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FOURTH-ORDER ACCURATE METHOD BASED ON HALF-STEP CUBIC SPLINE APPROXIMATIONS FOR THE 1D TIME-DEPENDENT QUASILINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, we discuss a fourth-order accurate scheme based on cubic spline approximations for the solution of quasilinear parabolic partial differential equations (PDE). The stability of the scheme is discussed using a model linear PDE. The proposed method is tested on Burgers' equations in polar coordinates and Burgers-Huxley equation.

Keywords: Quasi-linear parabolic equations; Uniform mesh; Cubic Spline approximations; Burgers-Huxley equations;

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1. Introduction

We study the following 1D quasi-linear parabolic partial differential equation (PDE) of the form

$$\frac{\partial^2 w}{\partial x^2} = f(x, t, w, w_x, w_t), \quad 0 < x < 1, \ t > 0 \tag{1}$$

subject to the initial and boundary conditions are prescribed by

$$w(x, 0) = w_0(x), \quad 0 \le x \le 1,$$

$$w(0, t) = g_0(t), \ w(1, t) = g_1(t), \ t > 0,$$

where we assume that f, $w_0(x)$, $g_0(t)$and $g_1(t)$ are the continuous functions of sufficient differentiability.

In this work, we attempt to capture the patterns of time dependent quasi-linear PDE in one dimensional model equations. The quasi-linear PDE arise in various mathematical models of physical phenomenon in science and engineering such as reaction mechanism, convection and diffusion transports. Major quasi-linear PDE (1) which occur in wide variety of physical problems are considered in our studies, which are Burgers-Huxley equation (BHE) and Burgers' equation.

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Mathematical modeling of nerve pulse propagation in nerve fibres and wall motion in liquid crystals is explained by BHE [16] whereas Burgers' equation explains the model of wave propagation in non-linear dissipative systems [3]. Various numerical and analytical methods for BHE have been found in the literature [1, 2, 4, 5, 6, 9, 10, 14]. High order finite difference methods for the solution of quasilinear parabolic equations have been discussed by Jain et al. [7, 8] and Mohanty et al. [12, 13]. Rashidinia et al. [15] have proposed non-polynomial cubic spline methods for the solution of parabolic equations. In the recent past, using the second order consistency condition, a polynomial cubic spline method for quasilinear parabolic PDE was proposed by Mohanty and Jain [11], and that method needs modification in order to solve singular parabolic PDE, which is a main drawback of that method. Using fourth order consistency condition, we have developed half-step cubic spline method which can solve singular parabolic PDE without any alteration in the scheme. To the authors' cognition, there is no polynomial cubic spline scheme of fourth-order accuracy has been seen in the literature so far.

Rest of the article is organized as follows: In section 2, cubic spline function and its properties on uniform mesh are given. Derivation of the implicit method is discussed in section 3. Unconditionally stability condition is proven for model linear problem in section 4. In section 5, we obtain the maximum absolute errors for BHE, Burgers' equation and Burgers' equation in polar coordinates and compare the numerical results with the results found in earlier research work. Concluding remarks of the article are presented in section 6.

2. Cubic spline function and its properties

For the approximate solution of the given PDE (1), we discretize the space interval $[0, 1]$ as $0 = x_0 < x_1 < .... < x_N < x_{N+1} = 1$. where N is a positive integer. The grid points of the interval is defined as $x_{l+1} = x_l + h,\ l = 0(1)N$, where h be spacing in x-direction and the time steps are given by $t_j = jk,\ j = 0(1)J$, where $k = t_{j+1} - t_j > 0,\ j = 0, 1, 2, ...$ be the spacing in t-direction. The half-step points are defined as $x_{l-1/2} = x_l - \frac{h}{2}$ and $x_{l+1/2} = x_l + \frac{h}{2},\ l = 1(1)N$. Let $W_l^j = w(x_l, t_j)$ be the exact solution value of $w(x, t)$ and is approximated by w_l^j.

A cubic spline polynomial of degree three interpolating the value w_l^j at jth-level is given by

$$S_j(x) = a_l^j + b_l^j(x - x_l) + c_l^j(x - x_l)^2 + d_l^j(x - x_l)^3,\ \ x_{l-1} \le x \le x_l,\ l = 1(1)N + 1,\ j > 0, \tag{2}$$

which satisfy the following conditions at jth-time level:

(i) In each subinterval $[x_{l-1}, x_l],\ S_j(x)$ coincides with a polynomial of degree three,

(ii) $S_j(x) \in C^2[0, 1]$, and

(iii) $S_j(x_l) = W_l^j,\ \ S_j(x_{l-1}) = W_{l-1}^j$.

We assume some notation of second order derivative of $S_j(x)$

$S''_j(x_l) = A_l^j = W_{xxl}^j, S''_j(x_{l\pm1}) = A_{l\pm1}^j = W_{xxl\pm1}^j$ and $S''_j(x_{l\pm1/2}) = A_{l\pm1/2}^j = W_{xxl\pm1/2,}^j\ l = 0(1)N + 1,\ j > 0$.

With the help of cubic spline properties, we get the coefficients

$$a_l^j = W_l^j, \quad b_l^j = \frac{W_l^j - W_{l-1}^j}{h} + \frac{h}{6}\left[A_l^j + 2A_{l-1/2}^j\right], \quad c_l^j = \frac{A_l^j}{2}, \quad d_l^j = \frac{A_l^j - A_{l-1/2}^j}{3h}.$$

Substituting the coefficients a_l^j, b_l^j, c_l^j, d_l^j, in the equation (2), we get the cubic spline function

$$\begin{aligned} S_j(x) = W_l^j &+ \left[\tfrac{W_l^j - W_{l-1}^j}{h} + \tfrac{h}{6}\left[A_l^j + 2A_{l-1/2}^j\right]\right](x - x_l) + \tfrac{A_l^j}{2}(x - x_l)^2 \\ &+ \tfrac{A_l^j - A_{l-1/2}^j}{3h}(x - x_l)^3, \quad x \in [x_{l-1}, x_l]. \end{aligned} \tag{3}$$

Similarly, we get

$$\begin{aligned} S_j(x) = W_l^j &+ \left[\tfrac{W_{l+1}^j - W_l^j}{h} - \tfrac{h}{6}\left[A_l^j + 2A_{l+1/2}^j\right]\right](x - x_l) + \tfrac{A_l^j}{2}(x - x_l)^2 \\ &+ \tfrac{A_{l+1/2}^j - A_l^j}{3h}(x - x_l)^3, \quad x \in [x_l, x_{l+1}]. \end{aligned} \tag{4}$$

From equation (3)-(4), we define the derivative of cubic spine function at $x_{l\pm1/2}$

$$S_j'(x_{l-1/2}) = \frac{W_l^j - W_{l-1}^j}{h} + \frac{h}{12}\left(A_{l-1/2}^j - A_l^j\right) = W_{xl-1/2}^j + O\left(h^4\right), \quad x \in [x_{l-1}, x_l], \tag{5}$$

$$S_j'(x_{l+1/2}) = \frac{W_{l+1}^j - W_l^j}{h} + \frac{h}{12}\left(A_l^j - A_{l+1/2}^j\right) = W_{xl+1/2}^j + O\left(h^4\right), \quad x \in [x_l, x_{l+1}], \tag{6}$$

Using the continuity of the first derivative, that is, $S_j'(x_l-) = S_j'(x_l+)$,we obtain the following consistency condition

$$W_{l+1}^j - 2W_l^j + W_{l-1}^j = \frac{h^2}{3}\left(A_{l+1/2}^j + A_l^j + A_{l-1/2}^j\right) + T_l^j, \tag{7}$$

where $T_l^j = O\left(h^6\right)$.

3. Derivation of the method

To formulate the method, we simply follow the approaches given by Mohanty [11].

At the grid point(x_l, t_j), let us define some notation

$$W_{pq} = \frac{\partial^{p+q} W}{\partial x^p \partial t^q}, \alpha_l^j = \left(\frac{\partial f}{\partial w}\right)_l^j, \beta_l^j = \left(\frac{\partial f}{\partial w_x}\right)_l^j, \gamma_l^j = \left(\frac{\partial f}{\partial w_t}\right)_l^j, \delta_l^j = \left(\frac{\partial f}{\partial t}\right)_l^j. \tag{8}$$

Partially differentiate the PDE (1) with respect to 't', we obtain

$$-\gamma_l^j W_{02} = \delta_l^j + W_{01}\alpha_l^j + W_{11}\beta_l^j - W_{21} \tag{9}$$

At the grid point(x_l, t_j), we can write the PDE (1) as

$$A_l^j = f\left(x_l, t_j, W_l^j, W_{xl}^j, W_{tl}^j\right) \tag{10}$$

Similarly,

$$A_{l\pm1/2}^j = f\left(x_{l\pm1/2}, t_j, W_{l\pm1/2}^j, W_{xl\pm1/2}^j, W_{tl\pm1/2}^j\right). \tag{11}$$

Since A_l^j and $A_{l\pm1/2}^j$contain the first derivative terms, then from the consistency condition (7) the cubic spline method for the parabolic equation (1) can be written as

$$\bar{W}_{l+1}^j - 2\bar{W}_l^j + \bar{W}_{l-1}^j = \frac{h_l^2}{3}\left(\hat{A}_{l+1/2}^j + \hat{A}_l^j + \hat{A}_{l-1/2}^j\right) + \hat{T}_l^j, \tag{12}$$

where $\hat{T}_l^j \equiv O\left(k^2h^2 + kh^4 + h^6\right)$,

for the development of above method (13), we use the following approximations:

$$\bar{t}_j = t_j + \theta k$$

$$\bar{W}_l^j = \theta W_l^{j+1} + (1-\theta)\, W_l^j = W_l^j + \theta k W_{01} + O\left(k^2\right),$$

$$\bar{W}_{l\pm1}^j = \theta W_{l\pm1}^{j+1} + (1-\theta)\, W_{l\pm1}^j = W_{l\pm1}^j + \theta k\left(W_{01} \pm hW_{11}\right) + O\left(k^2\right),$$

$$\bar{W}_{l\pm1/2}^j = \frac{1}{2}\left(\bar{W}_{l\pm1}^j + \bar{W}_l^j\right) = W_{l\pm1/2}^j + \theta k W_{01} + \frac{h^2}{8}W_{20} + O\left(k^2 + kh^2 + h^4\right),$$

$$\bar{W}_{tl}^j = \frac{1}{k}\left(W_l^{j+1} - W_l^j\right) = W_{01} + \frac{k}{2}W_{02} + O\left(k^2\right),$$

$$\bar{W}_{tl\pm1}^j = \frac{1}{k}\left(W_{l\pm1}^{j+1} - W_{l\pm1}^j\right) = W_{tl\pm1}^j + \frac{k}{2}W_{02} + O\left(kh + k^2\right),$$

$$\begin{aligned}\bar{W}_{tl\pm1/2}^j &= \frac{1}{2k}\left(W_{l\pm1}^{j+1} + W_l^{j+1} - W_{l\pm1}^j - W_l^j\right)\\ &= W_{tl\pm1/2}^j + \frac{k}{2}W_{02} + \frac{h^2}{8}W_{21} + O\left(k^2 + kh^2 + h^4\right),\end{aligned}$$

$$\bar{W}_{xl}^j = \frac{\bar{W}_{l+1}^j - \bar{W}_{l-1}^j}{2h} = W_{10} + \frac{h^2}{6}W_{30} + \theta k W_{11} + O\left(k^2 + h^4\right),$$

$$\bar{W}_{xl+1/2}^j = \frac{\bar{W}_{l+1}^j - \bar{W}_l^j}{h} = W_{xl+1/2}^j + \frac{h^4}{24}W_{30} + \theta k W_{11} + O\left(k^2 + kh^2 + h^4\right),$$

$$\bar{W}_{xl-1/2}^j = \frac{\bar{W}_l^j - \bar{W}_{l-1}^j}{h} = W_{xl-1/2}^j + \frac{h^2}{24}W_{30} + \theta k W_{11} + O\left(k^2 + kh^2 + h^4\right),$$

$$\bar{W}_{xxl}^j = \frac{\bar{W}_{l+1}^j - 2\bar{W}_l^j + \bar{W}_{l-1}^j}{h^2} = W_{xxl}^j + \theta k W_{21} + O(k^2 + h^2).$$

where $'\theta'$ is a parameter to be determined. By the help of above approximations, we can simplify the following approximations:

$$\begin{aligned}\bar{A}_l^j &= f\left(x_l, \bar{t}_j, \bar{W}_l^j, \bar{W}_{xl}^j, \bar{W}_{tl}^j\right)\\ &= A_l^j + \theta k\left(\delta_l^j + W_{01}\alpha_l^j + W_{11}\beta_l^j\right) + \frac{k}{2}W_{02}\gamma_l^j + \frac{h^2}{6}W_{30}\beta_l^j + O\left(k^2 + kh^2 + h^4\right),\end{aligned}$$

$$\bar{A}^j_{l\pm1/2} = f\left(x_{l\pm1/2}, \bar{t}_j, \bar{W}^j_{l\pm1/2}, \bar{W}^j_{xl\pm1/2}, \bar{W}^j_{tl\pm1/2}\right)$$
$$= A^j_{l\pm1/2} + \theta k\left(\delta^j_l + W_{01}\alpha^j_l + W_{11}\beta^j_l\right) + \tfrac{k}{2}W_{02}\gamma^j_l$$
$$+\tfrac{h^2}{24}\left(3W_{20}\alpha^j_l + W_{30}\beta^j_l + 3W_{21}\gamma^j_l\right) + O\left(k^2 + kh^2 + h^4\right).$$

From the properties of spline function given by (5) and (6), we define the approximations:

$$\hat{W}^j_{xl+1/2} = \frac{\bar{W}^j_{l+1} - \bar{W}^j_l}{h} + \frac{h}{12}\left(\bar{A}^j_l - \bar{A}^j_{l+1/2}\right), \quad (13)$$

$$\hat{W}^j_{xl-1/2} = \frac{\bar{W}^j_l - \bar{W}^j_{l-1}}{h} - \frac{h}{12}\left(\bar{A}^j_l - \bar{A}^j_{l-1/2}\right). \quad (14)$$

By the help of the approximations $\bar{W}^j_l$,$\bar{W}^j_{l\pm1}$,$\bar{A}^j_l$,$\bar{A}^j_{l\pm1/2}$, and simplifying (13)-14, we obtain

$$\hat{W}^j_{xl+1/2} = W^j_{xl+1/2} + \theta kW_{11} + O\left(k^2 + kh^2 + h^4\right), \quad (15)$$

$$\hat{W}^j_{xl-1/2} = W^j_{xl-1/2} + \theta kW_{11} + O\left(k^2 + kh^2 + h^4\right). \quad (16)$$

Now, we need $O\left(k^2 + kh^2 + h^4\right)$-approximations for W^j_l, W^j_{xl} and $O\left(k^2 + h^4\right)$-approximation for $\bar{W}^j_{tl}$. Let

$$\hat{W}^j_l = \bar{W}^j_l + ah^2\bar{W}^j_{xxl}, \quad (17)$$

$$\hat{W}^j_{xl} = \bar{W}^j_{xl} + bh\left(\bar{A}^j_{l+1/2} - \bar{A}^j_{l-1/2}\right), \quad (18)$$

$$\hat{W}^j_{tl} = \bar{W}^j_{tl} + c\left(\bar{W}^j_{tl+1} - 2\bar{W}^j_{tl} + \bar{W}^j_{tl-1}\right). \quad (19)$$

where a, b and c are parameters to be determined. With the help of the approximation $\bar{W}^j_{xl}$,$\bar{A}^j_{l\pm1/2}$, and from (18) we obtain

$$\hat{W}^j_{xl} = W^j_{xl} + \theta kW_{11} + \frac{h^2}{6}\left(1 + 6b\right)W_{30} + O\left(k^2 + kh^2 + h^4\right),$$

Equating the coefficient of h^2 to zero in above equation, we obtain $b = -\frac{1}{6}$ and the equation reduces to

$$\hat{W}^j_{xl} = W^j_{xl} + \theta kW_{11} + O\left(k^2 + kh^2 + h^4\right), \quad (20)$$

Similarly, simplifying (17) and (19), we obtain

$$\hat{W}^j_l = W^j_l + \theta kW_{01} + ah^2W_{20} + O\left(k^2 + kh^2 + h^4\right), \quad (21)$$

$$\hat{W}^j_{tl} = W^j_{tl} + \frac{k}{2}W_{02} + ch^2W_{21} + O\left(k^2 + h^4\right). \quad (22)$$

Further, we define

$$\hat{A}^j_l = f\left(x_l, \bar{t}_j, \hat{W}^j_l, \hat{W}^j_{xl}, \hat{W}^j_{tl}\right), \quad (23)$$

$$\hat{A}^j_{l\pm1/2} = f\left(x_{l\pm1/2}, \bar{t}_j, \bar{W}^j_{l\pm1/2}, \hat{W}^j_{xl\pm1/2}, \bar{W}^j_{tl\pm1/2}\right). \quad (24)$$

By the help of the approximations $\bar{t}_j, \bar{W}^j_{l\pm1/2}$, $\bar{W}^j_{tl\pm1/2}$, using the equation (15)-(16), and (20), we simplify the equation (23)-(24)

$$\hat{A}^j_l = A^j_l + \theta k \left(\delta^j_l + W_{01}\alpha^j_l + W_{11}\beta^j_l\right) + \frac{k}{2}W_{02}\gamma^j_l$$

$$+ \frac{h^2}{2}\left(2aW_{20}\alpha^j_l + 2cW_{21}\gamma^j_l\right) + O\left(k^2 + kh^2 + h^4\right),$$

$$\hat{A}^j_{l\pm1/2} = A^j_{l\pm1/2} + \theta k \left(\delta^j_l + W_{01}\alpha^j_l + W_{11}\beta^j_l\right) + \frac{k}{2}W_{02}\gamma^j_l$$

$$+ \frac{h^2}{8}\left(W_{20}\alpha^j_l + W_{21}\gamma^j_l\right) + O\left(k^2 + kh^2 + h^4\right),$$

Using the approximation $\bar{W}^j_l$, $\bar{W}^j_{l\pm1}$, $\hat{A}^j_l$, $\hat{A}^j_l$, from (12), we obtain

$$W^j_{l+1} - 2W^j_l + W^j_{l-1}$$

$$= \frac{h^2}{3}[A^j_{l+1/2} + A^j_l + A^j_{l-1/2} + 3\theta k(\delta^j_l + W_{01}\alpha^j_l + W_{11}\beta^j_l - W_{21})$$

$$+ \frac{3k}{2}W_{02}\gamma^j_l + \frac{h^2}{4}(1+4a)W_{20}\alpha^j_l + \frac{h^2}{4}(1+4c)W_{21}\gamma^j_l] + \hat{T}^j_l, \tag{25}$$

Now with the help of the consistency condition (7) and the relation (9), and using the equations (21)-(22), and (25), we obtain the local truncation error

$$\hat{T}^j_l = -\frac{h^2}{3}\left[3\left(\frac{1}{2} - \theta\right)kW_{02}\gamma^j_l + \frac{h^2}{8}(1+4a)W_{21}\alpha^j_l + \frac{h^2}{4}(1+4c)W_{30}\gamma^j_l\right]$$

$$+ O\left(k^2h^2 + kh^4 + h^6\right). \tag{26}$$

The proposed cubic spline method (12) to be of $O\left(k^2 + kh^2 + h^4\right)$, the coefficients of kh^2 and h^4 in (26) must be zero.

Thus we obtain $\theta = \frac{1}{2}$, $a = -\frac{1}{4}$, $c = -\frac{1}{4}$ and the local truncation error reduces to $\hat{T}^j_l \equiv O\left(k^2h^2 + kh^4 + h^6\right)$.

4. Application and stability consideration of the method

Now let us consider the quasilinear singular parabolic PDE in polar coordinates

$$\frac{1}{R_e}\left(w_{rr} + \frac{\tau}{r}w_r - \frac{\tau}{r^2}w\right) = w_t + ww_r + g(r,t), 0 < r < 1,\ t > 0, \tag{27}$$

where $R_e > 0$ denotes the Reynolds number. For $\tau = 1$ and 2, the above equation describes Burgers' equation in cylindrical and spherical symmetry, respectively. It represents the mathematical model of shock wave, turbulence and boundary layer in fluid dynamics. It establishes the similar characteristics with 1D Navier-Stokes equation because of the presence of viscosity and convection term. Due to the large fluctuations around the singularity in the numerical solution of Burgers' equation, half-step implicit cubic spline technique plays a crucial role for getting the high accurate numerical solution.

Re-writing equation (27) as

$$\varepsilon w_{rr} = w_t + Q(r)w_r + ww_r + S(r)w + g(r,t), \tag{28}$$

where $R_e = \varepsilon^{-1} > 0$ represents a Reynolds number and $Q(r) = \frac{-\tau\varepsilon}{r}$, $S(r) = \frac{\tau\varepsilon}{r^2}$.

Replacing the variable x by r and applying the method (12) to the differential equation (28), we obtain

$$\varepsilon\left(\bar{W}_{l+1}^{j}-2\bar{W}_{l}^{j}+\bar{W}_{l-1}^{j}\right)=\frac{h^{2}}{3}\left(\hat{A}_{l+1/2}^{j}+\hat{A}_{l}^{j}+\hat{A}_{l-1/2}^{j}\right)+\hat{T}_{l}^{j}, \tag{29}$$

where

$$\hat{A}_{l}^{j}=\hat{W}_{tl}^{j}+Q_{l}\hat{W}_{rl}^{j}+\hat{W}_{l}^{j}\hat{W}_{rl}^{j}+S_{l}\hat{W}_{l}^{j}+\bar{g}_{l}^{j},$$

$$\hat{A}_{l\pm1/2}^{j}=\bar{W}_{tl\pm1/2}^{j}+Q_{l\pm1/2}\hat{W}_{rl\pm1/2}^{j}+\bar{W}_{l\pm1/2}^{j}\hat{W}_{rl\pm1/2}^{j}+S_{l\pm1/2}\bar{W}_{l\pm1/2}^{j}+\bar{g}_{l\pm1/2}^{j},$$

where $\bar{W}_{l\pm1/2}^{j}$,$\bar{W}_{tl\pm1/2}^{j}$, $\hat{W}_{l}^{j}$, $\hat{W}_{rl}^{j}$,$\hat{W}_{rl\pm1/2}^{j}$ and $\hat{W}_{tl}^{j}$ are defined in section 3 and $Q_l = Q(r_l)$, $Q_{l\pm1/2} = Q(r_{l\pm1/2})$, $S_l = S(r_l)$, $S_{l\pm1/2} = S(r_{l\pm1/2})$, $\bar{g}_l = g\left(r_l, t_j + \frac{k}{2}\right)$, $\bar{g}_{l\pm1/2} = g\left(r_{l\pm1/2}, t_j + \frac{k}{2}\right)$.

We observe that, the method (29) is of fourth-order accuracy for the numerical solution of PDE (27) and is independent from the terms $1/(r_{l\pm1})$, i.e. there is no need of any other points to handle the singular problems. Hence, we can directly solve PDE (27) for $l = 1(1)N$; $j = 0, 1, 2, ...$ in the given domain.

To discuss the stability of the method, we consider the convection-diffusion equation

$$\nu w_{xx}=w_{t}+\rho w_{x},0<x<1,\ t>0 \tag{30}$$

where the coefficients ρ and $\nu > 0$ are constants and represents the convective velocity and the diffusivity respectively. Applying the method (12) to the differential equation (30), we obtain the following scheme

$$\left[1+\frac{1}{12}\left(1-6\lambda\nu-2\lambda\nu R^{2}\right)\delta_{x}^{2}-\frac{R}{12}\left(1-6\nu\lambda\right)\left(2\mu_{x}\delta_{x}\right)\right]w_{l}^{j+1}$$

$$=\left[1+\frac{1}{12}\left(1+6\lambda\nu+2\lambda\nu R^{2}\right)\delta_{x}^{2}-\frac{R}{12}\left(1+6\nu\lambda\right)\left(2\mu_{x}\delta_{x}\right)\right]w_{l}^{j}, \tag{31}$$

where $R = \frac{\rho h}{2\nu}$ and $\lambda = k/h^2$. Now, we consider the von Neumann stability method to establish the stability of the scheme (31) for which we consider the periodic data $u(0, t) = u(1, t)$. we define the error term $\varepsilon_l^j = \xi^j e^{i\mu l}$ at the grid point (x_l, t_j), where μ is a real number and ξ is a complex number. Now, put the value of ε_l^j into error equation of the scheme (31), we get the amplification factor ξ as

$$\xi=\frac{\left(1-\frac{1}{3}(1+6\nu\lambda+2\nu\lambda R^{2})\sin^{2}\frac{\eta}{2}-\frac{iR}{6}(1+6\nu\lambda)\sin\eta\right)}{\left(1-\frac{1}{3}(1-6\lambda\nu-2\nu\lambda R^{2})\sin^{2}\frac{\eta}{2}-\frac{iR}{6}(1-6\nu\lambda)\sin\eta\right)}=\frac{1+(C+iD)}{1-(C+iD)}, \tag{32}$$

where

$$C+iD=\frac{-2\lambda\nu\left[\left(1+\frac{R^{2}}{3}\right)\sin^{2}\frac{\eta}{2}+\frac{iR}{2}\sin\eta\right]}{\left[1-\frac{1}{3}\sin^{2}\frac{\eta}{2}-\frac{iR}{6}\sin\eta\right]}.$$

For stability, it is required that $|\xi|^2 \leq 1$. Imposing this condition on (32) yields $C \leq 0$, which satisfied for all variable angle η. Hence the scheme (31) is unconditionally stable.

5. Numerical Results

To check the efficiency and usefulness of the scheme, we solve BHE, Burgers' equation and Burgers equation in polar coordinates. The analytical solution (exact solution) is known as a test procedure. From the analytical solution of the problem, we can obtain the initial and boundary conditions for that problem. We use Newton-Raphson method to obtain the numerical solution of time dependent quasilinear PDE. In each example, we choose $\boldsymbol{0}$ as the initial guess.

Example 1. (Burgers-Huxley equation): The one-dimensional viscous Burgers-Huxley equation described in [3, 16] is given by the following form:

$$\varepsilon w_{xx} = w_t + \alpha w w_x + \beta(w^3 + \gamma_0 w^2 + \gamma w),\ a < x < b,\ t > 0$$

where $w = w(x,t)$ is sufficiently differentiable function, $\varepsilon > 0$ is a small positive parameter, $\alpha$is real parameter, $\beta \geq 0,\ \gamma \in (0,1)$ and $\gamma_0 = -(1+\gamma)$. The exact solution (See [10]) is given by

$$w(x,t) = \frac{1}{2}[1 + \tanh(c_1(x - c_2 t))],\ \ t \geq 0,$$

where

$$c_1 = \frac{-\alpha - \sqrt{\alpha^2 + 8\beta}}{8},\ c_2 = \frac{\alpha}{2} - \frac{(1-2\gamma)\left(\alpha - \sqrt{\alpha^2+8\beta}\right)}{4}.$$

In this example, we choose the interval $[a,b]$ as $[0,1]$ and $\beta = 1$. The maximum absolute errors are reported in Table 1 at $t = 1.0$ for different values of α and γ. The graphical results are depicted at $t = 1$ in Fig. 1.

Example 2. (Burgers' equation) In this example, we studied the problem considered in example 1 with the parameters $\alpha = 1$ and $\beta = 0$. The exact solution (See [8]) is given by

$$w(x,t) = \frac{2\varepsilon\pi \sin(\pi x)\exp(-\varepsilon\pi^2 t)}{2 + \cos(\pi x)\exp(-\varepsilon\pi^2 t)}.$$

where ε is the coefficient of viscosity and $R_e = \varepsilon^{-1} > 0$ is the Reynolds Number. In this case, we select the interval $[a,b]$ as $[0,1]$. The maximum absolute errors are reported in Table 2 at $t = 1.0$ for different values of R_e.. In Fig. 2, we portrayed the visual comparison of numerical and exact solution at $t = 1$.

Example 3. (Burgers' equation in polar coordinates): In this example, we solve the problem (28) with the exact solution $w(r,t) = e^{-t}\sinh r$. In this case; we select the parameters $\tau = 1$ and 2 with the time spacing $k = 1/100$. Maximum absolute errors at $t =$ 1 are presented in Table 3. In Fig. 3, we portrayed the two dimensional visual comparison between numerical and exact solution at $t = 1$.

6. Concluding Remarks

In this manuscript, we present a new fourth-order accurate implicit numerical method based on cubic spline approximations for the numerical solution of time-dependent quasi-linear parabolic PDE in one spatial dimension. The stability consideration for one-dimensional linear model problem has been demonstrated. The accuracy and applicability of the proposed method has been tested on BHE and Burgers' equation in polar coordinates. These have been verified by the maximum absolute error given in tables. It can be seen from table 1 and 2, fourth-order convergence have been achieved. We also see the proposed method works well for high Reynolds number.

$N+1$	Proposed Method (12)		Method given in [2]	
	$\alpha = 5, \gamma = 0.85$	$\alpha = 3, \gamma = 0.5$	$\alpha = 5, \gamma = 0.85$	$\alpha = 3, \gamma = 0.5$
8	1.6011(-05)	1.2921(-05)	2.2121(-04)	1.7151(-04)
16	1.0031(-06)	8.0703(-07)	5.0308(-05)	4.1196(-05)
32	6.1135(-08)	5.0420(-08)	1.2278(-05)	1.0147(-05)
64	3.8419(-09)	3.0595(-09)	3.0526(-06)	2.5274(-06)

Table 1. Table 1: Example 1: Maximum absolute errors at $t = 1$ with $\beta = 1$.

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References

[1] Bataineh, A. S., Noorani, M. S. M. and Hashim, I., (2009), Analytical treatment of generalized Burgers-Huxley equation by homotopy analysis method.Bull. Malaya Math. Sci. Soc. 32, pp. 233–243.

[2] Bratsos, A. G., (2011), A fourth order improved numerical scheme for the generalized Burgers-Huxley equation. Am. J. Comput. Math. 1, pp. 152–158.

[3] Burgers, J. M., (1948), A mathematical model illustrating the theory of turbulence. Adv. Appl. Mech. 1, pp. 171–199.

[4] Gao, H. and Zhao, R., (2010), New exact solutions to the generalized Burgers-Huxley equation, Appl. Math. Comput. 217, pp. 1598–1603.

[5] Hashim, I., Noorani, M. S. M. and Batiha, B., (2006), A note on the adomian decomposition method for the generalized Huxley equation, Appl. Math. Comput. 181, pp. 1439–1445.

[6] Ismail, H. N. A., Raslan, K. and Rabboh, A. A. A., (2004), Adomian decomposition method for Burgers-Huxley and Burgers-Fisher equations, Appl. Math. Comput. 159, pp. 291–301.

[7] Jain, M. K., Jain R. K. and Mohanty, R. K., (1990), Fourth order difference method for the one-dimensional general quasi-linear parabolic partial differential equation, Numer. Meth. Partial Diff. Eqn. 6, pp. 311–319.

[8] Jain, M. K., Jain, R. K. and Mohanty, R. K., (1990), High order difference methods for system of 1-D non-linear parabolic partial differential equations. Int. J. Comput. Math. 37, pp. 105–112.

[9] Javidi, M., (2006), A numerical solution of the generalized Burgers-Huxley equation by spectral collocation method. Appl. Math. Comput. 178, pp. 338–344.

[10] Mohanty, R. K., Dai, W. and Liu, D., (2015), Operator compact method of accuracy two in time and four in space for the solution of time independent Burgers-Huxley equation, Numer. Algor. 70, pp. 591–605.

[11] Mohanty, R. K. and Jain, M. K., (2009), High-accuracy cubic spline alternating group explicit methods for 1D quasilinear parabolic equations, Int. J. Comput. Math. 86, pp. 1556–1571.

[12] Mohanty, R. K. and Sharma, S., (2017), High accuracy quasi-variable mesh method for the system of 1D quasi-linear parabolic partial differential equations based on off-step spline in compression approximations, Adv. Diff. Eqn., 2017:212.

[13] Mohanty, R. K. and Sharma, S., (2019) A new two-level implicit scheme for the system of 1D quasi-linear parabolic partial differential equations using spline in compression approximations, Differ. Equ. Dyn. Syst. 27, pp. 327–356.

[14] Molabahrami, A. and Khami, F., (2009), The homotopy analysis method to solve the Burgers-Huxley equation, Nonlinear Anal. Real World Appl. 10, pp. 589–600.

[15] Rashidinia, J. and Mohammadi, R., (2008), Non-polynomial cubic spline methods for the solution of parabolic equations, Int. J. Comput. Math. 85, pp. 843–850.

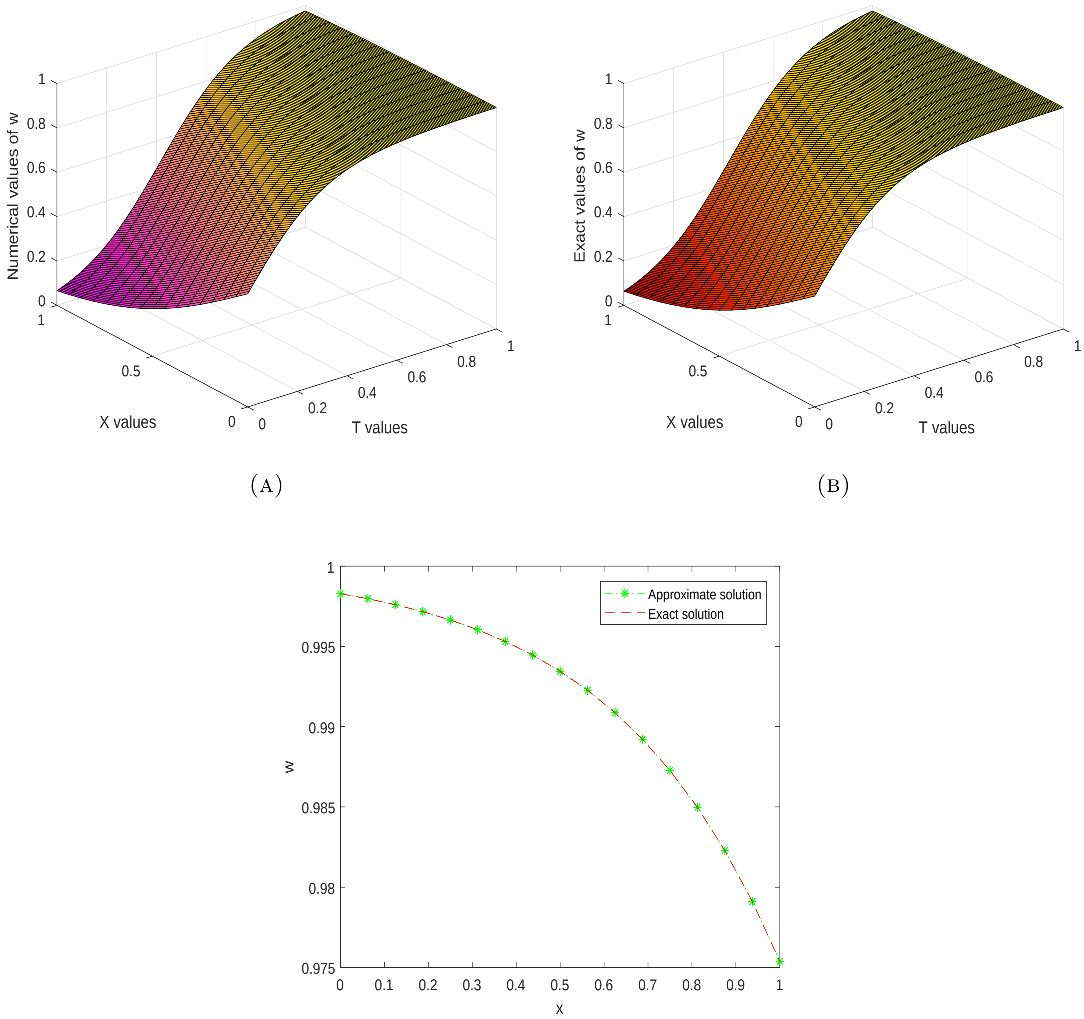
[16] Wang, X., (1985), Nerve propagation and wall in liquid crystals, Phys. Lett. 112A, pp. 402–406.

$N+1$	Proposed Method (12)			Method given in [2]		
	$R_e = 10^2$	$R_e = 10^4$	$R_e = 10^6$	$R_e = 10^2$	$R_e = 10^4$	$R_e = 10^6$
8	1.7214(-05)	3.8844(-09)	4.1139(-13)	4.1061(-04)	8.3710(-08)	8.4373(-12)
16	1.0144(-06)	2.4501(-10)	2.5384(-14)	1.1067(-04)	2.2489(-08)	2.2700(-12)
32	6.2387(-08)	1.5315(-11)	3.1595(-15)	2.7680(-05)	5.7437(-09)	5.7920(-13)
64	3.8844(-09)	1.0035(-11)	9.8998(-16)	6.9473(-06)	1.4564(-09)	1.4698(-13)

TABLE 2. Table 2: Example 2: Maximum absolute errors at $t = 1$ with $\alpha = 1,\ \beta = 0$.

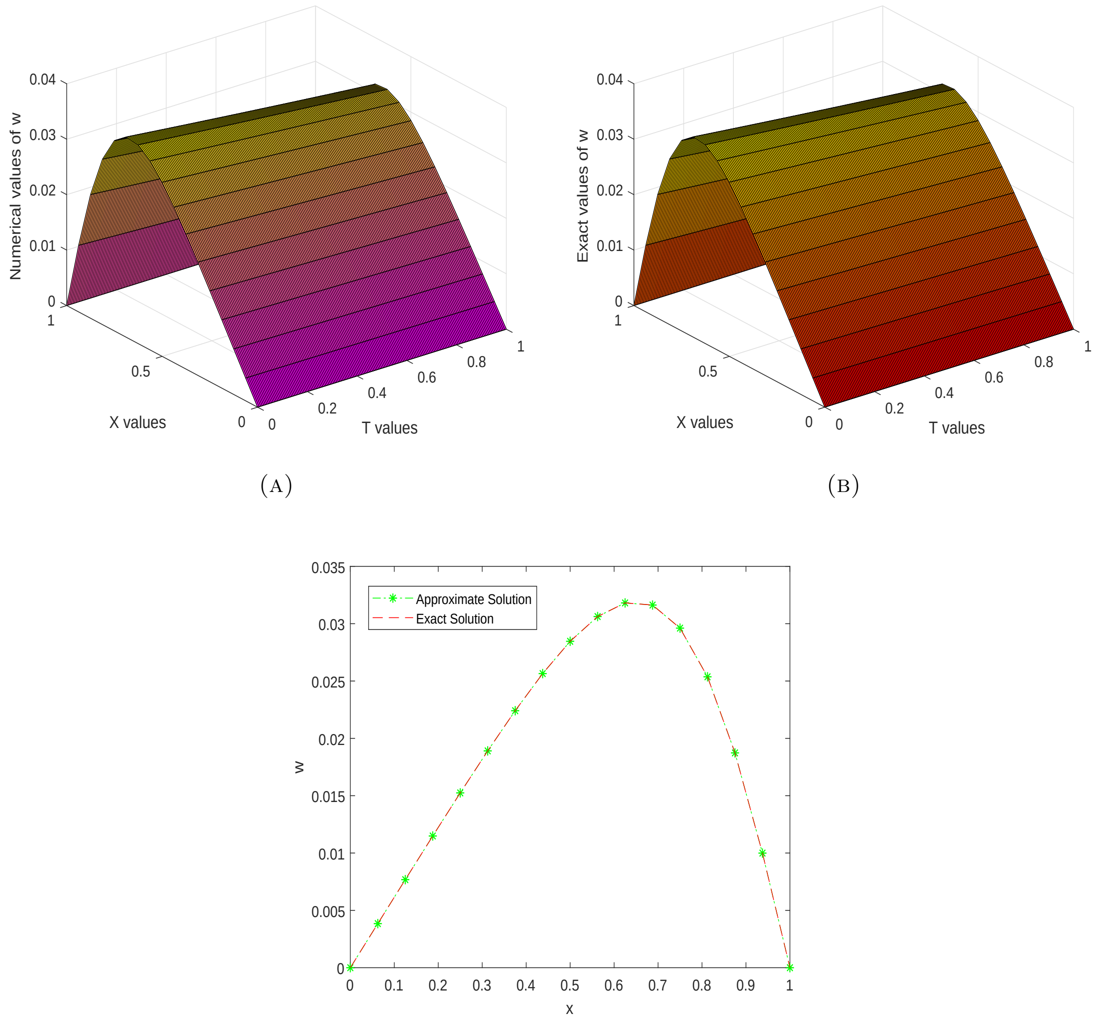
$N+1$	$\tau = 1$		$\tau = 2$	
	$R_e = 10$	$R_e = 100$	$R_e = 10$	$R_e = 100$
50	1.7780(-06)	4.6738(-06)	1.5605(-06)	4.6672(-06)
60	1.5303(-06)	4.6710(-06)	1.3393(-06)	4.6655(-06)
70	1.3135(-06)	4.6651(-06)	1.1353(-06)	4.6602(-06)
80	1.0137(-06)	4.6712(-06)	8.5868(-07)	4.6623(-06)
90	6.7447(-07)	4.6703(-06)	5.4159(-07)	4.6626(-06)

TABLE 3. Table 3: Example 3: Maximum absolute error at $t = 1,\ k = 0.01$.



(A) (B) (C)

Figure 1. Example 1: The graph of exact and numerical solutions for the values $\alpha = 5$, $\beta = 1.0$, $\gamma = 0.85$ at $t = 1$ (a)Numerical Solution,(b)Exact Solution and(c)Numerical vs Exact solution at $t = 1$



(A) (B) (C)

FIGURE 2. Example 2: The graph of exact and numerical solutions for the values $R_e = 100$, $N + 1 = 16$ at $t = 1$ (a)Numerical Solution,(b)Exact Solution and(c)Numerical vs Exact solution at $t = 1$

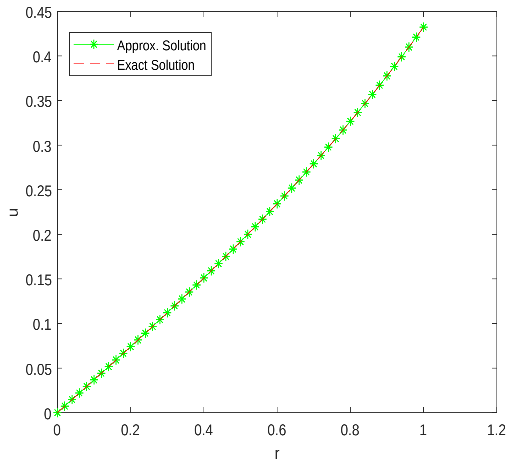


FIGURE 3. Example 3: The graphs of numerical and exact solution for the values $k = 1/100$, $N + 1 = 50$, and $R_e = 100$ at $t = 1$.

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