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LOCALLY AND WEAKLY CONTRACTIVE PRINCIPLE IN BIPOLAR METRIC SPACES

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ABSTRACT. In this article, we introduce concepts of (ϵ, λ)-uniformly locally contractive and weakly contractive mappings, which are generalizations of Banach contraction mapping, in bipolar metric spaces. Also, we express the results showing the existence and uniqueness of fixed point for these mappings.

bipolar metric space, ϵ-chainable, (ϵ, λ)-uniformly locally contractive, weakly contractive, fixed point.

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1. Introduction

In 2016, Mutlu and Gürdal [14] introduced the concept of bipolar metric space as a type of partial distance. Moreover, they stated the link between metric spaces and bipolar metric spaces, especially in the context of completeness, and gave some extensions of known fixed point theorems. After then, Mutlu, Özkan and Gürdal [15] extended coupled fixed point theorems to this new kind of metric space.

To generalize the Banach contraction principle, some researchers examinated several notions of locally contractive maps and weakly contractive maps, such that Banach theorem would still be satisfied. Some of the first major studies in this subject were by Edelstein [5], Rakoch [8–10] and many other authors [2, 3, 12]. Recently, some authors have been studied on this subject [1, 4, 6, 7, 11, 13, 16–20].

In this article, we introduce the notionss of (ϵ, λ)-uniformly locally contractive, which introduced by Edelstein and weakly contractive mappings, which introduced by Rakotch,

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in bipolar metric spaces. Moreover, we express the theorems which show the existence and uniqueness of fixed points for these mappings.

2. Bipolar Metric Spaces

In this paper, $\mathbb{R}^+$ is the set of all non-negative real numbers and $\mathbb{N}$ is the set of positive integers.

Definition 2.1. *[14] Let $X, Y \neq \emptyset$ and $d : X \times Y \to \mathbb{R}^+$ be a function. d is called a bipolar metric on (X, Y) if the following properties are satisfied*

(B0) $x = y$ *if* $d(x, y) = 0$,

(B1) $d(x, y) = 0$ *if* $x = y$,

(B2) $d(x, y) = d(y, x)$ *if* $x, y \in X \cap Y$,

(B3) $d(x, y) \leq d(x, y') + d(x', y') + d(x', y)$,

for all $(x, y), (x', y') \in X \times Y$. *Then the triple* (X, Y, d) *is called a bipolar metric space.*

Definition 2.2. *[14] Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be bipolar metric spaces. A function $f : X_1 \cup Y_1 \to X_2 \cup Y_2$ is called a covariant map if $f(X_1) \subseteq X_2$ and $f(Y_1) \subseteq Y_2$. Similarly, A function $f : X_1 \cup Y_1 \to X_2 \cup Y_2$ is called a contravariant map if $f(X_1) \subseteq Y_2$ and $f(X_2) \subseteq Y_1$. These maps are denoted as $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ and $f : (X_1, Y_1, d_1) \nrightarrow\!\!\!\!\!\leftarrow (X_2, Y_2, d_2)$, respectively.*

Definition 2.3. *[14] In a bipolar metric space (X, Y, d);*

(1) (a) *The points of the set X are called left points,*

(b) *The points of the set Y are called right points,*

(c) *The points of the set $X \cap Y$ are called central points,*

(2) (a) *A sequence of left points is called a left sequence,*

(b) *A sequence of right points is called a right sequence,*

(c) *The term "sequence" is used as a common for left sequences and right sequences,*

(3) (a) *If $\lim\limits_{n\to\infty} d(a_n, y) = 0$ for a left sequence (a_n) and a right point y, then (a_n) is called convergent to y,*

(b) *If $\lim\limits_{n\to\infty} d(x, b_n) = 0$ for a right sequence (b_n) and a left point x, then (b_n) is called convergent to x,*

(4) *A sequence (x_n, y_n) on the set $X \times Y$ is called a bisequence on (X, Y, d),*

(5) *A bisequence is called convergent, if both the left sequence (x_n) and the right sequence (y_n) converge,*

(6) *If (x_n) and (y_n) converge to a common point, then (x_n, y_n) is called biconvergent,*

(7) *A Cauchy bisequence is a bisequence (x_n, y_n) such that $\lim\limits_{n,m\to\infty} d(x_n, y_m) = 0$,*

(8) *A bipolar metric space in which every Cauchy bisequence converges, is called a complete bipolar metric space.*

It is shown in [14] that convergence of Cauchy bisequences implies biconvergence.

Definition 2.4. *[14]* (1) *A covariant map $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is called left-continuous at $x_0 \in X_1$ if and only if there exists a $\delta = \delta(x_0, \varepsilon) > 0$ such that*

$$d_1(x_0, y) < \delta \Rightarrow d_2(f(x_0), f(y)) < \varepsilon$$

for every $\varepsilon > 0$ and all $y \in Y_1$.

(2) *A covariant map $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ is right-continuous at $y_0 \in Y_1$ if and only if there exists a $\delta = \delta(y_0, \varepsilon) > 0$ such that*

$$d_1(x, y_0) < \delta \Rightarrow d_2(f(x), f(y_0)) < \varepsilon$$

for every $\varepsilon > 0$ *and all* $x \in X_1$.

(3) *if a covariant map* f *is left-continuous and right-continuous at each* $x \in X_1$ *and* $y \in Y_1$, *then it is called continuous.*

This definition implies that a contravariant or a covariant map f, which is defined from (X_1, Y_1, d_1) to (X_2, Y_2, d_2), is continuous, if and only if $(a_n) \to v$ on (X_1, Y_1, d_1) implies $(f(a_n)) \to f(v)$ on (X_2, Y_2, d_2).

3. Main Results

Definition 3.1. *Let* (X, Y, d) *be a bipolar metric space,* $\lambda \in (0,1)$ *and* $\epsilon > 0$. *A covariant map* $T : (X, Y, d) \rightrightarrows (X, Y, d)$ *is said to be* (ϵ, λ)*-uniformly locally contractive if*

$$d(x, y) < \epsilon \;\Rightarrow\; d(Tx, Ty) \leq \lambda d(x, y)$$

for all $(x, y) \in X \times Y$ *and a contravariant map* $T : (X, Y, d) \nearrow\!\!\!\!\swarrow (X, Y, d)$ *is said to be* (ϵ, λ)*-uniformly locally contractive if*

$$d(x, y) < \epsilon \;\Rightarrow\; d(Ty, Tx) \leq \lambda d(x, y)$$

for all $(x, y) \in X \times Y$.

Lemma 3.1. *Every* (ϵ, λ)*-uniformly locally contractive covariant (or contravariant) map on a bipolar metric space* (X, Y, d) *is continuous.*

Proof. Firstly, we consider the case where T is a (ϵ, λ)-uniformly locally contractive covariant map. Let $(u_n) \to v$. We assume that (u_n) is a sequence on X, and thus $v \in Y$. Let $\epsilon_0 > 0$. Define $0 < \epsilon_1 < \min\{\epsilon, \epsilon_0\}$. Since $(u_n) \to v$, we can take an $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $d(u_n, v) < \epsilon_1$ for all $n \in \mathbb{N}$. Thus, we get

$$d(u_n, v) < \epsilon_1 < \epsilon$$

which implies

$$d(Tu_n, Tv) \leq \lambda d(u_n, v) < \lambda \epsilon_1 < \lambda \epsilon_0 < \epsilon_0.$$

So, $Tu_n \to Tv$.

Now, we consider the case where T is a (ϵ, λ)-uniformly locally contractive contravariant map. Let $(u_n) \to v$. We assume that (u_n) is a sequence on X and thus $v \in Y$. Let $\epsilon_0 > 0$. Define $0 < \epsilon_1 < \min\{\epsilon, \epsilon_0\}$. Since, $(u_n) \to v$, we can take an $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies $d(u_n, v) < \epsilon_1$ for all $n \in \mathbb{N}$. Thus, we get

$$d(u_n, v) < \epsilon_1 < \epsilon$$

which implies

$$d(Tv, Tu_n) \leq \lambda d(u_n, v) < \lambda \epsilon_1 < \lambda \epsilon_0 < \epsilon_0.$$

Thus $Tu_n \to Tv$. □

Definition 3.2. *A bipolar metric space* (X, Y, d) *is said to be* ϵ*-chainable if there is a finite set of points*

$$a = x_0, y_0, x_1, y_1, \cdots, x_m, y_m = b$$

for every given points $a \in X$ *and* $b \in Y$, *such that* $d(x_i, y_i) < \epsilon$ *for* $0 \leq i \leq m$ *and* $d(x_i, y_{i-1}) < \epsilon$ *for* $1 \leq i \leq m$.

Theorem 3.1. *Let* (X, Y, d) *be an* ϵ*-chainable complete bipolar metric space and* $T : (X, Y, d) \rightrightarrows (X, Y, d)$ *be* (ϵ, λ)*-uniformly locally contractive covariant map. Then there exists a unique point* $u \in X \cap Y$ *such that* $Tu = u$.

Proof. We take two point $x \in X$ and $y \in Y$. In that case, there exists an ϵ-chain

$$x = x_0, y_0, x_1, y_1, \cdots, x_m, y_m = Ty.$$

By Definition 3.1, we have

$$\begin{array}{rcl} d(T^n x_i, T^n y_i) & \leq & \lambda d(T^{n-1}x_i, T^{n-1}y_i) \\ & \leq & \lambda^2 d(T^{n-2}x_i, T^{n-2}y_i) \\ & & \vdots \\ & \leq & \lambda^n d(x_i, y_i) \end{array}$$

for all integers $n \geq 1$ and $0 \leq i \leq m$. Similarly,

$$\begin{array}{rcl} d(T^n x_i, T^n y_{i-1}) & \leq & \lambda d(T^{n-1}x_i, T^{n-1}y_{i-1}) \\ & \leq & \lambda^2 d(T^{n-2}x_i, T^{n-2}y_{i-1}) \\ & & \vdots \\ & \leq & \lambda^n d(x_i, y_{i-1}) \end{array}$$

for all integers $n \geq 1$ and $1 \leq i \leq m$. Therefore,

$$\begin{array}{rcl} d(T^n x, T^{n+1} y) & = & d(T^n x_0, T^n y_m) \\ & \leq & d(T^n x_0, T^n y_0) + d(T^n x_1, T^n y_0) + d(T^n x_1, T^n y_m) \\ & & \vdots \\ & \leq & d(T^n x_0, T^n y_0) + d(T^n x_1, T^n y_0) + d(T^n x_1, T^n y_1) \\ & & +d(T^n x_2, T^n y_1) + \cdots + d(T^n x_m, T^n y_{m-1}) \\ & & +d(T^n x_m, T^n y_m) \\ & = & \displaystyle\sum_{i=0}^{m} d(T^n x_i, T^n y_i) + \sum_{i=1}^{m} d(T^n x_i, T^n y_{i-1}) \\ & \leq & \displaystyle\sum_{i=0}^{m} \lambda^n d(x_i, y_i) + \sum_{i=1}^{m} \lambda^n d(x_i, y_{i-1}) \\ & = & (2m+1)\lambda^n \epsilon. \end{array}$$

On the other hand, let

$$x = a_0, b_0, a_1, b_1, \cdots, a_k, b_k = y$$

be an ϵ-chain. Then similarly we have

$$d(T^n a_i, T^n b_i) \leq \lambda^n d(a_i, b_i)$$

for $n \geq 1$ and $1 \leq i \leq k$. Then we get

$$d(T^n x, T^n y) = d(T^n a_0, T^n b_k) \leq \cdots \leq (2k+1)\lambda^n \epsilon.$$

For any integers p, q such that $p < q$

$$\begin{aligned}
d(T^p x, T^q y) &\leq d(T^p x, T^{p+1} y) + d(T^{p+1} x, T^{p+1} y) + \cdots + d(T^{q-1} x, T^{q-1} y) \\
&\quad + d(T^{q-1} x, T^q y) + d(T^{q-1} x, T^q y) \\
&= \sum_{i=p}^{q-1} d(T^i x, T^{i+1} y) + \sum_{i=p+1}^{q-1} d(T^i x, T^i y) \\
&= \sum_{i=p}^{q-1} (2m+1)\lambda^i \epsilon + \sum_{i=p+1}^{q-1} (2k+1)\lambda^i \epsilon \\
&\leq \sum_{i=p}^{\infty} (2m+1)\lambda^i \epsilon + \sum_{i=p+1}^{\infty} (2k+1)\lambda^i \epsilon \\
&= \frac{(2m+1)\lambda^p \epsilon}{1-\lambda} + \frac{(2k+1)\lambda^{p+1}\epsilon}{1-\lambda}.
\end{aligned}$$

Thus, $d(T^p x, T^q y) \to 0$ as $p, q \to \infty$. We similarly obtain the same result for $p \geq q$ and conclude that $(T^n x, T^n y)$ is a Cauchy bisequence on (X, Y, d). Since (X, Y, d) is complete, $(T^n x, T^n y)$ converges (and in particular biconverges) to a point $u \in X \cap Y$. From Lemma 3.1, since any (ϵ, λ)-uniformly locally contractive covariant map is continuous, we get

$$Tu = T(\lim_{n\to\infty} T^n x) = \lim_{n\to\infty} T^{n+1} x = \lim_{n\to\infty} T^n x = u.$$

So, u is a fixed point of T.

Now, we examine the uniqueness of fixed point u for T. We assume that there exists $u' \neq u$ such that $Tu' = u'$, $u' \in X \cap Y$. And let

$$u = x_0, y_0, x_1, y_1, \cdots, x_k, y_k = u'$$

be an ϵ-chain. Then we have

$$\begin{aligned}
0 < d(u, u') &= d(Tu, Tu') \\
&= d(T^2 u, T^2 u') \\
&\ \ \vdots \\
&= d(T^n u, T^n u') \\
&= d(T^n(x_0), T^n(y_k)) \\
&\leq \sum_{i=0}^{k} d(T^n x_i, T^n y_i) + \sum_{i=1}^{k} d(T^n x_i, T^n y_{i-1}) \\
&\leq (2k+1)\lambda^n \epsilon \to 0 \text{ as } n \to \infty.
\end{aligned}$$

We obtain a contradiction. So, $u = u'$. □

Lemma 3.2. *Let (X, Y, d) be a bipolar metric space. If $T : (X, Y, d) \rightleftarrows (X, Y, d)$ is an (ϵ, λ)-uniformly locally contractive contravariant map, $T^2 : (X, Y, d) \rightrightarrows (X, Y, d)$ is an (ϵ, λ)-uniformly locally contractive map.*

Proof. We take two points $x \in X$ and $y \in Y$. In that case,

$$d(x, y) < \epsilon \Rightarrow d(Ty, Tx) \leq \lambda d(x, y) < \lambda\epsilon < \epsilon,$$

which in turn implies that

$$d(T^2x, T^2y) \leq \lambda d(Ty, Tx) \leq \lambda^2 d(x, y) \leq \lambda d(x, y)$$

as $\lambda \in (0, 1)$. $\square$

Theorem 3.2. *Let (X, Y, d) be an ϵ-chainable complete bipolar metric space and $T : (X, Y, d) \rightleftarrows (X, Y, d)$ be an (ϵ, λ)-uniformly locally contractive contravariant map. Then there exists a unique point $u \in X \cap Y$ such that $Tu = u$.*

Proof. Because of the fact that T is an (ϵ, λ)-uniformly locally contractive contravariant map, by Lemma 3.2, $S = T^2$ is an (ϵ, λ)-uniformly locally contractive covariant map and by Theorem 3.1, there exists a unique point $u \in X \cap Y$ such that $Su = u$.

Let $Tu = v$. Since, $u \in X$, $v \in Y$ and since $u \in Y$, $v \in X$,

$$Sv = T^2v = T^3u = TT^2u = TSu = Tu = v.$$

Then, $v \in X \cap Y$ is a fixed point of S. Since u is a unique fixed point, we get $v = u$, so $Tu = u$. Hence u is a fixed point of T.

Now, we examine the uniqueness of fixed point u for T. We assume that there exists $u' \in X \cap Y$ such that $u' \neq u$ and $Tu' = u'$. Then we get

$$Su' = T^2u' = TTu' = Tu' = u'.$$

This is a contradiction. Therefore, $u' = u$. $\square$

Definition 3.3. *Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be bipolar metric spaces. A covariant map $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ such that $d_2(Tx, Ty) < d_1(x, y)$ for all $x \in X_1$, $y \in Y_1$ or a contravariant map $T : (X_1, Y_1, d_1) \rightleftarrows (X_2, Y_2, d_2)$ such that $d_2(Ty, Tx) < d_1(x, y)$ for all $x \in X_1$, $y \in Y_1$, is called contractive.*

Incomplete bipolar metric spaces, contractive mappings may be without fixed points, but if they have a fixed point, this fixed point is unique.

Lemma 3.3. *Let T be a contractive, then T is continuous.*

Proof. Let $T : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$ be a covariant map and $x \in X_1$, $y \in Y_1$. Since T is a contractive, for given $\epsilon > 0$, there exists a $\delta = \epsilon > 0$ such that

$$d_1(x, y) < \delta \Rightarrow d_2(Tx, Ty) < d_1(x, y) < \delta = \epsilon.$$

Then, T is continuous.

Similarly, let $T : (X_1, Y_1, d_1) \rightleftarrows (X_2, Y_2, d_2)$ be a covariant map and $x \in X_1$, $y \in Y_1$. Since T is a contractive, for given $\epsilon > 0$, there exists a $\delta = \epsilon > 0$ such that

$$d_1(x, y) < \delta \Rightarrow d_2(Ty, Tx) < d_1(x, y) < \delta = \epsilon.$$

Then T is continuous. $\square$

Definition 3.4. *Let (X, Y, d) be a bipolar metric space and $T : (X, Y, d) \rightleftarrows (X, Y, d)$. If there exists a function $\lambda : (0, \infty) \to [0, 1)$ such that*

$$d(Ty, Tx) \leq \lambda(d(x, y))d(x, y) \tag{1}$$

for all $(x, y) \in X \times Y$ and

$$\sup\{\lambda(t) : 0 < k \leq t \leq l\} < 1$$

for all $k, l, t > 0$, then T is called a weakly contractive contravariant map.

Theorem 3.3. *Let (X, Y, d) be a complete bipolar metric space and $T : (X, Y, d) \rightleftarrows (X, Y, d)$ be a weakly contractive contravariant map. Then T has a unique fixed point.*

Proof. We take a left point $x \in X$. We consider the bisequence $(T^{2n}x, T^{2n+1}x)$ on (X, Y, d). If there exists $n \in \mathbb{N}$ such that $d(T^{2n}x, T^{2n+1}x) = 0$, then since

$$T^{2n}x = T^{2n+1}x = TT^{2n}x,$$

T has a fixed point and it is $T^{2n}x$. Similarly, if $d(T^{2n+2}x, T^{2n+1}x) = 0$, then $T^{2n+1}x$ is a fixed point of T.

Now, we assume that $d(T^{2n}x, T^{2n+1}x) > 0$ and $d(T^{2n+2}x, T^{2n+1}x) > 0$ for each non-negative integer n. Since, $\lambda(r) < 1$ for all $r \in (0, \infty)$, T is contractive. For each positive integer n, we get

$$\begin{aligned} d(T^{2n}x, T^{2n+1}x) &= d(TT^{2n-1}x, TT^{2n}x) \\ &\leq \lambda(d(T^{2n}x, T^{2n-1}x)).d(T^{2n}x, T^{2n-1}x) \\ &< d(T^{2n}x, T^{2n-1}x) \end{aligned}$$

and for each non-negative integer n, we get

$$\begin{aligned} d(T^{2n+2}x, T^{2n+1}x) &= d(TT^{2n+1}x, TT^{2n}x) \\ &\leq \lambda(d(T^{2n}x, T^{2n+1}x)).d(T^{2n}x, T^{2n+1}x) \\ &< d(T^{2n}x, T^{2n+1}x). \end{aligned}$$

Then we have

$$d(x, Tx) > d(T^2x, Tx) > d(T^2x, T^3x) > d(T^4x, T^3x) > \cdots \tag{2}$$

which means that the sequences $d(T^{2n}x, T^{2n+1}x)$ and $d(T^{2n+2}x, T^{2n+1}x)$ are monotone decreasing and bounded below 0 on $\mathbb{R}$. Then, these sequences are convergent and they converge to same point by (2). Let

$$\lim_{n\to\infty} d(T^{2n}x, T^{2n+1}x) = \lim_{n\to\infty} d(T^{2n+2}x, T^{2n+1}x) = \alpha.$$

Thus,

$$\alpha < d(T^{2n}x, T^{2n+1}x) \leq d(x, Tx)$$

and similarly

$$\alpha < d(T^{2n+2}x, T^{2n+1}x) \leq d(x, Tx).$$

We obtain that $\alpha = 0$. We assume the contrary. Let $\alpha > 0$. Set

$$\lambda_0 = \sup\{\lambda(t) : 0 < \alpha \leq t \leq d(x, Tx)\}.$$

Then we have

$$\lambda(d(T^{2n}x, T^{2n+1}x)) \leq \lambda_0 \quad \text{and} \quad \lambda(d(T^{2n+2}x, T^{2n+1}x)) \leq \lambda_0$$

for each non-negative integer n. So, we get

$$\begin{aligned} 0 < \alpha < d(T^{2n}x, T^{2n+1}x) &\leq \lambda_0 d(T^{2n}x, T^{2n-1}x) \\ &\vdots \\ &\leq \lambda_0^{2n} d(x, Tx) \to 0 \end{aligned}$$

is a contradiction. Hence, $\alpha = 0$.

Now, we show that $(T^{2n}x, T^{2n+1}x)$ is a Cauchy bisequence on (X, Y, d). Given a number $\epsilon > 0$. We set $\delta > 0$ as

$$\delta = \delta(\epsilon) = \sup\{\lambda(t) : 0 < \frac{\epsilon}{3} \leq t \leq \epsilon\} < 1. \tag{3}$$

Since, $\lim_{n\to\infty} d(T^{2n}x, T^{2n+1}x) = \lim_{n\to\infty} d(T^{2n+2}x, T^{2n+1}x) = \alpha = 0$ and $1 - \delta > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$d(T^{2n}x, T^{2n+1}x) < \frac{1-\delta}{3}\epsilon \quad \text{and} \quad d(T^{2n+2}x, T^{2n+1}x) < \frac{1-\delta}{3}\epsilon \tag{4}$$

for all $n \in \mathbb{N}$, $n \geq n_0$.

Let $m, n \geq n_0$ for $m, n \in \mathbb{N}$. If $m \geq n$ then $m = n + k$ where k is a non-negative integer. By induction, we show that

$$d(T^{2m}x, T^{2n+1}x) < \epsilon. \tag{5}$$

For k=0, we get

$$d(T^{2m}x, T^{2n+1}x) = d(T^{2n}x, T^{2n+1}x) < \frac{1-\delta}{3}\epsilon < \epsilon.$$

Thus, (5) is satisfied. Suppose $d(T^{2m}x, T^{2n+1}x) < \epsilon$ for $k > 0$. We consider $k+1$. If $d(T^{2m}x, T^{2n+1}x) \geq \frac{\epsilon}{3}$, then from (3) and (1) we get

$$d(T^{2n+2}x, T^{2m+1}x) \leq \lambda(d(T^{2m}x, T^{2n+1}x))d(T^{2m}x, T^{2n+1}x) < \delta\epsilon.$$

Thus, from (4) we get

$$\begin{aligned} d(T^{2(n+k+1)}x, T^{2n+1}x) &= d(T^{2m+2}x, T^{2n+1}x) \\ &\leq d(T^{2m+2}x, T^{2m+1}x) + d(T^{2n+2}x, T^{2m+1}x) \\ &\quad + d(T^{2n+2}x, T^{2n+1}x) \\ &< \tfrac{1-\delta}{3}\epsilon + \delta\epsilon + \tfrac{1-\delta}{3}\epsilon < \epsilon \end{aligned}$$

If $d(T^{2m}x, T^{2n+1}x) < \frac{\epsilon}{3}$, then from (4) we get

$$\begin{aligned} d(T^{2(n+k+1)}x, T^{2n+1}x) &= d(T^{2m+2}x, T^{2n+1}x) \\ &\leq d(T^{2m+2}x, T^{2m+1}x) + d(T^{2m}x, T^{2m+1}x) \\ &\quad + d(T^{2m}x, T^{2n+1}x) \\ &< \tfrac{1-\delta}{3}\epsilon + \tfrac{1-\delta}{3}\epsilon + \tfrac{\epsilon}{3} < \epsilon. \end{aligned}$$

If $m < n$ then $n = m + k$ where $k > 0$. By induction, we show that

$$d(T^{2m}x, T^{2n+1}x) < \epsilon.$$

Suppose $d(T^{2m}x, T^{2n+1}x) < \epsilon$ for $k > 0$. We consider $k+1$. If

$$d(T^{2m}x, T^{2n+1}x) \geq \frac{\epsilon}{3}$$

then from (3) and (1) we get

$$d(T^{2n+2}x, T^{2m+1}x) \leq \lambda(d(T^{2m}x, T^{2n+1}x))d(T^{2m}x, T^{2n+1}x) < \delta\epsilon.$$

Thus, from (4) we get

$$\begin{aligned} d(T^{2m}x, T^{2(m+k+1)+1}x) &= d(T^{2m}x, T^{2n+3}x) \\ &\leq d(T^{2m}x, T^{2m+1}x) + d(T^{2n+2}x, T^{2m+1}x) \\ &\quad + d(T^{2n+2}x, T^{2n+3}x) \\ &< \tfrac{1-\delta}{3}\epsilon + \delta\epsilon + \tfrac{1-\delta}{3}\epsilon < \epsilon \end{aligned}$$

If $d(T^{2m}x, T^{2n+1}x) < \frac{\epsilon}{3}$, then from (4) we get

$$\begin{aligned} d(T^{2m}x, T^{2(m+k+1)+1}x) &= d(T^{2m}x, T^{2n+3}x) \\ &\leq d(T^{2m}x, T^{2n+1}x) + d(T^{2n+2}x, T^{2n+1}x) \\ &\quad + d(T^{2n+2}x, T^{2n+3}x) \\ &< \tfrac{\epsilon}{3} + \tfrac{1-\delta}{3}\epsilon + \tfrac{1-\delta}{3}\epsilon < \epsilon. \end{aligned}$$

Hence, $(T^{2n}x, T^{2n+1}x)$ is a Cauchy bisequence on (X, Y, d). So, it biconverges to a point $u \in X \cap Y$, then

$$\lim_{n\to\infty} T^{2n}x = \lim_{n\to\infty} T^{2n+1}x = u.$$

Since, T is contractive, it is continuous. Then, we have

$$Tu = T(\lim_{n\to\infty} T^{2n}x) = \lim_{n\to\infty} TT^{2n}x = \lim_{n\to\infty} T^{2n+1}x = u.$$

So, u is a fixed point of T. Because of contractiveness of T, it is clear that the fixed point is unique. $\square$

Example 3.1. *Let $X = [0,1]$, $Y = [-1,1]$ and a function $d : X \times Y \to \mathbb{R}^+$ be defined such that $d(x,y) = |x-y|$ for $x \in X$, $y \in Y$. Then (X,Y,d) is a complete bipolar metric space. The contravariant $T : (X,Y,d) \rightleftarrows (X,Y,d)$ be defined as $Tz = \frac{z+1}{4}$ for all $z \in X \cup Y$ and the map $\lambda : (0,\infty) \to [0,1)$ be defined as $\lambda(t) = \frac{t}{t+1}$ for $t > 0$. We obtain that*

$$d(Ty,Tx) \le \lambda(d(x,y))d(x,y)$$

is satisfied for all $x \in X$, $y \in Y$. And, it is obvious that

$$sup\{\lambda(t) : 0 < k \le t \le l\} < 1$$

for all $k,l,t \in \mathbb{R}^+$. So, T is a weakly contractive contravariant map. Therefore, from Theorem (3.3), T has a unique fixed point and it is $\frac{1}{3} \in \mathbb{R}$.

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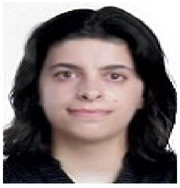

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