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SEIDEL BORDERENERGETIC GRAPHS

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Abstract. A graph G of order n is said to be Seidel borderenergetic if its Seidel energy equals the Seidel energy of the complete graph K_n. Let G be graph on n vertices with two distinct Seidel eigenvalues. In this paper, we prove that G is Seidel borderenergetic if and only if $G \cong K_n$ or $G \cong \overline{K}_n$ or $G \cong K_i \cup K_j$ or $G \cong K_{i,j}$, where $i + j = n$. We also, show that if G is a connected k-regular graph on $n \geq 3$ vertices with three distinct eigenvalues, then G is Seidel borderenergetic if and only if $G \cong K_{\frac{n}{2},\frac{n}{2}}$ where n is even. Finally, we determine all Seidel borderenergetic graphs with at most 10 vertices.

Keywords: Seidel matrix, Seidel eigenvalue, Seidel borderenergetic graph.

AMS Subject Classification: 05C50.

1. Introduction

Here, we recall some definitions that will be used in the paper. Let G be a simple graph with n vertices, m edges and $A(G)$ denotes the adjacency matrix of G. The eigenvalues of graph G are the roots of charateristic polynomial $\chi_G(\lambda) = det(\lambda I - A(G))$, where I is the identity matrix of order n. The energy of a graph is defined as the sum of absolute value of the eigenvalues of $A(G)$, see [10]. The rank of the matrix $A(G)$ denoted by $rank(A(G))$ is equal to the number of linearly independent columns of $A(G)$.

For given graph G its complement is denoted by $\overline{G}$. For two graphs G_1 and G_2, the graph $G_1 \cup G_2$ is the disjoint union of G_1 and G_2. The graph $G - \{v\}$ is a graph obtaining from G by removing the vertex v with all edges connected to v. The complete graph on n vertices is denoted by K_n. A complete bipartite graph with a bipartition of sizes n_1 and n_2 is denoted by K_{n_1,n_2}.

Suppose $L = D - A$ is the Laplacian matrix of graph G, where $D = [d_{ij}]$ is a diagonal matrix with $d_{ii} = deg_G(v_i)$, and $d_{ij} = 0$; otherwise. The spectra of L is a sequence of its eigenvalues an displayed in increasing order, denoted by $LSepc(G) = \{0 = \delta_n, \delta_{n-1}, \ldots, \delta_1\}$.

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The Laplacian energy of the graph G is defined as

$$LE(G) = \sum_{i=1}^{n} |\delta_i - \overline{d}|,$$

where δ_i's are the Laplacian eigenvalues of G and $\overline{d}$ is the average degree of G . For the Laplacian energy, we have $LE(K_n) = 2n-2$. Details on the properties of Laplacian energy can be found in [11, 16].

Recently, Gong et al. [9] proposed the concept of borderenergetic graphs, namely graphs of order n satisfying $E(G) = 2n - 2$. Tura in [23] proposed the concept of Laplacian borderenergetic graphs. In this way, we say G is Laplacian borderenergetic if $LE(G) = LE(K_n)$. More details on borderenergetic and Laplacian borderenergetic graphs can found in [6, 15, 17, 18, 22] as well as [5, 14, 13].

In 1966, Van Lint and Seidel in [24] introduced a symmetric $(0, -1, 1)$-adjacency matrix for a graph G called the Seidel matrix of G as $S(G) = J - I - 2A(G)$, where J is the matrix with entries 1 in every position. Let $\mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_s(G)$ be the distinct Seidel eigenvalues of G with multiplicity $t_1, t_2, \ldots, t_s$, respectively. The multiset $Spec_S(G) = \{[\mu_1(G)]^{t_1}, [\mu_2(G)]^{t_2}, \ldots, [\mu_s(G)]^{t_s}\}$ is called the Seidel spectrum of G. Two non-isomorphic graphs are said to be Seidel co-spectral if their Seidel spectra coincide. In [12] Haemers defined the Seidel energy of G as

$$E_S(G) = \sum_{i=1}^{n} |\mu_i(G)|, \tag{1}$$

where $\mu_i(G)$'s are the Seidel eigenvalues of G. Two graphs G and G' are said to be Seidel equienergetic if $E_S(G) = E_S(G')$, see [20]. In a trivial manner, co-spectral graphs are equienergetic. If the Seidel eigenvalues of a graph G are $\mu_i(G)$'s, $(1 \leq i \leq n)$, then the Seidel eigenvalues of $\overline{G}$ are $-\mu_i(G)$'s, $(1 \leq i \leq n)$ and so $E_S(G) = E_S(\overline{G})$. A graph G of order n is said to be Seidel borderenergetic if its Seidel energy equals the Seidel energy of the complete graph K_n, i.e., if $E_S(G) = 2(n - 1)$.

Let U_1 and $U_2 = V(G) \setminus U_1$ be the partitioned sets of the vertex set $V(G)$ of a graph G. Let G' be the graph obtained from G by deleting all edges between U_1 and U_2 and inserting all edges between U_1 and U_2 that were not presented in G. Then G' and G are said to be Seidel switching, with respect to U_1. If G' and G are Seidel switching then $S(G')$ and $S(G)$ are similar and therefore G' and G have the same Seidel eigenvalues, see [12].

Given a set V of m vectors (points in $\mathbb{R}^n$), the Gram matrix Γ is a real symmetric $(n \times n)$-matrix of all possible inner products of V, i.e., $\gamma_{ij} = x_i^t x_j$, where x^t denotes the transposed vector of x. The Gram matrix can be written as $\Gamma = H^t H$, where H is $(m \times n)$-matrix and m is the rank of Γ. Let θ be the smallest eigenvalue of $S(G)$. Then $\theta < 0$ since $S(G) \neq 0$ and $trace(S(G)) = 0$. The $\Gamma = I - \frac{1}{\theta} S(G)$ is the Gram matrix of a set of vectors in $\mathbb{R}^d$, where $d = rank(S(G) - \theta I) = n - m(\theta)$, n is the number of vertices of the graph and $m(\theta)$ is the multiplicity of θ as eigenvalue of $S(G)$, see [2].

Lemma 1.1. [2]. For any graph G on $n \geq 2$ vertices, we have

i) $\sum_{i=1}^{n} \mu_i(G) = 0$,

ii) $\sum_{i=1}^{n} \mu_i^2(G) = n(n - 1)$.

Lemma 1.2. [3]. Let G be a k-regular graph on n vertices. Then the Seidel spectrum of G is $\{n-1-2k, -1-2\lambda_{n-1}(G), \ldots, -1-2\lambda_1(G)\}$, where $\lambda_i(G)$'s $(1 \leq i \leq n)$ are eigenvalues of G.

Lemma 1.3. [8]. Let G be a connected k-regular graph on n vertices with adjacency matrix $A(G)$. Assume that $A(G)$ has exactly t distinct eigenvalues. Then the Seidel matrix $S(G)$ has at most t distinct eigenvalues.

Lemma 1.4. [8]. Suppose that $S(G)$ is a Seidel matrix of order $n \geq 2$ with spectrum $\{[\mu_1(G)]^{n-t}, [\mu_2(G)]^t\}$ for some t where $1 \leq t \leq n-1$. Let $S(G')$ be a principal $(n-1) \times (n-1)$ submatrix of $S(G)$. Then the spectrum of $S(G')$ is

$$\Big\{[\mu_1(G)]^{n-t-1}, [\mu_2(G)]^{t-1}, [\mu_1(G)+\mu_2(G)]^1\Big\}.$$

Lemma 1.5. [1]. Let G be a connected graph with least eigenvalue $\lambda(G)$. Then if G is neither complete nor null, then $\lambda(G) \leq -\sqrt{2}$ with equality if and only if $G \cong K_{1,2}$.

2. Main Results

Here, we characterize all Seidel borderenergetic graphs with at most three Seidel eigenvalues. The following Lemma is essential in the proof of Proposition 2.1.

Lemma 2.1. Let G be graph on $n \geq 2$ vertices with two distinct Seidel eigenvalues. Then

$$Spec_S(G) = \Big\{\Big[\sqrt{\frac{t_2}{t_1}(n-1)}\Big]^{t_1}, \Big[-\sqrt{\frac{t_1}{t_2}(n-1)}\Big]^{t_2}\Big\},$$

where $t_1 + t_2 = n$.

Proof. Let G be graph on n vertices with two distinct Seidel eigenvalues $[\mu_1(G)]^{t_1}$, $[\mu_2(G)]^{t_2}$, where $t_1 + t_2 = n$. By Lemma 1.1 (i), we have $t_1\mu_1(G) + t_2\mu_2(G) = 0$, then

$$\mu_2(G) = -\frac{t_1}{t_2}\mu_1(G). \tag{2}$$

By Lemma 1.1 (ii), we have $t_1\mu_1^2(G) + t_2\mu_2^2(G) = n(n-1)$, then

$$\mu_1(G) = -\sqrt{\frac{t_2}{t_1}(n-1)}. \tag{3}$$

Therefore by using Equations 2, 3, we have

$$Spec_S(G) = \Big\{\Big[\sqrt{\frac{t_2}{t_1}(n-1)}\Big]^{t_1}, \Big[-\sqrt{\frac{t_1}{t_2}(n-1)}\Big]^{t_2}\Big\}.$$

□

Proposition 2.1. Let G be graph on n vertices with two distinct Seidel eigenvalues. Then G is Seidel borderenergetic if and only if $G \cong K_n$ or $G \cong \overline{K}_n$ or $G \cong K_i \cup K_j$ or $G \cong K_{i,j}$, where $i + j = n$.

Proof. By Lemma 2.1, we have $E_S(G) = 2\sqrt{t_1t_2(n-1)}$. Thus G is Seidel borderenergetic if $n-1 = t_1t_2$. Since $t_1 + t_2 = n$, then t_1, t_2 are integeres. Withouth loss of generallity, we can suppose that $t_1 = n-1$ and $t_2 = 1$. Then $Spec_S(G) = \{[n-1]^1, [-1]^{n-1}\}$ and so

Gram matrix $\Gamma = I + S(G)$ is of rank 1. Thus, by [21] there are column vectors $v, w \in \mathbb{R}^n$ such that $vw^t = A$. Let $x_1 = [1, s_{1,2}, \dots, s_{1,n}]$, since

$$\Gamma = \begin{bmatrix} 1 & s_{1,2} & s_{1,3} & \dots & s_{1,n} \\ s_{1,2} & 1 & s_{2,3} & \dots & s_{2,n} \\ s_{1,3} & s_{2,3} & 1 & \dots & s_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{1,n} & s_{2,n} & s_{3,n} & \cdots & 1 \end{bmatrix},$$

then

$$x_1^t x_1 = \begin{bmatrix} 1 & s_{1,2} & s_{1,3} & \dots & s_{1,n} \\ s_{1,2} & s_{1,2}^2 & s_{1,2}s_{1,3} & \dots & s_{1,2}s_{2,n} \\ s_{1,3} & s_{12}s_{1,3} & s_{1,3}^2 & \dots & s_{1,3}s_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{1,n} & s_{1,2}s_{1,n} & s_{1,3}s_{1,n} & \cdots & s_{1,n}^2 \end{bmatrix}.$$

By comparing two matrices Γ and $x_1^t x_1$, we have $s_{1u} = \pm 1$ and $s_{uv} = s_{1u}s_{1v}$, $(2 \leq u \neq v \leq n)$. It not difficult to see that

$$\Gamma = \begin{bmatrix} J_{i\times i} & -J_{(n-i)\times i} \\ -J_{i\times(n-i)} & J_{(n-i)\times(n-i)} \end{bmatrix},$$

and

$$A = \frac{1}{2}(S + I - J) = \begin{bmatrix} 0_{l\times l} & -J_{(n-l)\times l} \\ -J_{l\times(n-l)} & 0_{(n-l)\times(n-l)} \end{bmatrix}.$$

Since A is the adjacency matrix of $K_{i,n-i}$, by a Seidel switching we have $G \cong \overline{K}_n$ or $K_{i,j}$, where $i + j = n$. By Eq. 1, $E_S(G) = E_S(\overline{G})$ and so $G \cong K_n$ or $K_i \cup K_j$, where $i + j = n - 1$. Conversely, we have

$$Spec_s(K_n) = Spec_s(K_i \cup K_j) = \{[1-n]^1, [1]^{n-1}\},$$
$$Spec_s(\overline{K}_n) = Spec_s(K_{i,j}) = \{[n-1]^1, [-1]^{n-1}\}.$$

This yields that $E_s(K_n) = E_s(\overline{K}_n) = E_s(K_i \cup K_j) = E_s(K_{i,j}) = 2n - 2$. □

Corollary 2.1. Let G be a graph on n vertices with two distinct Seidel eigenvalues. Then graph $G - \{v\}$ is Seidel borderenergetic if and only if $G - \{v\} \cong K_{n-1}$ or $\overline{K}_{n-1}$ or $K_i \cup K_j$ or $K_{i,j}$, where $i + j = n - 1$.

Proof. Let G be graph on n vertices with two distinct Seidel eigenvalues $[\mu_1]^{t_1}$, $[\mu_2]^{t_2}$, where $t_1 + t_2 = n$ and $t_1, t_2 \in \mathbb{N}$. By Lemmas 1.4, 2.1, the Seidel spectrum of graph $G - \{v\}$ can be computed as follows:

$$\Big\{\Big[\sqrt{\frac{t_2}{t_1}(n-1)} - \sqrt{\frac{t_1}{t_2}(n-1)}\Big]^1, \Big[\sqrt{\frac{t_2}{t_1}(n-1)}\Big]^{t_1-1}, \Big[-\sqrt{\frac{t_1}{t_2}(n-1)}\Big]^{t_2-1}\Big\}.$$

The following cases hold:

Case 1: If $t_2 \geq t_1$, then

$$E_S(G - \{v\}) = 2\sqrt{n-1}\Big(\sqrt{t_1 t_2} - \sqrt{\frac{t_1}{t_2}}\Big).$$

Thus $G - \{v\}$ is Seidel borderenergetic if $n - 2 = \sqrt{n-1}\Big(\sqrt{t_1 t_2} - \sqrt{\frac{t_1}{t_2}}\Big)$. Then

$$\frac{t_1}{t_2}(n-1)(t_2-1)^2 - (n-2)^2 = 0.$$

Since $t_2 = n - t_1$, we have

$$\frac{t_1-1}{n-t_1}\Big(t_1\big(n(5-2n)+t_1(n-1)-3\big)+n(n-2)^2\Big)=0.$$

If $t_1-1=0$ then $t_1=1$ and $t_2=n-1$. If $t_1\big(n(5-2n)+t_1(n-1)-3\big)+n(n-2)^2=0$, then

$$t_1=\frac{1}{2(n-1)}(2n^2-5n+3+\sqrt{5n^2-14n+9}),$$

or

$$t_1=\frac{-1}{2(n-1)}(-2n^2+5n-3+\sqrt{5n^2-14n+9}),$$

and both of them are impossible.

Case 2: If $t_2<t_1$, then

$$E_S(G-\{v\})=2\sqrt{n-1}\Big(\sqrt{t_1t_2}-\sqrt{\frac{t_2}{t_1}}\Big). \tag{4}$$

A similar argument shows that $t_1=n-1$ and $t_2=1$. Then

$$Spec_S(G-\{v\})=\{[n-2]^1,[-1]^{n-2}\}.$$

This completes the proof. □

A strongly regular graph $srg(n,k,e,f)$ is a k-regular graph of order n whenever it is not complete or edgeless and every pair of adjacent (non-adjacent) vertices has $e(f)$ common neighbours. It is a well-known fact that every regular graph with exactly three distinct eigenvalues is strongly regular, see [2].

Proposition 2.2. Let G be connected k-regular graph on $n\geq 3$ vertices with three distinct eigenvalues. Then G is Seidel borderenergetic if and only if $G\cong K_{\frac{n}{2},\frac{n}{2}}$ where n is even.

Proof. Suppose that G is connected k-regular graph on n vertices with three distinct eigenvalues $k>\lambda_1>\lambda_2$. Thus G is strongly regular. By [4], the spectrum of G is $\{[k]^1,[\lambda_1]^{t_1},[\lambda_2]^{t_2}\}$ where t_1, t_2 satisfy in the following equations:

$$t_1+t_2=n-1, \tag{5}$$

$$t_1\lambda_1+t_2\lambda_2=-k, \tag{6}$$

$$t_1=\frac{-(n-1)\lambda_2+k}{\lambda_1-\lambda_2}, \tag{7}$$

$$t_2=\frac{(n-1)\lambda_1+k}{\lambda_1-\lambda_2}. \tag{8}$$

By Lemma 1.2, the Seidel spectrum of G is $\{[n-1-2k]^1,[-1-2\lambda_1]^{t_1},[-1-2\lambda_2]^{t_2}\}$ and by Lemma 1.3, the Seidel matrix $S(G)$ has at most three distinct eigenvalues. If the Seidel matrix $S(G)$ has two distinct eigenvalues, then by Proposition 2.1, we have $G\cong K_{\frac{n}{2},\frac{n}{2}}$ where n is even. Let Seidel matrix $S(G)$ has three distinct eigenvalues. Then the Seidel energy of G is

$$E_S(G)=|n-1-2k|+t_1|-1-2\lambda_1|+t_2|-1-2\lambda_2|.$$

By Lemma 1.5, $\lambda_2<-\sqrt{2}$ and the following cases hold:

Case 1: Suppose $\lambda_1\geq 0$ and $k\leq\frac{1}{2}(n-1)$. By Equations 5, 6 we have

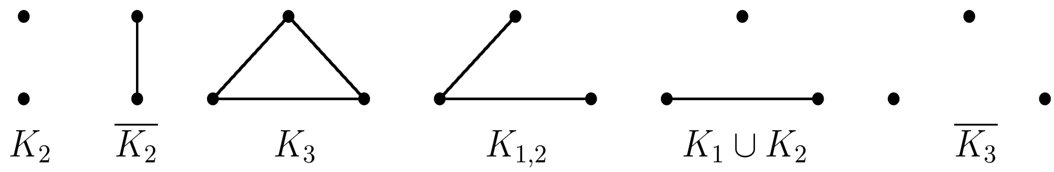

FIGURE 1. Seidel borderenergetic graphs of order 2, 3.

$$\begin{aligned} E_S(G) &= (n-1-2k) + t_1(1+2\lambda_1) + t_2(-1-2\lambda_2) \\ &= 2t_1(1+\lambda_1). \end{aligned}$$

Thus, G is Seidel borderenergetic if $n-1 = t_1(1+2\lambda_1)$. By Eq. 5, $\lambda_1 = \frac{t_2}{2t_1}$. Then by Eq. 6, $k = -t_2(\frac{1}{2}+\lambda_2)$ and by Eq. 7, we have $\lambda_2 = -\frac{1}{2}$, which is impossible.

Case 2: Suppose $\lambda_1 \geq 0$ and $k > \frac{1}{2}(n-1)$. By Equations 5, 6 we have

$$\begin{aligned} E_S(G) &= (2k-n+1) + t_1(1+2\lambda_1) + t_2(-1-2\lambda_2) \\ &= -2t_2(1+2\lambda_2). \end{aligned}$$

Thus, G is Seidel borderenergetic if $n-1 = -2t_2(1+2\lambda_2)$ and this yields that $\lambda_2 = -(1+\frac{t_1}{2t_2})$. By Eq. 6, $k = t_1(\frac{1}{2}-\lambda_1)+t_2$. By Eq. 7, $\lambda_1 = \frac{1}{2}+\frac{t_2}{t_1}$ which yields that $k=0$, a contradiction.

Case 3: Suppose that $\lambda_1 < 0$ and $k \leq \frac{1}{2}(n-1)$. By Equations 5, 6 we have

$$E_S(G) = (n-1-2k) + t_1(-1-2\lambda_1) + t_2(-1-2\lambda_2) = 0.$$

Thus, G is Seidel borderenergetic if $2(n-1)=0$ and so $n=1$, a contradiction.

Case 4: Suppose $\lambda_1 < 0$ and $k > \frac{1}{2}(n-1)$. By Equations 5, 6 we have

$$\begin{aligned} E_S(G) &= (2k-n+1) + t_1(1+2\lambda_1) + t_2(-1-2\lambda_2) \\ &= 2(2k-n+1). \end{aligned}$$

Thus, G is Seidel borderenergetic if $k = n-1$, a contradiction. Hence, if G has three distinct Seidel eigenvalues, then G is not Seidel borderenergetic. This completes the proof. □

Corollary 2.2. If G is a k-regular graph with exactly distinct three Seidel eigenvalues, then G is not Seidel borderenergetic.

2.1. **The smallest Seidel borderenergetic graphs.** Here, we introduce all non-isomorphic Seidel borderenergetic graphs of order n where $2 \leq n \leq 10$ and we determine their Seidel eigenvalues. Our computations are done by software package nauty developed by McKay [19] and the The GNU MPFR library [7]. See Table 3 and Figures 3-10.

Conjecture 2.1. Let G be graph on n vertices. Then G is Seidel borderenergetic if and only if $G \cong K_n$ or $G \cong \overline{K}_n$ or $G \cong K_i \cup K_j$ or $G \cong K_{i,j}$, where $i+j=n$.

Table 1. Seidel borderenergetic graphs of order n and their Seidel Spectra.

n	Graphs	S-Spectra
2	K_2, $\overline{K_2}$	$\{[1]^1, [-1]^1\}$
3	K_3, $K_1 \cup K_2$	$\{[2]^1, [-1]^2\}$
	$K_{1,2}$,$\overline{K_3}$	$\{[1]^2, [-2]^1\}$
4	K_4, $K_1 \cup K_3$, $K_2 \cup K_2$	$\{[-3]^1, [1]^3\}$
	$\overline{K_4}$, $K_{1,3}$, $K_{2,2}$	$\{[-1]^3, [3]^1\}$
5	K_5, $K_1 \cup K_4$, $K_2 \cup K_3$	$\{[-4]^1, [1]^4\}$
	$\overline{K_5}$, $K_{1,4}, K_{2,3}$	$\{[-1]^4, [4]^1\}$
6	$K_6, K_1 \cup K_5, K_2 \cup K_4, K_3 \cup K_3$	$\{[-5]^1, [1]^5\}$
	$\overline{K_6}, K_{1,5}, K_{2,4}, K_{3,3}$	$\{[-1]^5, [5]^1\}$
7	$K_7, K_1 \cup K_6, K_2 \cup K_5, K_3 \cup K_4$	$\{[-6]^1, [1]^6\}$
	$\overline{K_7}, K_{1,6}, K_{2,5}, K_{3,4}$	$\{[-1]^6, [6]^1\}$
8	K_8, $\overline{K_8}$, $K_{1,7}$, $K_{2,6}$, $K_{3,5}$, $K_{4,4}$	$\{[-7]^1, [1]^7\}$
	$K_1 \cup K_7$, $K_2 \cup K_6$, $K_3 \cup K_5$, $K_4 \cup K_4$	$\{[7]^1, [-1]^7\}$
9	K_9, $\overline{K_9}$, $K_{1,8}$, $K_{2,7}$, $K_{3,6}$, $K_{4,5}$	$\{[-8]^1, [1]^8\}$
	$K_1 \cup K_8$, $K_2 \cup K_7$, $K_3 \cup K_6$, $K_4 \cup K_5$	$\{[8]^1, [-1]^8\}$
10	K_{10}, $\overline{K_{10}}$, $K_{1,9}$, $K_{2,8}$, $K_{3,7}$, $K_{4,6}$, $K_{5,5}$	$\{[-9]^1, [1]^9\}$
	$K_1 \cup K_9$, $K_2 \cup K_8$, $K_3 \cup K_7$, $K_4 \cup K_6$, $K_5 \cup K_5$	$\{[9]^1, [-1]^9\}$

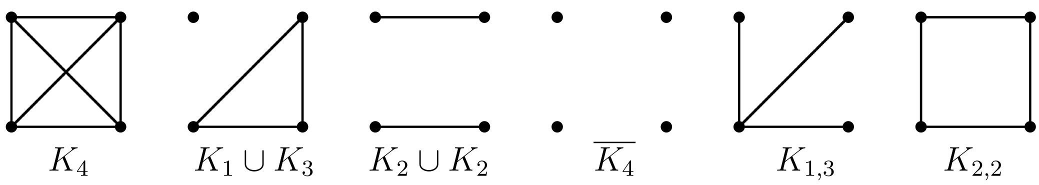

Figure 2. Seidel borderenergetic graphs of order 4.

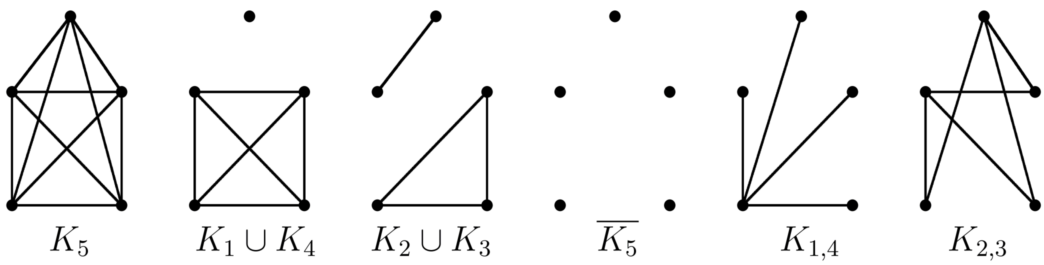

Figure 3. Seidel borderenergetic graphs of order 5.

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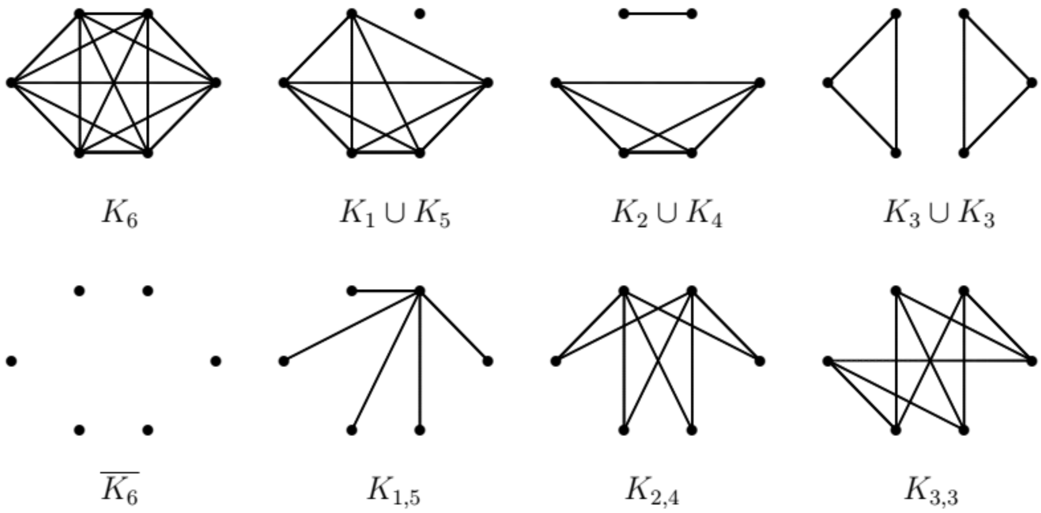

FIGURE 4. Seidel borderenergetic graphs of order 6.

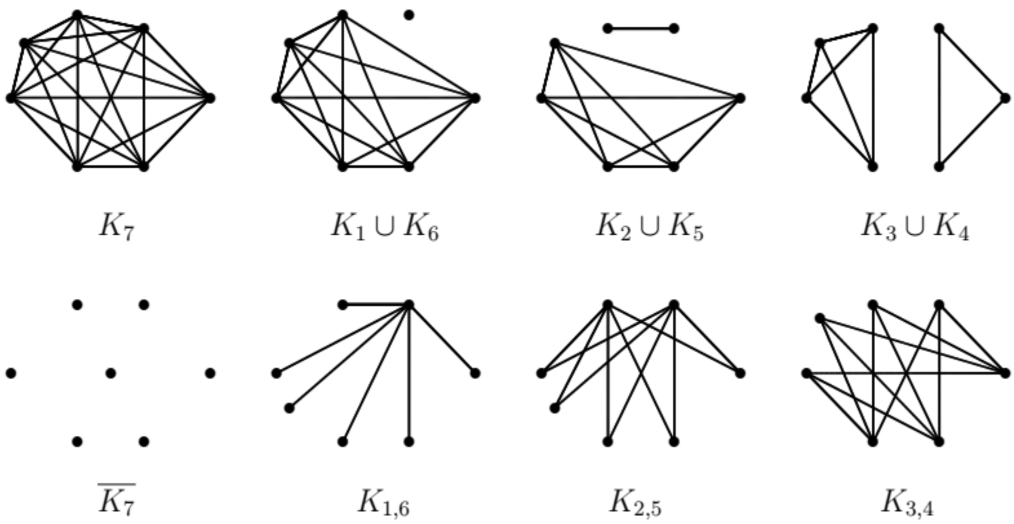

FIGURE 5. Seidel borderenergetic graphs of order 7.

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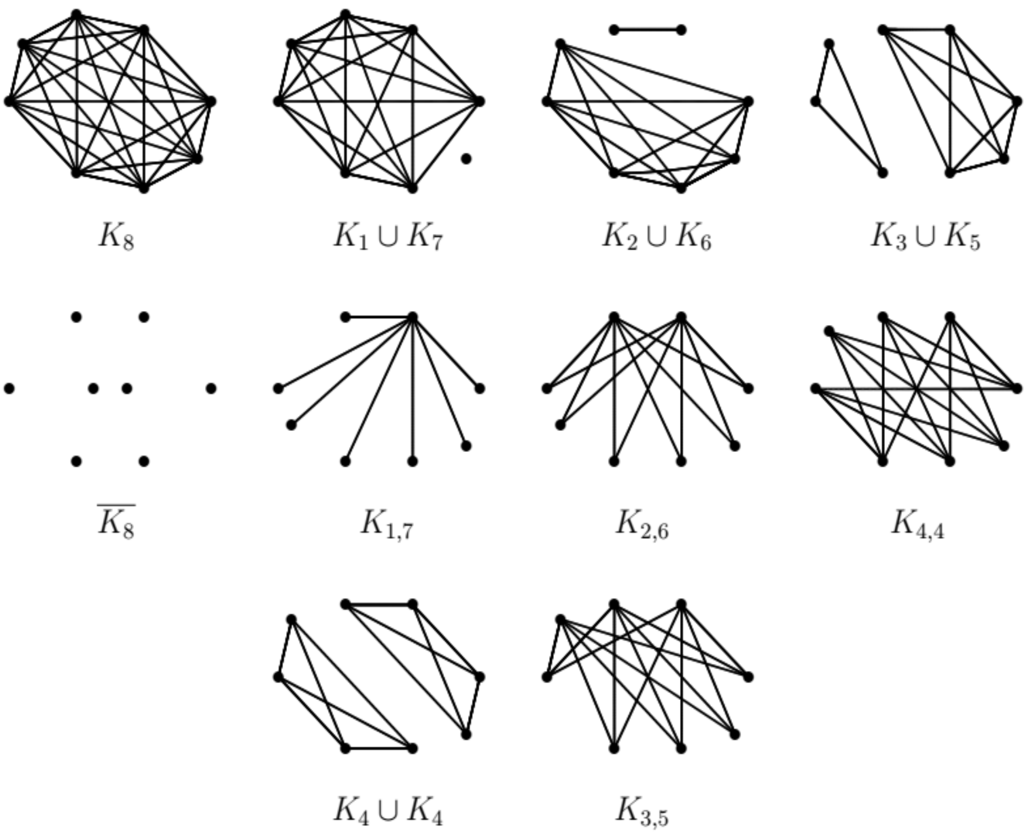

Figure 6. Seidel borderenergetic graphs of order 8.

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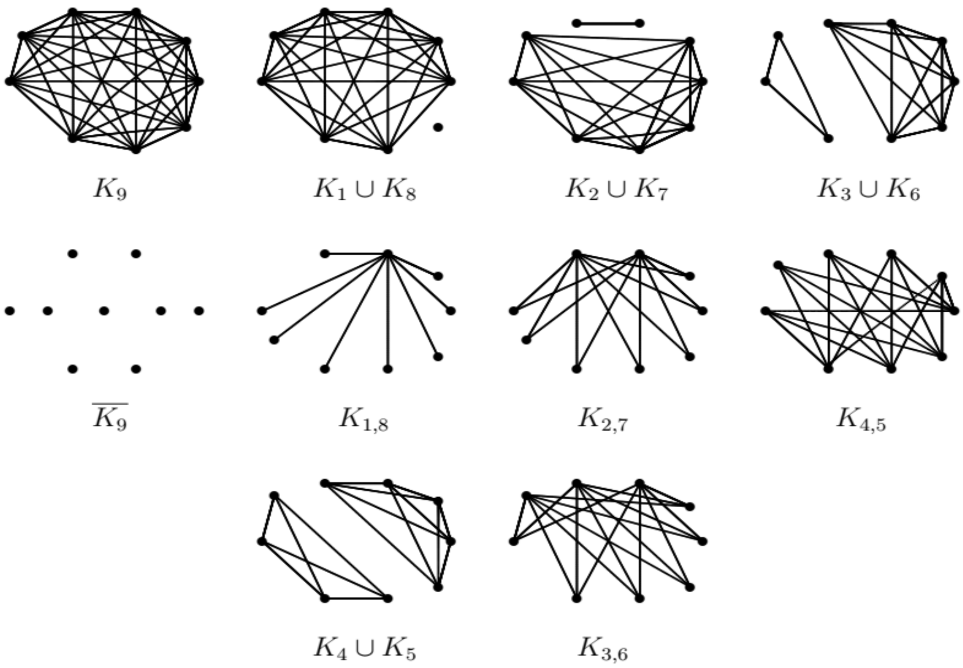

FIGURE 7. Seidel borderenergetic graphs of order 9.

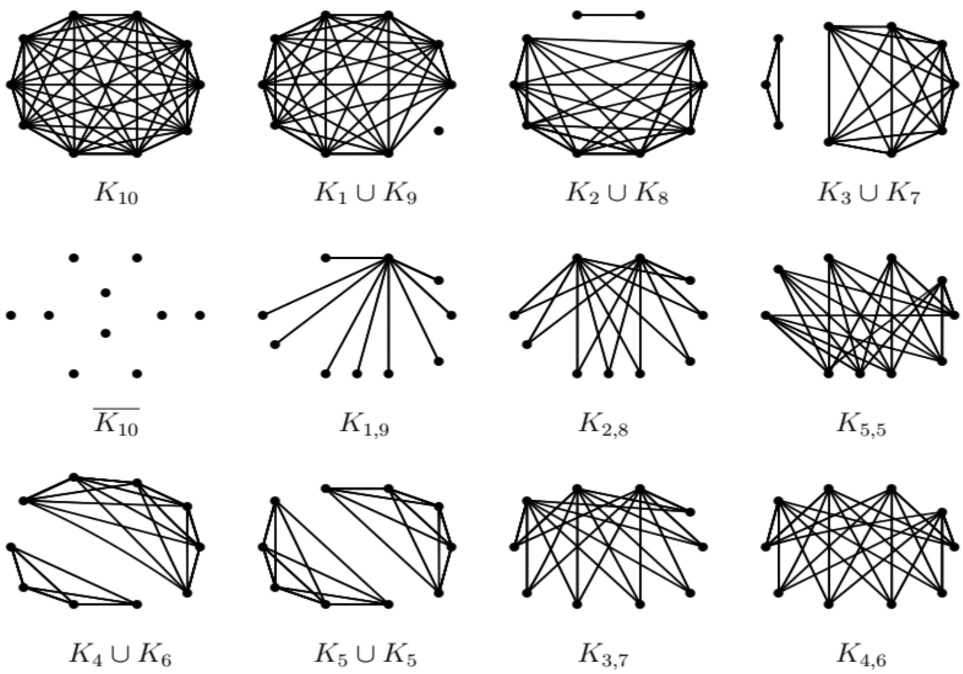

FIGURE 8. Seidel borderenergetic graphs of order 10.

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