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NUMERICAL TREATMENT OF NEWELL-WHITEHEAD-SEGEL EQUATION

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Abstract. In this paper, a comparative study of recently developed numerical methods in solving nonlinear Newell-–Whitehead—Segel equations is carried out. Results computed using Variational iteration method (VIM) are compared with Adomian decomposition method (ADM), iterative method and the exact results. The numerical results obtained by VIM are discussed with the help of different figures and tables. The plotted graph and numerical results show accuracy and efficiency of this method in solving nonlinear equations.

Keywords:Variational iteration method, Newell-–Whitehead-–Segel equation, He's polynomials, Lagrange's multiplier

AMS Subject Classification: 44A99, 35Q99

1. Introduction

Nonlinear phenomena are always visible in the study of applied Mathematics, Physics, Chemistry and many related fields of engineering. In daily life, we come across many real life models of Mathematics for numerical solution of nonlinear differential equations. The significance of obtaining their exact solution, if available, facilitates the authentication of numerical solvers as well as supports in stability analysis of the solution. Analytical solutions to nonlinear PDEs play a vital role in physical science as they may offer more physical information as well as better insight into its physical aspects [1]. Various methods are used for the solution of nonlinear PDEs by researchers in the last few decades. For details of notable methods, we direct readers to the papers [2-20].

The Newell—Whitehead—Segel (NWS) equations have broad pertinence in bio engineering, biology, ecology, chemical and mechanical engineering. They are one of the most significant amplitude equations that discover the visual aspect of the streak pattern in 2–dimensional systems and applied in Faraday instability, Rayleigh–Benard convection, nonlinear optics, Taylor-–Couette flow, biological systems etc. Also, the interaction of

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effect of diffusion term with effect of the nonlinear reaction term is modeled as,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + au - bu^q, \tag{1}$$

where the constants a, b, k ∈ R, with k > 0 and q being a positive integer. u(x, t) is a function of temporal variable t and spatial variable x, where t ≥ 0, x ∈ R. Here u may be considered as the nonlinear distribution in a thin infinitely long rod and also be seen as the fluid flow velocity in an infinitely long pipe having small diameter. The derivative on the LHS of Eq.(1), $\frac{\partial u}{\partial t}$ represents the partial rate of change of u with respect to t at a set position. The derivative on RHS of Eq.(1), $\frac{\partial^2 u}{\partial x^2}$ represents partial rate of change of u with respect to x at a fixed time. The effect of source term is shown by the term $au - bu^q$ on the RHS of Eq.(1).

So far, LADM [21], Differential transform method [22], ADM [23], Multi-–quadric quasi-–interpolation methods [24], HPM [25] etc. have been used for the solution of Eq.(1). Reliability of solution schemes is also a very important aspect than modeling dimensions of equations. VIM has a thoroughness in mathematical derivation of Lagrange's multiplier by variational theory for fractional calculus. It leads to solution converging to exact one. Recently, fractional complex transform is developed to build a simpler variational iteration algorithm for fractional calculus.

In the present paper, we propose to study Eq.(1) using VIM. This method directly attacks the nonlinear partial differential equation without a need to find certain polynomials for nonlinear terms and gives result in an infinite series. It rapidly converges to analytical solution. This method also requires no linearization, discretization, little perturbations or restrictive assumptions. It lessens mathematical computations significantly. A comparative study is made of the results obtained with ADM and a new iterative method (NIM). The numerical results acquired show that VIM is efficient, easier and more convenient for solving NWS equation.

This paper is structured in the following manner. Section 1, is introduction. In section 2, the working algorithm of VIM is proposed by taking the problem under consideration. Section 3, presents some numerical test examples on which VIM is applied to find the approximate analytic solution. Section 4, deals with the discussion of obtained numerical results and their comparison with other methods. In last section 5, the conclusions are drawn after summarizing the results.

2. Implementation of VIM

Consider the nonlinear PDE

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + au - bu^q, \tag{2}$$

A correction functional is formed as follows,

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left[\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2} - au_n(x,\xi) + b\tilde{u}_n^3(x,\xi)\right]d\xi, \tag{3}$$

where λ is Lagrange's multiplier. Optimally, λ may be found using theory of variations. $\tilde{u}_n$ is restricted variation such that $\delta\tilde{u}_n = 0$.

Taking variations on both sides of Eq.(2), we acquire

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda(\xi) \left[\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2} - a u_n(x,\xi) + b \tilde{u}_n^3(x,\xi) \right] d\xi. \tag{4}$$

This gives stationary conditions

$$\begin{aligned} -\lambda^{'}(\xi) - a\lambda(\xi)|_{\xi=t} &= 0, \\ 1 + \lambda(\xi)|_{\xi=t} &= 0. \end{aligned} \tag{5}$$

We quickly get,

$$\lambda = -e^{-a(t-\xi)}. \tag{6}$$

Consecutive approximations u_{n+1}, n $\geq$ 0, are promptly found out using a discriminating function u_0. u_0 is the zeroth approximation.
Finally, the result is achieved as $u(x,t) = \lim_{n\to\infty} u_n(x,t)$, which will be the solution of Eq.(2).

3. Test Examples

In this section, we show the applicability and efficiency of VIM to examine the NWS equation by taking few test examples.
Example 1. Taking a, b $>$ 0 and q = 3, Eq.(2) becomes,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + au - bu^3, \tag{7}$$

with initial condition

$$u(x,0) = \sqrt{\frac{a}{b}} \left(\frac{e^{\frac{\sqrt{2a}x}{2}}}{e^{\frac{\sqrt{2a}x}{2}} + 1} \right), \tag{8}$$

Correction functional for Eq. (7) is given as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi) \left[\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial x^2} - a u_n(x,\xi) + b \tilde{u}_n^3(x,\xi) \right] d\xi, \tag{9}$$

where $\tilde{u_n}$ is restricted variation such that $\delta\tilde{u_n} = 0$.
Lagrange's multiplier is obtained as

$$\lambda = -e^{-a(t-\xi)}. \tag{10}$$

By applying VIM, we will form a sequence. Putting λ in Eq. (9), we get

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t e^{-a(t-\xi)} \left[\frac{\partial u_n(x,\xi)}{\partial \xi} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2} - a u_n(x,\xi) + b u_n^3(x,\xi) \right] d\xi. \tag{11}$$

Taking u_0 = u(x,0), we get

$$u_1(x,t) = \sqrt{\tfrac{a}{b}}\left[\frac{e^{\frac{\sqrt{2a}x}{2}}}{e^{\frac{\sqrt{2a}x}{2}}+1} + \frac{3(1-e^{-at})}{4\left(1+\cosh(\frac{\sqrt{2a}x}{2})\right)}\right],$$

$$u_2(x,t) = \sqrt{\frac{a}{b}}\left[\frac{e^{\frac{\sqrt{2a}x}{2}}}{e^{\frac{\sqrt{2a}x}{2}}+1} - \frac{3e^{-3at+\frac{\sqrt{2a}x}{2}}}{16(e^{\frac{\sqrt{2a}x}{2}}+1)^2}\left(-20e^{3at} + 9e^{\sqrt{2a}x} - 56e^{3at+\frac{\sqrt{2a}x}{2}} - 36e^{at+3\frac{\sqrt{2a}x}{2}} + 28e^{3at+3\frac{\sqrt{2}}{}}\right.\right.$$
$$-90e^{at+\sqrt{2a}x} + 6e^{3at+\sqrt{2a}x} + 4e^{3at+2\sqrt{2a}x} + e^{2at+\sqrt{2a}x}(75-78at) + e^{2at+3\frac{\sqrt{2a}x}{2}}(8-64at) + 20e^{2}$$
$$\left.\left.(1+at) + 56e^{2at+\frac{\sqrt{2a}x}{2}}(1+at) - 4e^{2at+2\sqrt{2a}x}(1+at) + \frac{3(1-e^{-at})}{4\left(1+\cosh(\frac{\sqrt{2a}x}{2})\right)}\right)\right],$$

and so on. The next approximations can be found easily using Maple package.

So as n tends to ∞, u_n(x, t)will tends to $u(x,t) = \sqrt{\frac{a}{b}}\ \left(\frac{e^{\frac{\sqrt{2a}x}{2}}}{e^{\frac{\sqrt{2a}x}{2}}+e^{-\frac{3}{2}at}}\right)$,

which is the exact solution of Eqs. (7)—(8).

Example 2. Taking a< 0, b > 0 and q = 3, Eq. (2) becomes,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + au - bu^3, \tag{12}$$

with

$$u(x,0) = \sqrt{\frac{|a|}{b}}\ \ tan\left(\frac{\sqrt{2|a|}x}{4} + \frac{1}{2}\right), \tag{13}$$

We obtain,

$$\lambda = -e^{-a(t-\xi)}. \tag{14}$$

Taking the initial approximation u_0 = u(x,0), and using λ, we get

$$u_1(x,t) = \frac{1}{4}\sqrt{\frac{|a|}{b}}\ \ \left[4tan\left(\frac{\sqrt{2|a|}x}{4}+\frac{1}{2}\right) + \frac{1}{a}\left\{(1-e^{-at})(4a+|a|\{4-3sec\left(\frac{\sqrt{2|a|}x}{4}+\frac{1}{2}\right)^2\})\right\}\right],$$

Similarly, the next approximations u_2, u_3,... can be found easily using Maple package.

As n tends to ∞, u_n(x, t)will tends to $u(x,t) = \sqrt{\frac{|a|}{b}}\ \ \dfrac{sin\left(\frac{\sqrt{2|a|}x}{2}+1\right)}{cos\left(\frac{\sqrt{2|a|}x}{2}+1\right)+e^{\frac{-3}{2}at}}$,

which is the exact solution of Eqs. (12)—(13).

Example 3. Taking a = 0, b > 0 and q = 3, Eq. (2) becomes,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - bu^3, \tag{15}$$

with

$$u(x,0) = u_0(x,t) = \sqrt{\frac{2}{b}}\ \ (\frac{2x}{x^2+1}), \tag{16}$$

We obtain, $\lambda = -1$.

Using λ and taking u (x,0) $= u_0$, we get the successive approximations as,

$$u_1(x,t) = \sqrt{\frac{2}{b}}\ (\frac{2x}{x^2+1}) - \sqrt{\frac{2}{b}}\ (\frac{12tx}{(x^2+1)^2}),$$

$$u_2(x,t) = \sqrt{\frac{2}{b}}\ (\frac{2x}{x^2+1}) - \sqrt{\frac{2}{b}}\ (\frac{12tx}{(x^2+1)^2}) + \sqrt{\frac{2}{b}}\frac{72t^2x}{(x^2+1)^6}(12t^2x^2 - 8tx^2(x^2+1) + (x^2+1)^3),$$

& so on. The next approximations can be found easily using Maple package.

As n tends to ∞, u_n(x, t) will tends to $u(x,t) = \sqrt{\frac{2}{b}}\ \left(\frac{2x}{x^2+6t+1}\right)$,

which is the exact solution of Eqs. (15)-–(16).

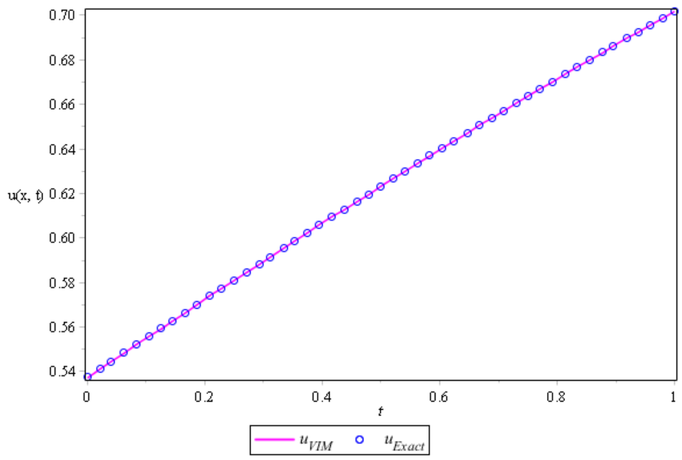

FIGURE 1. Comparison of exact and approximate solutions for Ex. 1

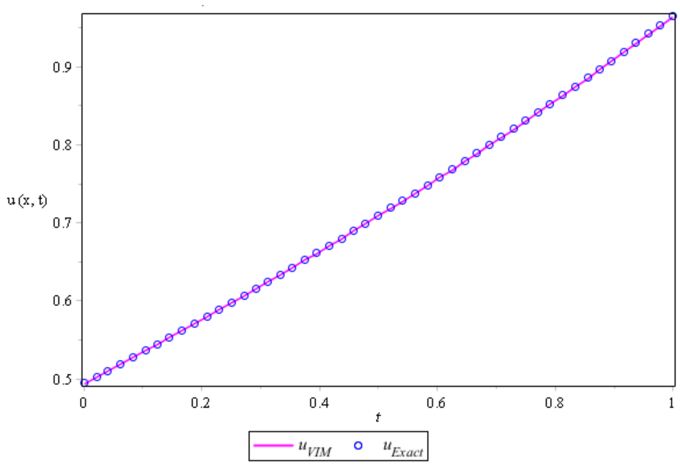

FIGURE 2. Comparison of exact and approximate solutions for Ex. 2

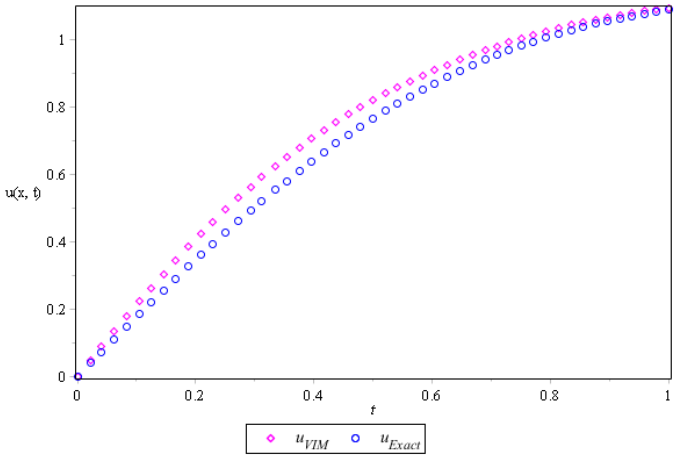

FIGURE 3. Comparison of exact and approximate solutions for Ex. 3

TABLE 1. Comparison of percentage relative errors among VIM, ADM and NIM for Ex.1

x	VIM solution	ADM solution[26]	NIM solution[26]
	$\left\|\frac{u-u_2}{u(x,t)}\right\| \times 100$	$\left\|\frac{u-u_2}{u(x,t)}\right\| \times 100$	$\left\|\frac{u-u_2}{u(x,t)}\right\| \times 100$
0	0.6493899524	0.8901699823	4.356623184
2	0.0076272041	0.0198264374	15.07592524
4	0.0008455034	0.0067615547	19.20443264
6	0.0001057104	0.0010194110	19.88949384
8	0.0000141370	0.0001399741	19.98499124
10	0.0000019155	0.0000189890	19.99796781

TABLE 2. Percentage relative error between exact and approximate solutions for Ex. 2

x	Approximate solution by VIM	Exact solution	Percentage Relative Error
	u_2	u(x, t)	$\left\|\frac{u-u_2}{u(x,t)}\right\| \times 100$
0	0.5931335317	0.5931335457	2.360345e—6
2	0.0426715120	0.0426715239	2.788756e—5
4	0.0021726115	0.0021726255	6.443827e—4
6	0.0001082868	0.0001082908	3.693765e—3
8	0.0000053890	0.0000053917	5.007697e—2
10	0.0000000265	0.0000000268	3.731343e—1

4. NUMERICAL RESULTS AND DISCUSSION

The results by VIM are found in accordance with the exact result which shows the effectiveness of this method. Table 1 shows comparison of percentage relative errors among VIM, ADM and NIM for Ex. 1. It shows that error obtained by VIM is less than recently

TABLE 3. Percentage relative error between exact and approximate solutions for Ex. 3

x	Approximate solution by VIM	Exact solution	Percentage Relative Error
	u_2	u(x, t)	$\left\lvert\frac{u-u_2}{u(x,t)}\right\rvert \times 100$
2	1.009875306	1.010152545	2.744525e-2
4	0.642816215	0.642824347	1.265011e-3
6	0.451344152	0.451344754	1.302552e-4
8	0.34493005	0.344930137	2.530947e-5
10	0.278388478	0.278388497	6.824995e-6

developed NIM and ADM. Table 2 and 3 depict the negligible percentage relative errors between exact and approximate solutions by VIM which itself is an indication of the rapid convergence of this method for Ex. 2 and Ex. 3 respectively. Figures are drawn using Maple package. Figs. 1-–3 show the variation of u (x,t) with t at x = 0.1 where 0 < t <1 at an interval of 0.2 for the cases (i) a > 0, b > 0 (ii) a < 0, b > 0 (iii) a = 0, b > 0 respectively. From fig. 1, it is clear that u increases with increase in t. In fig. 2, u slowly increases with increase in t initially but after some time, it increases rapidly. In fig. 3, u increases up to 1 as t increases from 0 to 1.

5. Conclusion

In this work, He's VIM is successfully implemented to get numerical solutions for some general cases of Newell—Whitehead—Segel equation. In each case, results are found to be compatible with the exact solution in very few iterations. Results are also compared with NIM as well as ADM. The result shows that VIM is more accurate, powerful, efficient and promising method in finding the approximate solutions for nonlinear differential equations. Hence, VIM is convenient and easier than other methods.

References

[1] Soliman, A. A. and Abdou, M. A., (2007), Numerical solutions of nonlinear evolution equations using variational iteration method, J. Comput. Appl. Math., 207, pp. 111-120.

[2] Ablowitz, M. J. and Clarkson, P. A., (1991), Solitons, Nonlinear evolution equations and inverse scattering, Cambridge University Press, Cambridge.

[3] Wazwaz, A. M., (2000), A new algorithm for calculating Adomian polynomials for nonlinear operators, Appl. Math. Comput., 111, pp. 53-69.

[4] Fadaei, J., (2011), Application of Laplace-Adomian decomposition method on linear and nonlinear system of PDEs, Appl. Math. Sci., 5, pp. 1307-1315.

[5] He, J. H., (2000), A coupling method of homotopy technique and perturbation technique for nonlinear problems, Int. J. Nonlin. Mech., 35, pp. 37–43.

[6] Saravanan, A. and Magesh, N., (2013), A comparison between the reduced differential transform method and the Adomian decomposition method for the Newell—Whitehead—Segel equation, J. Egyptian Math. Soc., 21, pp. 259-265.

[7] Bataineh, A. S., Noorani, M. S. M. and Hashim, I., (2008), Approximate analytical solutions of systems of PDEs by homotopy analysis method, Comput. Math. Appl., 55, pp. 2913-2923.

[8] Parand, K. and Rad, J. A., (2012), Exp-function method for some nonlinear PDE's and a nonlinear ODE's, J. King Saud Univ. Sci., 24, pp. 1-10.

[9] He, J. H., (1999), Variational iteration method-a kind of non–linear analytical technique: Some examples, Int. J. Nonlin. Mech., 34, pp. 699-708.

[10] Prakash, A. and Kumar, M., (2016), He's Variational Iteration Method for solution of non-linear Newell-–Whitehead-–Segel Equation, J. Appl. Anal. Comput., 6, pp. 738-748.

[11] He, J. H. and Wu, X. H., (2007), Variational iteration method: new development and applications, Comput. Math. Appl., 54, pp. 881-894.

[12] Sulaiman, T. A., Yokus, A. and Gulluoglu, N., (2018), H.M.Baskonus, Regarding the Numerical and Stability Analysis of the Sharma-Tosso-Olver Equation, 3rd International Conference on Computational Mathematics and Engineering Sciences (CMES-2018), Girne/Cyprus

[13] Düşünceli, F. and Çelik, E., (2015), An Effective Tool: Numerical Solutions by Legendre Polynomials for High-Order Linear Complex Differential Equations, Br. J. Appl. Sci. Tech., 8(4), pp. 348-355.

[14] Yokus, A., Baskonus, H. M., Sulaiman, T. A. and Bulut, H., (2018), Numerical simulation and solutions of the two-component second order KdV evolutionary system, Numer Methods Partial Differ Equ., 34(1), pp. 211-227.

[15] Belgacem, F. B. M., Baskonus, H. M. and Bulut, H., (2015), Variational Iteration Method for Hyperchaotic Nonlinear Fractional Differential Equations Systems, Advances in Mathematics and Statistical Sciences, pp. 445-453.

[16] Düşünceli, F. and Çelik, E., (2017), Numerical Solution for High-Order Linear Complex Differential Equations By Hermite Polynomials, Iğdır University Journal of the Institute of Science and Technology,7(4), pp. 189-201.

[17] Baskonus, H. M., Belgacem, F. B. M. and Bulut, H., (2015), Solutions of Nonlinear Fractional Differential Equations Systems through an Implementation of the Variational Iteration Method, Fractional Dynamics, De Gruyter, ISBN: 333-342, 2015, ISBN(Online): 9783110472097; DOI(Chapter): 10.1515/9783110472097-020; DOI(Book): 10.1515/9783110472097.

[18] Kumar, D., Singh, J., Baleanu, D. and Rathore, S., (2018), Analysis of a fractional model of Ambartsumian equation, Eur. Phys. J. Plus, 133, pp. 259.

[19] Singh, J., Kumar, D., Baleanu, D. and Rathore, S., (2018), An efficient numerical algorithm for the fractional Drinfeld-Sokolov-Wilson equation, Appl. Math. Comput., 335, pp. 12-24.

[20] Kumar, D., Agarwal, R. P. and Singh, J., (2018), A modified numerical scheme and convergence analysis for fractional model of Lienard's equation, J. Comput. Appl. Math. 339, pp. 405-413.

[21] Pue-on, P., (2013), Laplace Adomian Decomposition Method for Solving Newell–Whitehead–Segel Equation, Appl. Math. Sci., 7, pp. 6593-6600.

[22] Aasaraai, A., (2011), Analytic solution for Newell-–Whitehead—Segel equation by differential transform method, Middle–East J. Sci. Res., 10, pp. 270-273.

[23] Manaa, S. A., (2011), An Approximate solution to the Newell-Whitehead-Segel equation by the Adomian decomposition method, Raf. J. Comp. Math. 8, pp. 171-180.

[24] Ezzati, C. R. and Shakibi, K., (2011), Using adomian's decomposition and multi–quadric quasi-interpolation methods for solving Newell-–Whitehead—Segel equation, Procedia Comput. Sci., 3, pp. 1043-1048.

[25] Mahgoub, M. A., (2016), Homotopy Perturbation Method for Solving Newell—Whitehead-–Segel Equation, Adv. Theor. Appl. Math., 11, pp. 399-406.

[26] Patade, J. and Bhalekar, S., (2015), Approximate analytical solution of Newell–Whitehead–Segel Equation using a new iterative method, World J. Model Simul., 11, pp. 94-103.

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