligned} d_H(\phi_L(x_1), \phi_L(x_2)) &= d_H(\overline{q_1} + \phi_L(r_1), \overline{q_2} + \phi_L(r_2)) \\ &= p^{k-1} + r_2 - r_1, \end{aligned}$$

since when $1 \leq i \leq r_2$ $x_{2i} = q_2 + 1 = q_1 + 2$ $x_{1i} = q_1 + 1$, and when $r_2 + 1 \leq i \leq r_1$ $x_{2i} = q_2 = q_1 + 1$ $x_{1i} = q_1 + 1$, and when $r_1 + 1 \leq i \leq p^{k-1}$ $x_{2i} = q_2 = q_1 + 1$ $x_{1i} = q_1$. Namely, $\phi_L(x_1)$ and $\phi_L(x_2)$ differ in $p^{k-1} + r_2 - r_1$ coordinates. Second assume that $q_2 \geq q_1 + 2$. Then,

$$\begin{aligned} d_L(x_1, x_2) &= w_L(x_2 - x_1) \\ &= w_L(q_2 p^{k-1} + r_2 - (q_1 p^{k-1} + r_1)) \\ &= w_L(q_2 p^{k-1} + r_2 - q_1 p^{k-1} - r_1)) \\ &= w_L((q_2 - q_1 - 1)p^{k-1} + p^{k-1} + r_2 - r_1)) \\ &= p^{k-1}. \end{aligned}$$

Now,

$$d_H(\phi_L(x_1), \phi_L(x_2)) = d_H(\overline{q_1} + \phi_L(r_1), \overline{q_2} + \phi_L(r_2)) = p^{k-1},$$

since when $1 \le i \le r_2$ $x_{2i} = q_2 + 1$ $x_{1i} = q_1 + 1$, and when $r_2 + 1 \le i \le r_1$ $x_{2i} = q_2 \ne q_1 + 1$ $x_{1i} = q_1 + 1$, and when $r_1 + 1 \le i \le p^{k-1}$ $x_{2i} = q_2$ $x_{1i} = q_1$. Namely, $\phi_L(x_1)$ and $\phi_L(x_2)$ differ in p^{k-1} coordinates.

Case 3: In this case assume that $q_2 - q_1 = p - 1$, then $q_2 = p - 1$, $q_1 = 0$. So $x_1 = r_1$, $x_2 = (p-1)p^{k-1} + r_2$. Then,

$$d_L(x_1, x_2) = w_L(x_2 - x_1) = w_L((p-1)p^{k-1} + r_2 - r_1).$$

We have two subcases here.

(i) Assume that $r_2 \le r_1$, then,

$$w_L(x_2 - x_1) = w_L((p-1)p^{k-1} + r_2 - r_1) = p^{k-1}.$$

Also,

$$d_H(\phi_L(x_1), \phi_L(x_2)) = d_H(\overline{(p-1)} + \phi_L(r_2), \phi_L(r_1)) = p^{k-1},$$

since when $1 \le i \le r_2$ $x_{2i} = (p-1)+1 = 0$, $x_{1i} = 1$, and when $r_2+1 \le i \le r_1$ $x_{2i} = (p-1)$, $x_{1i} = 1$, and when $r_1 + 1 \le i \le p^{k-1}$ $x_{2i} = (p-1)$, $x_{1i} = 0$. In other words $\phi_L(x_1)$ and $\phi_L(x_2)$ differ in p^{k-1} coordinates.

(ii) Now assume that $r_2 > r_1$, then,

$$\begin{aligned} w_L(x_2 - x_1) &= w_L((p-1)p^{k-1} + r_2 - r_1) \\ &= p^{k-1} - (r_2 - r_1) \\ &= p^{k-1} + r_1 - r_2. \end{aligned}$$

Also,

$$d_H(\phi_L(x_1), \phi_L(x_2)) = d_H(\overline{(p-1)} + \phi_L(r_2), \phi_L(r_1)) = p^{k-1} + r_1 - r_2,$$

since when $1 \le i \le r_1$ $x_{2i} = (p-1) + 1 = 0$, $x_{1i} = 1$, and when $r_1 + 1 \le i \le r_2$ $x_{2i} = (p-1) + 1 = 0$, $x_{1i} = 0$, and when $r_2 + 1 \le i \le p^{k-1}$ $x_{2i} = (p-1)$, $x_{1i} = 0$. In other words $\phi_L(x_1)$ and $\phi_L(x_2)$ differ in $p^{k-1} - (r_2 - r_1) = p^{k-1} + r_1 - r_2$ coordinates.

So we have seen that in each case we get $d_L(x_1, x_2) = d_H(\phi_L(x_1), \phi_L(x_2))$. □

Corollary 2.1. *If C is a linear code over $\mathbb{Z}_{p^k}$ of length n, size M and minimum Lee distance d, then $\phi_L(C)$ is a (possibly non-linear) code over $\mathbb{F}_p$ of length np^{k-1}, size M and minimum Hamming distance d.*

3. The Extended Lee Weight over $GR(p^k, m)$

In this section we will first introduce the Galois rings and the extended Lee weight over Galois rings. The introduction given here is taken mainly from [14]. Let p be a prime number and $P(x) \in \mathbb{Z}_{p^k}[x]$ be a basic irreducible polynomial of degree m. Then the Galois ring $GR(p^k, m)$ is defined as the quotient $\mathbb{Z}_{p^k}[x]/(P(x))$. If m_1 is a positive integer such that $m_1 \mid m$, then $GR(p^k, m_1)$ is a subring of $GR(p^k, m)$. A very important property of Galois rings is that it is a finite chain ring and it also has a unique maximal ideal which is given by $(p) = pGR(p^k, m)$ and the quotient field is

$$\frac{GR(p^k, m)}{pGR(p^k, m)} \cong \mathbb{F}_{p^m}.$$

All the ideals of $GR(p^k, m)$ can be ordered as

$$\{0\} = p^k GR(p^k, m) \subset p^{k-1} GR(p^k, m) \subset \cdots \subset pGR(p^k, m) \subset GR(p^k, m).$$

Since $GR(p^k, m) = \mathbb{Z}_{p^k}[x]/(P(x))$ with $P(x)$ basic irreducible of degree m, every element $u \in GR(p^k, m)$ can be written uniquely as

$$u \equiv u_0 + u_1 x + \cdots u_{m-1} x^{m-1} \pmod{P(x)}, \quad u_i \in \mathbb{Z}_{p^k}. \tag{4}$$

A linear code C over the Galois ring $GR(p^k, m)$ of length n is a submodule of $GR(p^k, m)^n$. The following theorem from [7] gives us information about the type and dimension for linear over Galois rings:

Theorem 3.1. *(Huffman[7]) A $GR(p^k, m)$-linear code C is permutationally equivalent to a code with generating matrix of the form*

$$G = \begin{bmatrix} I_{s_1} & A_1 & \cdot & \cdot & \cdot & A_k \\ 0 & pI_{s_2} & pB_1 & \cdot & \cdot & pB_{k-1} \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 & p^{k-1} I_{s_k} & p^{k-1} C \end{bmatrix}$$

where the matrices A_i's, B_j's and so on matrices over $GR(p^k, m)$ and the columns are grouped into blocks of size $s_1, s_2, \ldots, s_k$. The size of C is $p^{m\alpha}$, where

$$\alpha = \sum_{i=1}^{k} s_i (k + 1 - i).$$

In this case, we say that C is of type

$$(p^{km})^{s_1} (p^{(k-1)m})^{s_2} \cdots (p^m)^{s_k}.$$

The homogeneous weight for linear codes over Galois rings is given as:

$$w_{\text{hom}}(x) := \begin{cases} 0 & \text{if } x = 0, \\ p^{m(k-1)} & \text{if } 0 \neq x \in p^{k-1} GR(p^k, m), \\ (p^m - 1) p^{m(k-2)} & \text{otherwise.} \end{cases}$$

The weight is naturally extended to codes by letting, for $\overline{c} = (c_1, c_2, \ldots, c_n) \in GR(p^k, m)^n$,

$$w_{\text{hom}}(\overline{c}) = \sum_{i=1}^{n} w_{\text{hom}}(c_i).$$

Algebraic constructions for the Gray map for the homogeneous weight were given in [5] and [9]. In 2009, Yildiz gave a combinatorial construction for the gray map of the homogeneous weight over Galois rings, by using Affine geometries.

We can extend the Lee weight w_L defined for $\mathbb{Z}_{p^k}$ to Galois rings, in the following way: With $u \in GR(p^k, m)$ given by

$$u \equiv u_0 + u_1 x + \cdots u_{m-1} x^{m-1} \pmod{P(x)}, \quad u_i \in \mathbb{Z}_{p^k},$$

we define

$$w_L(u) = \sum_{i=0}^{m-1} w_L(u_i).$$

where $w_L(u_i)$ is the Lee weight in $\mathbb{Z}_{p^k}$.

The Gray map of this weight can be defined naturally in a similar way:

$$\phi_L(u) = (\phi_L(u_0), \phi_L(u_1), \ldots, \phi_L(u_{m-1}))$$

where $\phi_L(u_i)$ is the Gray image of $u_i \in \mathbb{Z}_{p^k}$ as was defined previously. Note that this is a distance preserving map from $GR(p^k, m)$ to $\mathbb{F}_p^{p^{(k-1)m}}$.

Remark 3.1. *The Gray map for the homogeneous weight in Galois rings $GR(p^k, m)$ is a distance-preserving map from $GR(p^k, m)$ to $\mathbb{F}_p^{p^{(k-1)m}}$. This means that, for an $\mathbb{F}_p$-code C to be the image of a code over $GR(p^k, m)$ under this map, the length of C must be a multiple of $p^{(k-1)m}$, which is highly restrictive. On the other hand, the Gray map for the Lee weight on $GR(p^k, m)$, ϕ_L, is a map from $GR(p^k, m)$ to $\mathbb{F}_p^{p^{k-1}m}$.*

4. Conclusion

The homogeneous weight has been commonly used in extending the Lee weight on $\mathbb{Z}_4$ due to its relation to different areas of mathematics such as Finite Geometries and Number Theory. However, it has only two non-zero weights, which can be restrictive in obtaining different codes from linear codes over rings. We gave a different extension of the Lee weight together with a Gray map that was simply defined. The Gray maps in both cases take $\mathbb{Z}_{p^k}$ to $\mathbb{F}_p^{p^{k-1}}$, hence the two weights are comparable. However in the case of the Lee weight over $\mathbb{Z}_{p^k}$, we have p^{k-1} different non-zero weights, ranging from 1 to p^{k-1}. This gives us more diversity in obtaining good codes whose weights are not divisible.

Besides, for codes over $\mathbb{Z}_{p^k}$ the number of the codewords in a code that are of Lee weight p^{k-1} is $p^{k-1}(p-2)+1$ in our case, while this number is $p-1$ for the homogeneous weight.

Also while the Gray map of an element of $GR(p^k, m)$ with the homogeneous weight gives a vector of length $p^{m(k-1)}$, the length of the resulting vector is $p^{k-1}m$ for the extended Lee weight.

Possible directions for future work include finding connections between the extended Lee weight and other mathematical structures in a similar way that was done for the homogeneous weight. One can also consider the properties of the Lee weight enumerators of codes over Galois rings similar to what was done in [14] for the homogeneous weight.

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Assist. Prof. Bahattin Yıldız graduated from Department of Mathematics, Bilkent University, Ankara, Turkey in 2001. He became a PhD student at Department of Mathematics, California Institute of Technology, Pasadena (CA), USA in 2001. During his PhD he worked as an instructor and as a teaching assistant. He graduated from this university in 2006. He is a member of Department of Mathematics in Fatih University, Istanbul, Turkey since 2006. His research interests focus mainly in number theory, algebra, combinatorics, and coding theory.

Zeynep Ödemiş Özger graduated from Department of Mathematics, Boğaziçi University, Istanbul, Turkey in 2008 and received her M.S. degree in Mathematics from Fatih University, Istanbul, Turkey in 2010. Presently she is a PhD student and a research assistant at Department of Mathematics, Fatih University, Istanbul, Turkey. Her research interests focus mainly in algebra, combinatorics, and coding theory.