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NEIGHBOURHOODS OF A CERTAIN SUBCLASS OF STRONGLY STARLIKE FUNCTIONS

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ABSTRACT. In this paper we introduce and study a new subclass of strongly starlike functions of order α defined by convolution structure. We investigate neighbourhoods and coefficient bounds of this class.

Keywords: strongly starlike, Hadamard product, neighborhood.

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1. INTRODUCTION

Let A be the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions and satisfy the following usual normalization condition $f(0) = f'(0) - 1 = 0$. We denote S by the subclass of A consisting of functions which are all univalent in E. Let $ST(\alpha), 0 < \alpha \leq 1$, be denoted the class of functions in A that are starlike of order α and CV be denote the class of convex functions. Then we have the classical analytic characterizations

$$f \in ST(\alpha) \Leftrightarrow Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, z \in E \tag{1}$$

and

$$f \in CV \Leftrightarrow Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, z \in E. \tag{2}$$

Any $f \in A$ has the Taylor's expansion $f(z) = z + a_2 z^2 + \cdots$ in E. The convolution or Hadamard product of $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$ is defined as

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$(f*g)(z) = z + \sum\limits_{n=2}^{\infty} a_n b_n z^n.$

Clearly $f(z) * \frac{z}{(1-z)^2} = zf'(z)$ and $f(z) * \frac{z}{1-z^2} = \frac{f(z)-f(-z)}{2}$.

Strongly starlike and strongly convex functions were introduced and discussed by Brannan and Kirwan [1] and also by Stankiewincz [4] and [5]. The notion of $\delta-$neighbourhood was introduced by Ruscheweyh [2]. In 1973, Rusheweyh and Sheil-Small [3] proved the Polya-Schoenberg conjecture that the class of convex functions is preserved under convolution.

In this paper we introduce the class $STS_s(\alpha), 0 < \alpha \leq 1$, satisfying the condition $\left|arg\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)\right| < \frac{\alpha\pi}{2}$. We study neighbourhoods of this class and also prove a necessary and sufficient condition in terms of convolution for a function f to be $STS_s(\alpha)$. Furthermore, it is shown that class $STS_s(\alpha)$ is closed under convolution with function f which are convex univalent in E.

Definition 1.1. *For $\delta \geq 0$, the $\delta-$neighbourhood of $f(z) \in A$ is defined by*

$$N_\delta(f) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n : \sum_{n=2}^{\infty} n|a_n - b_n| \leq \delta \right\},\ z \in E. \tag{3}$$

To prove our results we need the following lemma.

Lemma 1.1. [3] *If ϕ is a convex univalent function with $\phi(0) = \phi'(0) - 1$ in the unit disk E and g is starlike univalent in E then for each analytic function F in E, the image of E under $\frac{(\phi * Fg)(z)}{(\phi * g)(z)}$ is a subset of the convex hull of $F(E)$.*

2. Main Results

In this section we give the definitions of $STS_s(\alpha), 0 < \alpha \leq 1$ and study the neighbourhoods of this class and also prove a necessary and sufficient condition in terms of convolution for a function f to be $STS_s(\alpha)$. Furthermore, it is shown that class $STS_s(\alpha)$ is closed under convolution with function f which are convex univalent in E.

Definition 2.1. *A function $f(z)$ is said to be in the class $STS_s(\alpha), 0 < \alpha \leq 1$ if all $z \in E$*

$$\left| arg\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)\right| < \frac{\alpha\pi}{2}. \tag{4}$$

$f \in STS_s(\alpha)$ means that the image of E under $w = \frac{2zf'(z)}{f(z)-f(-z)}$ lies in the region $\Omega = |arg\ w| < \frac{\alpha\pi}{2}$, equivalently $\frac{2zf'(z)}{f(z)-f(-z)} \neq t\ e^{\pm i\frac{\alpha\pi}{2}}, t \in \mathbb{R}^+$.

Now let us give a characterization for a function $f \in A$ to be in $STS_s(\alpha)$ by means of convolution.

Definition 2.2. *The class of all analytic functions $STS'_s(\alpha), 0 < \alpha \leq 1$ is defined in E by*

$$H(z) = \frac{1}{1 - t\ e^{\pm i\frac{\alpha\pi}{2}}}\left[\frac{z}{(1-z)^2} - t\ e^{\pm i\frac{\alpha\pi}{2}}\left(\frac{z}{1-z^2}\right)\right], t \in \mathbb{R}^+. \tag{5}$$

Theorem 2.1. *Let $0 < \alpha \leq 1$ and $z \in E$. Then $f \in STS_s(\alpha)$ if and only if $\frac{(f*H)(z)}{z} \neq 0$, for all $H(z) \in STS'_s(\alpha)$.*

Proof. Let us assume that $\frac{(f*H)(z)}{z} \neq 0,\ z \in E$ and for all $H(z) \in STS'_s(\alpha)$. Then we have

$$\begin{aligned}\frac{(f*H)(z)}{z} &= \frac{1}{z(1-t\ e^{\pm i\frac{\alpha\pi}{2}})}\left[f(z)*\frac{z}{(1-z)^2}-(t\ e^{\pm i\frac{\alpha\pi}{2}})\left(f(z)*\frac{z}{1-z^2}\right)\right]\\ &= \frac{1}{z(1-t\ e^{\pm i\frac{\alpha\pi}{2}})}\left[zf'(z)-t\ e^{\pm i\frac{\alpha\pi}{2}}\left(\frac{f(z)-f(-z)}{2}\right)\right]\neq 0, t\in\mathbb{R}^+.\end{aligned}$$

Equivalently $\frac{2zf'(z)}{f(z)-f(-z)} \neq t\ e^{\pm i\frac{\alpha\pi}{2}}$. But $t \in \mathbb{R}^+$ then $t\ e^{\pm i\frac{\alpha\pi}{2}}$ covers the half lines $arg\ w = \pm\frac{\alpha\pi}{2}$.
Then $\frac{2zf'(z)}{f(z)-f(-z)} = 1$ at $z = 0$. Hence $\frac{2zf'(z)}{f(z)-f(-z)} \in \Omega = \{z \in C : |arg\ w| < \frac{\alpha\pi}{2}$ or $f \in STS_s(\alpha)$.

Conversely let us assume that $f \in STS_s(\alpha)$. Then $\frac{2zf'(z)}{f(z)-f(-z)} \neq t\ e^{\pm i\frac{\alpha\pi}{2}}$.
Or equivalently $f(z) * \left[\frac{z}{(1-z)^2} - t\ e^{\pm i\frac{\alpha\pi}{2}}\left(\frac{z}{1-z^2}\right)\right] \neq 0$, for $z \neq 0$.
Normalizing the function with in the brackets, we get $\frac{(f*H)(z)}{z} \neq 0$ in E, where $H(z)$ is the function defined (5) □

Lemma 2.1. *Let* $H(z) = z + \sum_{n=2}^{\infty} c_n z^n \in STS'_s(\alpha), 0 < \alpha \leq 1$. *Then*

$$|c_n| \leq \frac{n}{sin(\frac{\alpha\pi}{2})}.$$

Proof. Let $H(z) \in STS'_s(\alpha)$. Then by Definition 2.2, for $t \in \mathbb{R}^+$,

$$\begin{aligned}H(z) &= \frac{1}{1-t\ e^{\pm i\frac{\alpha\pi}{2}}}\left[\frac{z}{(1-z)^2}-t\ e^{\pm i\frac{\alpha\pi}{2}}\left(\frac{z}{1-z^2}\right)\right]\\ &= \frac{1}{1-t\ e^{\pm i\frac{\alpha\pi}{2}}}\left[(z+2z^2+\cdots)-(t\ e^{\pm i\frac{\alpha\pi}{2}})(z+z^3+\cdots)\right]\\ &= z+\sum_{n=2}^{\infty}c_nz^n.\end{aligned}$$

Then comparing the coefficients on either side, we get

$$c_n = \begin{cases} \frac{n}{1-t\ e^{\pm i\frac{\alpha\pi}{2}}}, & \text{when } n \text{ is an even}\\ \\ \frac{n-t\ e^{\pm i\frac{\alpha\pi}{2}}}{1-t\ e^{\pm i\frac{\alpha\pi}{2}}}, & \text{when } n \text{ is an odd}\end{cases}$$

case (i): If n is an even then

$$\begin{aligned}
|c_n|^2 &= \left|\frac{n}{1-t\ e^{\pm i\frac{\alpha\pi}{2}}}\right|^2 = \frac{n^2}{(1-tcos(\frac{\alpha\pi}{2}))^2+(tsin(\frac{\alpha\pi}{2}))^2}\\
&= \frac{n^2}{1-2tcos(\frac{\alpha\pi}{2})+t^2}\\
&= 1+\frac{n^2-1+2tcos(\frac{\alpha\pi}{2})-t^2}{1-2tcos(\frac{\alpha\pi}{2})+t^2}\\
&\le \max_t\left[1+\frac{n^2-1}{1-2tcos(\frac{\alpha\pi}{2})+t^2}\right],\ \text{since } t\ge 0\\
&\le \left[1+\frac{n^2-1}{sin^2(\frac{\alpha\pi}{2})}\right] = \frac{n^2-cos^2(\frac{\alpha\pi}{2})}{sin^2(\frac{\alpha\pi}{2})}.
\end{aligned}$$

Therefore $|c_n| \le \dfrac{n}{sin(\frac{\alpha\pi}{2})}$.

case (ii): If n is an odd then

$$\begin{aligned}
|c_n|^2 &= \left|\frac{n-t\ e^{\pm i\frac{\alpha\pi}{2}}}{1-t\ e^{\pm i\frac{\alpha\pi}{2}}}\right|^2 = \frac{(n-tcos(\frac{\alpha\pi}{2}))^2+(tsin(\frac{\alpha\pi}{2}))^2}{(1-tcos(\frac{\alpha\pi}{2}))^2+(tsin(\frac{\alpha\pi}{2}))^2}\\
&= \frac{n^2-2ntcos(\frac{\alpha\pi}{2})+t^2}{1-2tcos(\frac{\alpha\pi}{2})+t^2}\\
&= 1+\frac{n^2-1+2t(n-1)cos(\frac{\alpha\pi}{2})}{1-2tcos(\frac{\alpha\pi}{2})+t^2}\\
&\le \max_t\left[1+\frac{n^2-1}{1-2tcos(\frac{\alpha\pi}{2})+t^2}\right],\ \text{since } t\ge 0\\
&= \left[1+\frac{n^2-1}{sin^2(\frac{\alpha\pi}{2})}\right] = \frac{n^2-cos^2(\frac{\alpha\pi}{2})}{sin^2(\frac{\alpha\pi}{2})}.
\end{aligned}$$

Therefore $|c_n| \le \dfrac{n}{sin(\frac{\alpha\pi}{2})}$.

□

Lemma 2.2. *For $f \in A$ and and for every $\varepsilon \in C$ such that $|\varepsilon| < \delta$, if $F_\varepsilon(z) = \frac{f(z)+\varepsilon z}{1+\varepsilon} \in STS_s(\alpha)$ then for every $H \in STS'_s(\alpha)$, $\left|\frac{(f*H)(z)}{z}\right| \ge \delta,\ z \in E$.*

Proof. Let $F_\varepsilon(z) = \frac{f(z)+\varepsilon z}{1+\varepsilon}$. Then by Theorem 2.1, $\frac{(f*H)(z)}{z} \ne 0$, for all $f \in STS_s(\alpha), z \in E$. Equivalently, $\frac{(f*H)(z)+\varepsilon z}{(1+\varepsilon)z} \ne 0$ in E or $\frac{(f*H)(z)}{z} \ne -z$, which show that $\left|\frac{(f*H)(z)}{z}\right| \ge \delta$. □

Theorem 2.2. *For $f \in A$ and $\varepsilon \in C, |\varepsilon| < \delta < 1$, assume $F_\varepsilon(z) \in STS_s(\alpha)$. Then $N_{\delta'}(f) \subset STS_s(\alpha)$, where $\delta' = \delta sin(\frac{\alpha\pi}{2})$.*

Proof. Let $H \in STS'_s(\alpha)$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in N_{\delta'}(f)$. Then

$$\left|\frac{(g*H)(z)}{z}\right| = \left|\frac{(f*H)(z)}{z} + \frac{((g-f)*H)(z)}{z}\right|$$
$$\geq \left|\frac{(f*H)(z)}{z}\right| - \left|\frac{(g-f)(z)*H(z)}{z}\right|$$
$$\geq \delta - \left|\sum_{n=2}^{\infty} \frac{(b_n - a_n)c_n z^n}{z}\right|, \text{ by Lemma 2.2.}$$

Thus

$$\left|\frac{(g*H)(z)}{z}\right| \geq \delta - |z| \sum_{n=2}^{\infty} |b_n - a_n||c_n|$$
$$> \delta - \frac{1}{sin(\frac{\alpha\pi}{2})} \sum_{n=2}^{\infty} n|b_n - a_n|, \text{ by Lemma 2.1}$$
$$> \delta - \frac{\delta'}{sin(\frac{\alpha\pi}{2})} = 0, \quad \text{for } \delta' = \delta sin(\frac{\alpha\pi}{2}).$$

Thus $\frac{(g*H)(z)}{z} \neq 0$ in E for all $H \in STS'_s(\alpha)$ which means by Theorem 2.2, $g \in STS_s(\alpha)$, in other words, $N_{\delta sin(\frac{\alpha\pi}{2})} \subset STS_s(\alpha)$. □

Lemma 2.3. *If $0 < \alpha \leq 1$ and $g \in STS_s(\alpha)$ then $G(z) = \frac{g(z)-g(-z)}{2} \in STS(\alpha) \subset ST(\alpha)$.*

Proof. Let $0 < \alpha \leq 1$ and $g \in STS_s(\alpha)$. Then $\frac{2zg'(z)}{g(z)-g(-z)} \in \Omega$. Now

$$\frac{zG'(z)}{G(z)} = \frac{zg'(z)}{2G(z)} + \frac{-zg'(-z)}{2G(-z)}.$$

There exist ζ_1, ζ_2 in Ω such that $\frac{zG'(z)}{G(z)} = \frac{\zeta_1}{2} + \frac{\zeta_2}{2} = \zeta_3.$

Since Ω is convex sector $\zeta_3 \in \Omega$ and hence $\frac{zG'(z)}{G(z)} \in \Omega$.
It can be easily seen that $STS(\alpha) \subset ST(\alpha)$.
Thus $G(z) \in STS(\alpha) \subset ST(\alpha)$. □

Theorem 2.3. *Let $f \in CV$ and $g \in STS_s(\alpha)$. Then $(f*g)(z) \in STS_s(\alpha)$.*

Proof. Let $f(z) \in CV, g(z) \in STS_s(\alpha), G(z) = \frac{g(z)-g(-z)}{2}$ and Ω is convex domain.
Since $g(z) \in STS_s(\alpha), G(z) = \frac{g(z)-g(-z)}{2} \in ST(\alpha)$, by Lemma 2.3.
Hence, by an application of Lemma 1.1, we get

$$\frac{z(f*g)'(z)}{(f*G)(z)} = \frac{(f*zg')(z)}{(f*G)(z)}$$
$$= \frac{f * \frac{zg'(z)}{G(z)} G(z)}{(f*G)(z)}$$
$$\subset \overline{C_0}\left(\frac{zg'(z)}{G(z)}\right).$$

Since Ω is convex and $g \in STS_s(\alpha)$. This proves that $(f*g)(z) \in STS_s(\alpha)$. □

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