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BOUNDS FOR INITIAL MACLAURIN COEFFICIENTS FOR A NEW SUBCLASSES OF ANALYTIC AND M-FOLD SYMMETRIC BI-UNIVALENT FUNCTIONS

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Abstract. In the present paper, we introduce and study two new subclasses of the function class Σ_m consisting of analytic and m-fold symmetric bi-univalent functions in the open unit disk U. We establish upper bounds for the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in these subclasses. Certain special cases are also indicated.

Keywords: Analytic functions, univalent functions, bi-univalent functions, m-fold symmetric bi-univalent functions, coefficient bounds.

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1. Introduction

Let $\mathcal{A}$ stand for the class of functions f that are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$ and having the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

Let S be the subclass of $\mathcal{A}$ consisting of the form (1) which are also univalent in U. The Koebe one-quarter theorem (see [4]) states that "the image of U under every function $f \in S$ contains a disk of radius $\frac{1}{4}$. Therefore, every function $f \in S$ has an inverse f^{-1} which satisfies $f^{-1}(f(z)) = z$, $(z \in U)$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})$", where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. We denote by Σ the class of bi-univalent functions in U given by (1). In fact, Srivastava et al. [17] has apparently revived the study of analytic and bi-univalent functions in recent years, it was followed by such works as those by Frasin and Aouf [6], Goyal and Goswami [7], Srivastava and Bansal [11] and others (see, for example [3, 12, 13, 14, 16]).

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For each function $f \in S$, the function $h(z) = (f(z^m))^{\frac{1}{m}}$, $(z \in U, m \in \mathbb{N})$ is univalent and maps the unit disk U into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [8]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \ (z \in U, m \in \mathbb{N}). \tag{3}$$

We denote by S_m the class of m-fold symmetric univalent functions in U, which are normalized by the series expansion (3). In fact, the functions in the class S are one-fold symmetric.

In [18] Srivastava et al. defined m-fold symmetric bi-univalent functions analogues to the concept of m-fold symmetric univalent functions. They gave some important results, such as each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each $m \in \mathbb{N}$. Furthermore, for the normalized form of f given by (3), they obtained the series expansion for f^{-1} as follows:

$$\begin{aligned} g(w) &= w - a_{m+1}w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1}\right] w^{2m+1} \\ &- \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots , \end{aligned} \tag{4}$$

where $f^{-1} = g$. We denote by Σ_m the class of m-fold symmetric bi-univalent functions in U. It is easily seen that for $m = 1$, the formula (4) coincides with the formula (2) of the class Σ. Some examples of m-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}}, \ \left[\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)\right]^{\frac{1}{m}} \ \textit{and} \ \left[-\log\left(1-z^m\right)\right]^{\frac{1}{m}}$$

with the corresponding inverse functions

$$\left(\frac{w^m}{1+w^m}\right)^{\frac{1}{m}}, \ \left(\frac{e^{2w^m}-1}{e^{2w^m}+1}\right)^{\frac{1}{m}} \ \textit{and} \ \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{\frac{1}{m}},$$

respectively.

Recently, many authors investigated bounds for various subclasses of m-fold bi-univalent functions (see [1, 2, 5, 15, 18, 19, 20]).

The aim of the present paper is to introduce the new subclasses $\mathcal{Z}_{\Sigma_m}(\lambda;\alpha)$ and $\mathcal{Z}^*_{\Sigma_m}(\lambda;\beta)$ of Σ_m and find estimates on the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new subclasses.

In order to prove our main results, we require the following lemma.

Lemma 1.1. ([4]) *If $h \in \mathcal{P}$, then $|c_k| \le 2$ for each $k \in \mathbb{N}$, where $\mathcal{P}$ is the family of all functions h analytic in U for which*

$$Re\,(h(z)) > 0, \quad (z \in U),$$

where

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots , \quad (z \in U).$$

2. Coefficient Estimates for the Functions Class $\mathcal{Z}_{\Sigma_m}(\lambda;\alpha)$

Definition 2.1. *A function $f \in \Sigma_m$ given by (3) is said to be in the class $\mathcal{Z}_{\Sigma_m}(\lambda;\alpha)$ if it satisfies the following conditions:*

$$\left|arg\left[1 + \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} - \frac{\lambda z^2 f''(z) + zf'(z)}{\lambda z f'(z) + (1-\lambda)f(z)}\right]\right| < \frac{\alpha\pi}{2}, \quad (z \in U) \tag{5}$$

and

$$\left|arg\left[1+\frac{wg'(w)}{g(w)}+\frac{wg''(w)}{g'(w)}-\frac{\lambda w^2g''(w)+wg'(w)}{\lambda wg'(w)+(1-\lambda)g(w)}\right]\right|<\frac{\alpha\pi}{2},\quad (w\in U), \tag{6}$$

$$(0<\alpha\leq 1,\ 0\leq\lambda\leq 1,\ m\in\mathbb{N}),$$

where the function $g=f^{-1}$ is given by (4).

Theorem 2.1. *Let $f\in\mathcal{Z}_{\Sigma_m}(\lambda;\alpha)$ $(0<\alpha\leq 1,\ 0\leq\lambda\leq 1,\ m\in\mathbb{N})$ be given by (3). Then*

$$|a_{m+1}|\leq\frac{\sqrt{2\alpha}}{m\sqrt{\lambda m(\alpha\lambda-2)+m^2\left(1-\lambda\right)^2(1-\alpha)+4\alpha\lambda(m+1)+m(2-\alpha)}} \tag{7}$$

and

$$|a_{2m+1}|\leq\frac{2\alpha^2(m+1)}{m^2\left(1+m(1-\lambda)\right)^2}+\frac{\alpha}{m\left(1+2m(1-\lambda)\right)}. \tag{8}$$

Proof. It follows from conditions (5) and (6) that

$$1+\frac{zf'(z)}{f(z)}+\frac{zf''(z)}{f'(z)}-\frac{\lambda z^2f''(z)+zf'(z)}{\lambda zf'(z)+(1-\lambda)f(z)}=[p(z)]^{\alpha} \tag{9}$$

and

$$1+\frac{wg'(w)}{g(w)}+\frac{wg''(w)}{g'(w)}-\frac{\lambda w^2g''(w)+wg'(w)}{\lambda wg'(w)+(1-\lambda)g(w)}=[q(w)]^{\alpha}, \tag{10}$$

where $g=f^{-1}$ and p, q in $\mathcal{P}$ have the following series representations:

$$p(z)=1+p_mz^m+p_{2m}z^{2m}+p_{3m}z^{3m}+\cdots \tag{11}$$

and

$$q(w)=1+q_mw^m+q_{2m}w^{2m}+q_{3m}w^{3m}+\cdots. \tag{12}$$

Comparing the corresponding coefficients of (9) and (10) yields

$$m\left(1+m(1-\lambda)\right)a_{m+1}=\alpha p_m, \tag{13}$$

$$m\left[2\left(1+2m(1-\lambda)\right)a_{2m+1}-\left(1+(m+1)^2-(\lambda m+1)^2\right)a_{m+1}^2\right]$$
$$=\alpha p_{2m}+\frac{\alpha(\alpha-1)}{2}p_m^2, \tag{14}$$

$$-m\left(1+m(1-\lambda)\right)a_{m+1}=\alpha q_m \tag{15}$$

and

$$m\left[\left(m^2(3-4\lambda)+4m(1-\lambda)+(\lambda m+1)^2\right)a_{m+1}^2-2\left(1+2m(1-\lambda)\right)a_{2m+1}\right]$$
$$=\alpha q_{2m}+\frac{\alpha(\alpha-1)}{2}q_m^2. \tag{16}$$

Making use of (13) and (15), we obtain

$$p_m=-q_m \tag{17}$$

and

$$2m^2\left(1+m(1-\lambda)\right)^2a_{m+1}^2=\alpha^2(p_m^2+q_m^2). \tag{18}$$

Also, from (14), (16) and (18), we find that

$$m\left[-1-(m+1)^2+m^2(3-4\lambda)+4m(1-\lambda)+2\left(\lambda m+1\right)^2\right]a_{m+1}^2$$
$$=\alpha(p_{2m}+q_{2m})+\frac{\alpha(\alpha-1)}{2}\left(p_m^2+q_m^2\right)$$
$$=\alpha(p_{2m}+q_{2m})+\frac{2m^2(\alpha-1)\left(1+m(1-\lambda)\right)^2}{\alpha}a_{m+1}^2.$$

Therefore, we have

$$a_{m+1}^2=\frac{\alpha^2(p_{2m}+q_{2m})}{2m^2\left[\lambda m(\alpha\lambda-2)+m^2\left(1-\lambda\right)^2(1-\alpha)+4\alpha\lambda(m+1)+m(2-\alpha)\right]}. \tag{19}$$

Now, taking the absolute value of (19) and applying Lemma 1.1 for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}|\leq\frac{\sqrt{2\alpha}}{m\sqrt{\lambda m(\alpha\lambda-2)+m^2\left(1-\lambda\right)^2(1-\alpha)+4\alpha\lambda(m+1)+m(2-\alpha)}}.$$

This gives the desired estimate for $|a_{m+1}|$ as asserted in (7).
In order to find the bound on $|a_{2m+1}|$, by subtracting (16) from (14), we get

$$2m\left(1+2m(1-\lambda)\right)\left[2a_{2m+1}-(m+1)a_{m+1}^2\right]=\alpha\left(p_{2m}-q_{2m}\right)+\frac{\alpha(\alpha-1)}{2}\left(p_m^2-q_m^2\right). \tag{20}$$

It follows from (17), (18) and (20) that

$$a_{2m+1}=\frac{\alpha^2(m+1)\left(p_m^2+q_m^2\right)}{4m^2\left(1+m(1-\lambda)\right)^2}+\frac{\alpha\left(p_{2m}-q_{2m}\right)}{4m\left(1+2m(1-\lambda)\right)}. \tag{21}$$

Taking the absolute value of (21) and applying Lemma 1.1 once again for the coefficients p_m, p_{2m}, q_m and q_{2m}, we obtain

$$|a_{2m+1}|\leq\frac{2\alpha^2(m+1)}{m^2\left(1+m(1-\lambda)\right)^2}+\frac{\alpha}{m\left(1+2m(1-\lambda)\right)},$$

which completes the proof of Theorem 2.1. □

3. Coefficient Estimates for the Functions Class $\mathcal{Z}^*_{\Sigma_m}(\lambda;\beta)$

Definition 3.1. *A function $f\in\Sigma_m$ given by (3) is said to be in the class $\mathcal{Z}^*_{\Sigma_m}(\lambda;\beta)$ if it satisfies the following conditions:*

$$Re\left\{1+\frac{zf'(z)}{f(z)}+\frac{zf''(z)}{f'(z)}-\frac{\lambda z^2f''(z)+zf'(z)}{\lambda zf'(z)+(1-\lambda)f(z)}\right\}>\beta,\quad(z\in U) \tag{22}$$

and

$$Re\left\{1+\frac{wg'(w)}{g(w)}+\frac{wg''(w)}{g'(w)}-\frac{\lambda w^2g''(w)+wg'(w)}{\lambda wg'(w)+(1-\lambda)g(w)}\right\}>\beta,\quad(w\in U), \tag{23}$$

$$\left(0\leq\beta<1,\ 0\leq\lambda\leq1,\ m\in\mathbb{N}\right),$$

where the function $g=f^{-1}$ is given by (4).

Theorem 3.1. *Let* $f \in \mathcal{Z}^{*}_{\Sigma_m}(\lambda;\beta)$ $(0 \le \beta < 1,\ 0 \le \lambda \le 1,\ m \in \mathbb{N})$ *be given by (3). Then*

$$|a_{m+1}| \le \sqrt{\frac{2(1-\beta)}{m\left[(\lambda m+1)^2 + m(m+1)(1-2\lambda) - 1\right]}} \tag{24}$$

and

$$|a_{2m+1}| \le \frac{2(m+1)\,(1-\beta)^2}{m^2\,(1+m(1-\lambda))^2} + \frac{1-\beta}{m\,(1+2m(1-\lambda))}. \tag{25}$$

Proof. It follows from conditions (22) and (23) that there exist $p, q \in \mathcal{P}$ such that

$$1 + \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} - \frac{\lambda z^2 f''(z) + zf'(z)}{\lambda z f'(z) + (1-\lambda) f(z)} = \beta + (1-\beta)p(z) \tag{26}$$

and

$$1 + \frac{wg'(w)}{g(w)} + \frac{wg''(w)}{g'(w)} - \frac{\lambda w^2 g''(w) + wg'(w)}{\lambda w g'(w) + (1-\lambda) g(w)} = \beta + (1-\beta)q(w), \tag{27}$$

where $p(z)$ and $q(w)$ have the forms (11) and (12), respectively. Equating coefficients (26) and (27) yields

$$m\,(1+m(1-\lambda))\,a_{m+1} = (1-\beta)p_m, \tag{28}$$

$$m\left[2\,(1+2m(1-\lambda))\,a_{2m+1} - \left(1+(m+1)^2-(\lambda m+1)^2\right)a_{m+1}^2\right] = (1-\beta)p_{2m}, \tag{29}$$

$$-m\,(1+m(1-\lambda))\,a_{m+1} = (1-\beta)q_m \tag{30}$$

and

$$m\left[\left(m^2(3-4\lambda)+4m(1-\lambda)+(\lambda m+1)^2\right)a_{m+1}^2 - 2\,(1+2m(1-\lambda))\,a_{2m+1}\right] = (1-\beta)q_{2m}. \tag{31}$$

From (28) and (30), we get

$$p_m = -q_m \tag{32}$$

and

$$2m^2\,(1+m(1-\lambda))^2\,a_{m+1}^2 = (1-\beta)^2\,(p_m^2+q_m^2). \tag{33}$$

Adding (29) and (31), we obtain

$$2m\left[(\lambda m+1)^2 + m(m+1)(1-2\lambda) - 1\right]a_{m+1}^2 = (1-\beta)(p_{2m}+q_{2m}). \tag{34}$$

Therefore, we have

$$a_{m+1}^2 = \frac{(1-\beta)(p_{2m}+q_{2m})}{2m\left[(\lambda m+1)^2 + m(m+1)(1-2\lambda) - 1\right]}.$$

Applying Lemma 1.1 for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}| \le \sqrt{\frac{2(1-\beta)}{m\left[(\lambda m+1)^2 + m(m+1)(1-2\lambda) - 1\right]}}.$$

This gives the desired estimate for $|a_{m+1}|$ as asserted in (24).
In order to find the bound on $|a_{2m+1}|$, by subtracting (31) from (29), we get

$$2m\,(1+2m(1-\lambda))\left[2a_{2m+1} - (m+1)a_{m+1}^2\right] = (1-\beta)\,(p_{2m}-q_{2m}),$$

or equivalently

$$a_{2m+1} = \frac{m+1}{2}a_{m+1}^2 + \frac{(1-\beta)\,(p_{2m}-q_{2m})}{4m\,(1+2m(1-\lambda))}.$$

Upon substituting the value of a_{m+1}^2 from (33), it follows that

$$a_{2m+1} = \frac{(m+1)(1-\beta)^2 (p_m^2 + q_m^2)}{4m^2 (1+m(1-\lambda))^2} + \frac{(1-\beta)(p_{2m} - q_{2m})}{4m(1+2m(1-\lambda))}.$$

Applying Lemma 1.1 once again for the coefficients p_m, p_{2m}, q_m and q_{2m}, we obtain

$$|a_{2m+1}| \leq \frac{2(m+1)(1-\beta)^2}{m^2 (1+m(1-\lambda))^2} + \frac{1-\beta}{m(1+2m(1-\lambda))}.$$

which completes the proof of Theorem 3.1. □

Remark 3.1. *For one-fold symmetric bi-univalent functions, if we set $\lambda = 1$ in Theorems 2.1 and 3.1, we obtain the results given by Liu and Wang* [9]. *In addition, for one-fold symmetric bi-univalent functions, if we set $\lambda = 0$ in Theorems 2.1 and 3.1, we obtain the results given by Murugusundaramoorthy et al.* [10]. *Furthermore, for m-fold symmetric bi-univalent functions, if we set $\lambda = 1$ in our Theorems, we have the results which were proven earlier by Altinkaya and Yalçin* [1].

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