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DEGREE EQUIVALENCE GRAPH OF A GRAPH

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Abstract. Given a set S and an equivalence relation R on S, one can define an equivalence graph with vertex set S. Given a graph with vertex set V, we can define an equivalence relation on V using the concept of degree of a vertex as follows: two vertices a and b in V are related if and only if they are of same degree. The degree equivalence graph of a graph G is the equivalence graph with vertex set V with respect to the above equivalence relation. In this paper, we study some properties of degree equivalence graph of a graph.

Keywords: Equivalence relation, graph, energy of a graph.

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1. Introduction

For standard terminology and notion in graphs and matrices, we refer the reader to the text-books of Harary [2] and Bapat [1]. The non-standard will be given in this paper as and when required.

Throughout this paper, for a graph G, $V(G)$ and $E(G)$ denote vertex set and edge set of G, respectively. The adjacency matrix of a graph G is denoted by A_G and n represents a positive integer. If A_G is an $n \times n$ matrix and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A_G, the energy of G is defined as

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

A binary relation R on a set S is called an equivalence relation if it is reflexive, symmetric, and transitive.

Let S be a non-empty set. Let R be an equivalence relation on S with respect to the relation R, we can draw a graph (undirected) G_R as follows: For $a, b \in S, a \neq b$,

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$$a \text{ and } b \text{ are adjacent in } G_R \Leftrightarrow aRb.$$

The graph G_R is called *equivalence graph* on S with respect to the relation R. We have the following observations:

(1) If there are two or more equivalence classes in the partition of S with respect to the relation R, then G_R is disconnected and the number of components is the number of distinct equivalence classes. Each component is a complete graph. If there is only one equivalence class, then G_R is the complete graph with $|S|$ vertices.
(2) Given a graph $G = (V, E)$, we can define new graphs with a vertex set V by defining equivalence relations on V with respect to some property of elements of V in G.

2. An equivalence relation with respect to the degrees of vertices

Let $G = (V, E)$ be a graph and $|V| = n$. We define a relation $\sim$ on V as follows: for $a, b \in V$,

$$a \sim b \Leftrightarrow deg(a) = deg(b).$$

It is easy to see that $\sim$ is an equivalence relation on V. Let $V_1, V_2, \ldots, V_k$ be the partition of V into disjoint classes by the relation $\sim$. Let $|V_i| = n_i, 1 \leq i \leq k$ so that $n_1+n_2+\ldots+n_k = n$. The equivalence class graph on V defined by $\sim$ is called *degree equivalence graph* of G and is denoted by $D(G)$. Note that two distinct vertices a and b in $D(G)$ are adjacent if and only if $deg(a) = deg(b)$. We observe that $D(G)$ is a simple graph. By the definition of degree equivalence graph, we have the following proposition.

Proposition 2.1. *The degree equivalence graph $D(G)$ of a graph G is the disjoint union of the complete graphs $K_{n_1}, K_{n_2}, \ldots, K_{n_k}$ on the vertex sets $V_1, V_2, \ldots, V_k$ respectively, where $V_1, V_2, \ldots, V_k$ are the cells in the partition of V in to disjoint classes by the relation $\sim$.*

Adjacency matrix of $D(G)$: Rearranging the vertices $v_1, \ldots, v_n$ of V such that $v_{11}, \ldots, v_{1n_1}$, $v_{21}, \ldots, v_{2n_2}$, ..., $v_{k1}, \ldots, v_{kn_k}$, where $v_{i1}, \ldots, v_{in_i}$ are the vertices of V_i, the adjacency matrix of $D(G)$ can be written as

$$A_{D(G)} = \begin{bmatrix} Y_{n_1} - I_{n_1} & & & \\ & Y_{n_2} - I_{n_2} & & \\ & & \ddots & \\ & & & Y_{n_r} - I_{n_r} \end{bmatrix}$$

where Y_{n_i} is the $n_i \times n_i$ matrix with all it entries equal to 1, and I_{n_i} is the $n_i \times n_i$ identity matrix.

Eigenvalues of $A_{D(G)}$: First, we find the eigenvalues of $Y_{n_i} - I_{n_i}$. By the elementary linear algebra of matrices, the eigenvalues of Y_{n_i} are n_i and 0, the latter with multiplicity $n_i - 1$. We have,

$$\begin{aligned} det(Y_{n_i} - I_{n_i} - \lambda I_{n_i}) = 0 \quad &\Leftrightarrow \quad det(Y_{n_i} - (\lambda+1)I_{n_i}) = 0 \\ &\Leftrightarrow \quad \lambda + 1 = n_i \text{ (once), and } \lambda + 1 = 0, (n_i - 1) \text{ times} \\ &\Leftrightarrow \quad \lambda = n_i - 1 \text{ (once), and } \lambda = -1, \ (n_i - 1) \text{ times} \end{aligned}$$

Also, $\sum_{i=1}^{k}(n_i - 1) = n - k$. The eigenvalues of $A_{D(G)}$ are given below:

$$\begin{matrix} \text{eigenvalue} \rightarrow \\ \text{multiplicity} \rightarrow \end{matrix} \begin{pmatrix} n_1 - 1 & n_2 - 1 & \ldots & n_k - 1 & -1 \\ 1 & 1 & & 1 & n - k \end{pmatrix}$$

The energy of $D(G)$: By the definition of energy of a graph, we have,

$$\begin{aligned}\mathcal{E}(D(G)) &= \sum_{i=1}^{k}|n_i-1|+\sum_{j=1}^{n-k}|-1| \\ &= \sum_{i=1}^{k}(n_i-1)+\sum_{j=1}^{n-k}1 \\ &= (n-k)+(n-k) \\ &= 2(n-k).\end{aligned}$$

Thus, we have the following theorem:

Theorem 2.1. *The energy of the degree equivalence graph $D(G)$ of a graph G with n vertices is*

$$\mathcal{E}(D(G)) = 2(n-k),$$

where k is the number of cells in the partition of the vertex set V of G in to disjoint classes with respect to the relation $\sim$.

Corollary 2.1. *The energy of the degree equivalence graph $D(G)$ is twice the rank of $D(G)$.*

Proof. Note that the number of cells in the partition of the vertex set V of a graph G in to disjoint classes with respect to the relation $\sim$ is nothing but the number of components in the degree equivalence graph $D(G)$. Therefore by Theorem 2.1, the corollory follows. □

Proposition 2.2. *For a regular graph G with n vertices $D(G) \cong K_n$.*

Proof. Let G be a r-regular graph. Then all vertices are of degree r. So, in $D(G)$, every vertex is adjacent to every other vertex. Therefore $D(G) \cong K_n$. □

Corollary 2.2. $D(K_{n,n}) \cong K_{2n}$.

Proof. Since $K_{n,n}$ contains $2n$ vertices of degree n, the proof follows by The Proposition 2.2. □

Proposition 2.3. *Let G_1 and G_2 be two graphs. If $G_1 \cong G_2$, then $D(G_1) \cong D(G_2)$.*

Proof. Obvious. □

Remark 2.1. *Converse of the above proposition is not true. Consider the complete graph K_3 on 3 vertices graphs and the null graph N_3 on 3 vertices. Note that, K_3 and N_3 are not isomorphic. Since K_3 is 3-regular and N_3 is 0-regular, by the Proposition 2.2, it follows that, $D(K_3) \cong K_3 \cong D(N_3)$.*

Proposition 2.4. *$D(K_{m,n})$ is the disjoint union of K_m and K_n*

Proof. In $K_{m,n}$, there are m vertices of degree n and n vertices of degree m. Then the equivalence relation $\sim$ partitions the vertex set $V(K_{m,n})$ in to two disjoint classes V_1 and V_2 with $|V_1| = m$, $|V_2| = n$. Therefore, by definition of $D(G), D(K_{m,n})$ is the disjoint union of K_m and K_n. □

Proposition 2.5. *For any graph, $D(G) = D(\overline{G})$, where $\overline{G}$ is the complement of G.*

Proof. We know that, for any graph G, $V(G) = V(\overline{G})$. For a vertex v, we denote the degree of v in G by $deg_G(v)$ and we denote the degree of v in $\overline{G}$ by $deg_{\overline{G}}(v)$. Since $G \cup \overline{G} = K_n$, a complete graph with n vertices, it follows that, if $v \in V(G)$ with $deg_G(v) = d$, then $deg_{\overline{G}}(v) = n-1-d$. Hence, two vertices u and v are adjacent in G if and only if u and v are adjacent in $\overline{G}$. Therefore $D(G) = D(\overline{G})$. □

Corollary 2.3. *Let G be a graph and $L(G)$ be the line graph of G. Then* $D(L(G)) = D(\overline{L(G)})$

Proof. Follows by Proposition 2.5. □

3. Conclusions

In this paper, we have defined the degree equivalence graph of a graph G. It is shown that the energy of the degree equivalence graph $D(G)$ is twice the rank of $D(G)$. In future one may one may discover further properties and applications of degree equivalence graph.

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