TWMS J. App. and Eng. Math. V.10, N.2, 2020, pp. 492-498

NUMERICAL RANGE AND SUB-SELF-ADJOINT OPERATORS

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Abstract. In this paper, we show that the numerical range of a bounded linear operator T on a complex Hilbert space is a line segment if and only if there are scalars λ and μ such that $T^* = \lambda T + \mu I$, and we determine the equation of the straight support of this numerical range in terms of λ and μ. An operator T is called sub-self-adjoint if their numerical range is a line segment. The class of sub-self-adjoint operators contains every self-adjoint operator and contained in the class of normal operators. We show that this class is uniformly closed, invariant under unitary equivalence and invariant under affine transformation. Some properties of the sub-self-adjoint operators and their numerical ranges are investigated.

Keywords: Numerical range, self-adjoint operator, normal operator.

AMS Subject Classification: 47A12, 47B15.

1. Introduction

Let H be a complex Hilbert space, with inner product $\langle \cdot, \cdot \rangle$, and denote by $B(H)$ the algebra of bounded linear operators on H. The numerical range of an operator $T \in B(H)$ is defined by

$$W(T) = \{\langle Tx, x \rangle : x \in H, \| x \| = 1\}.$$

The numerical radius $w(T)$ of an operator $T \in B(H)$ is given by

$$w(T) = \sup\{|z| : z \in W(T)\}.$$

It is well known that $W(T)$ is a nonempty bounded convex subset of $\mathbb{C}$ containing the set of eigenvalues of T. In particular, the numerical range of a normal matrix is the convex hull of its eigenvalues. The geometrical properties of $W(T)$ often provide useful information about the analytic and algebraic properties of T. For example $W(T)$ is a point $\{\delta\}$ if and only if $T = \delta I$, $W(T)$ is real if and only if T is self-adjoint [2, p. 7], and if $W(T)$ is a line segment, then T is normal [2, p. 15]. We have also that if T is normal then $w(T) = \|T\|$ [3, p. 117]. For the numerical range properties, we refer the reader to the books [2,3].

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§ Manuscript received: December 25, 2018; accepted: May 12, 2019.
TWMS Journal of Applied and Engineering Mathematics, Vol.10, No.2;

In the following we will denote the spectrum and the approximate spectrum of T by $\sigma(T)$ and $\sigma_a(T)$ respectively.

In Kippenhahn's article [4] (see its English translation [5]), the numerical range $W(A)$ of a matrix A is a line segment exactly when A and a Hermitian matrix are affine equivalents. In the present paper we generalize this result, we give a necessary and sufficient condition for an operator T so that his numerical range is a line segment and we determine their straight support in this case. This paper initiates a study of the class $\mathbb{S}(H)$ of operators T on H which have the property that $T^* = \lambda T + \mu I$, for $\lambda, \mu \in \mathbb{C}$, that we called sub-self-adjoint (for matrices, this definition coincides with essentially Hermitian). We show that the class $\mathbb{S}(H)$ is uniformly closed, invariant under unitary equivalence and invariant under affine transformation. It is known that if $\overline{W(T)} = [m, M] \subset \mathbb{R}$ (identically T is self-adjoint) then $\|T\| = \max\{|m|, |M|\}$ and $m, M \in \sigma(T)$ [2, p. 7]. We generalize this result to sub-self-adjoint operator. In the last we study the parallelism and the orthogonality of the straight supports numerical ranges of two sub-self-adjoint operators.

2. Main Results

Theorem 2.1. *Let $T \in B(H)$. Then $W(T)$ is a line segment if and only if there are $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$ such that $\alpha T + \beta I$ is self-adjoint.*

Proof. Suppose that $W(T)$ is a line segment. Let θ the inclination of $W(T)$ and let γ a point of $W(T)$. Then $W\left(e^{i\theta}(T - \gamma I)\right)$ is included in the real axis. Therefore, for $\alpha = e^{i\theta}$ and $\beta = -e^{i\theta}\gamma$, $\alpha T + \beta I$ is self-adjoint.

Conversely, let $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$ such that $\alpha T + \beta I$ is self-adjoint. It is clear that $W(\alpha T + \beta I)$ is a line segment of the real axis. Since the affine transformation preserves the line segments, then $W(T) = \frac{1}{\alpha} W(\alpha T + \beta I) - \frac{\beta}{\alpha}$ is a line segment of $\mathbb{C}$. □

Theorem 2.2. *Let $T \in B(H)$. Then $W(T)$ is a line segment if and only if there are $\lambda, \mu \in \mathbb{C}$ with $\lambda \neq 0$ such that $T^* = \lambda T + \mu I$.*

Proof. According to the previous theorem, it suffices to show that the existence of $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$ such that $\alpha T + \beta I$ is self-adjoint is equivalent to the existence of $\lambda, \mu \in \mathbb{C}$ with $\lambda \neq 0$ such that $T^* = \lambda T + \mu I$.

If $(\alpha T + \beta I)^* = \alpha T + \beta I$, then

$$T^* = \lambda T + \mu I, \text{ with } \lambda = \frac{\alpha}{\overline{\alpha}}, \text{ and } \mu = \frac{2i\Im\beta}{\overline{\alpha}}.$$

Conversely, if $T^* = \lambda T + \mu I$, then

$$T = (T^*)^* = \overline{\lambda} T^* + \overline{\mu} I = |\lambda|^2 T + \left(\overline{\lambda}\mu + \overline{\mu}\right) I.$$

We have two cases:

(1) $|\lambda| \neq 1$. In this case $T = \delta I$, where

$$\delta = \frac{\overline{\lambda}\mu + \overline{\mu}}{1 - |\lambda|^2}.$$

Then for all $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$ such that $\alpha\delta + \mu \in \mathbb{R}$, the operator $\alpha T + \beta I$ is self-adjoint.

(2) $|\lambda| = 1$. In this case $T = T + \left(\overline{\lambda}\mu + \overline{\mu}\right) I$, then $\overline{\lambda}\mu + \overline{\mu} = 0$, identically, $\mu = 0$ or $(\mu/\overline{\mu}) = -\lambda$.

It is clear that if there are $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$ such that

$$\frac{\alpha}{\overline{\alpha}} = \lambda, \text{ and } \frac{2i\Im(\beta)}{\overline{\alpha}} = \mu,$$

then $\alpha T + \beta I$ is self-adjoint.

For $\lambda = e^{i\theta}$, with $\theta \in [0, 2\pi]$, the system

$$\begin{cases} \dfrac{\alpha}{\overline{\alpha}} = \lambda \\ \dfrac{2i\Im(\beta)}{\overline{\alpha}} = \mu \end{cases}$$

has solutions

$$\begin{cases} \alpha = \rho e^{i\left(\frac{\theta}{2}+k\pi\right)} \\ \beta = \eta + \frac{1}{2}\rho\mu e^{-i\left(\frac{\theta}{2}+k\pi\right)} \end{cases}$$

for all $\rho > 0$, $\eta \in \mathbb{R}$ and $k \in \mathbb{Z}$. □

Remark 2.1. *We give in the following corollary a short proof that in a two-dimensional space an operator T is normal if and only if their numerical range is a line segment without using the elliptic theorem* [1, *p.* 262].

Corollary 2.1. *If* $\dim H = 2$ *then* $T \in B(H)$ *is normal if and only if* $T \in \mathbb{S}(H)$.

Proof. Let T be a normal operator and λ_1,λ_2 their eigenvalues. If $\lambda_1 = \lambda_2$, then $W(T) = \{\lambda_1\}$ and $T \in \mathbb{S}(H)$. If $\lambda_1 \neq \lambda_2$ then the operator $S = \left(\dfrac{1}{\lambda_2 - \lambda_1}\right)(T - \lambda_1 I)$ is normal with eigenvalues 0 and 1. Therefore, $W(S) = [0, 1]$, hence S is self-adjoint and by the first theorem $W(T)$ is a line segment, identically $T \in \mathbb{S}(H)$. □

Theorem 2.3. *Let $T \in \mathbb{S}(H)$ and let $\lambda, \mu \in \mathbb{C}$ such that $T^* = \lambda T + \mu I$. Then,*

(1) *If $|\lambda| \neq 1$, then $W(T)$ is the point $\{\delta\}$, where*

$$\delta = \frac{\overline{\lambda}\mu + \overline{\mu}}{1 - |\lambda|^2}.$$

(2) *If $\lambda = 1$, then $W(T)$ is an horizontal line segment whose the equation of their straight support is*

$$Y = \frac{\mu}{2}i, \text{ with } \Re\mu = 0.$$

(3) *If $\lambda = -1$, then $W(T)$ is a vertical line segment whose the equation of their straight support is*

$$X = \frac{\mu}{2}, \text{ with } \mu \in \mathbb{R}.$$

(4) *Otherwise, $W(T)$ is an inclined line segment whose the equation of their straight support is*

$$Y = \left(\frac{-1 + \Re\lambda}{\Im\lambda}\right)X + \frac{\Re\mu}{\Im\lambda}.$$

Proof. (1) It is evident from Theorem 2.2.

(2) If $\lambda = 1$, then $T = T + (\mu + \overline{\mu})I$, identically $\Re\mu = 0$.

Since $T^* = T + \mu I$, then for all unit vector $x \in H$, we have

$$\langle T^* x, x\rangle = \langle Tx, x\rangle + \mu.$$

Therefore,

$$\Im\langle Tx, x\rangle = \mu i/2.$$

Thus, the equation of the straight support is

$$Y = \frac{\mu}{2}i.$$

(3) If $\lambda = -1$, then $T = T + (-\mu + \overline{\mu})I$, identically $\mu \in \mathbb{R}$.

Since $T^* = -T + \mu I$, then for all unit vector $x \in H$, we have

$$\langle T^* x, x\rangle = -\langle Tx, x\rangle + \mu.$$

Therefore,

$$\Re \langle Tx, x\rangle = \mu/2.$$

Thus, the equation of the straight support is

$$X = \frac{\mu}{2}.$$

(4) We have for all unit vector $x \in H$

$$\langle T^* x, x\rangle = \lambda \langle Tx, x\rangle + \mu.$$

Then

$$\begin{cases} \Re \langle Tx, x\rangle = (\Re\lambda)(\Re \langle Tx, x\rangle) - (\Im\lambda)(\Im \langle Tx, x\rangle) + \Re\mu, \\ -\Im \langle Tx, x\rangle = (\Re\lambda)(\Im \langle Tx, x\rangle) + (\Im\lambda)(\Re \langle Tx, x\rangle) + \Im\mu. \end{cases}$$

Therefore,

$$\begin{cases} \Im \langle Tx, x\rangle = \left(\dfrac{-1+\Re\lambda}{\Im\lambda}\right)(\Re \langle Tx, x\rangle) + \dfrac{\Re\mu}{\Im\lambda}, \\ \Im \langle Tx, x\rangle = \left(\dfrac{-\Im\lambda}{1+\Re\lambda}\right)(\Re \langle Tx, x\rangle) - \dfrac{\Im\mu}{1+\Re\lambda}. \end{cases}$$

Since $|\lambda| = 1$, it can be easly verified that

$$\frac{-1+\Re\lambda}{\Im\lambda} = \frac{-\Im\lambda}{1+\Re\lambda}.$$

If $\mu \neq 0$, then $\lambda = -(\mu/\overline{\mu}) = -\left(\mu^2/|\mu|^2\right)$. Thus,

$$1 + \Re\lambda = \frac{2(\Im\mu)^2}{|\mu|^2} \text{ and } \Im\lambda = -\frac{2(\Re\mu)(\Im\mu)}{|\mu|^2}.$$

Hence

$$\frac{\Re\mu}{\Im\lambda} = -\frac{\Im\mu}{1+\Re\lambda},$$

then, we have for all unit vector $x \in H$

$$\Im \langle Tx, x\rangle = \left(\frac{-1+\Re\lambda}{\Im\lambda}\right)(\Re \langle Tx, x\rangle) + \frac{\Re\mu}{\Im\lambda}.$$

We conclude that the equation of the straight support is

$$Y = \left(\frac{-1+\Re\lambda}{\Im\lambda}\right) X + \frac{\Re\mu}{\Im\lambda}.$$

□

Example 2.1. *Let*

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1-i \\ 0 & 1-i & 1 \end{pmatrix}.$$

Then

$$iT + (1-i)I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1+i \\ 0 & 1+i & 1 \end{pmatrix} = T^*.$$

Therefore, T is sub-self-adjoint and not self-adjoint, the equation of the straight support of $W(T)$ is $Y = -X + 1$.

On the other hand T is normal with eigenvalues 1, i and $(2-i)$, then $W(T)$ is the convex hull of the eigenvalues, identically $W(T)$ is the line segment connecting the two points (i) and $(2-i)$ which belongs to the right of the equation $Y=-X+1$.

Theorem 2.4. *The class $\mathbb{S}(H)$ of sub-self-adjoint operators is uniformly closed.*

Proof. Let (T_n) a sequence of sub-self-adjoint operators converges uniformly to an operator $T \in B(H)$.

If T is scalar operator, then it is sub-self-adjoint. Otherwise, the set of indices k where T_k is scalar operator is finite. By elimination of scalar operators from the sequence (T_n), we obtain a sub-sequence (T_n') converges to same operator T . Let (λ_n) and (μ_n) two scalars sequences such that, for all $n \in \mathbb{N}$, $(T_n')^* = \lambda_n T_n' + \mu_n I$.

According to theorem 2 case (1), for all $n \in \mathbb{N}$, $|\lambda_n| = 1$ because T_n' is not scalar operator. Then, for all $n \in \mathbb{N}$

$$\begin{aligned} \|\lambda_n T + \mu_n I - T^*\| &= \left\|\lambda_n\left(T - T_n'\right) + \left(\lambda_n T_n' + \mu_n I\right) - T^*\right\| \\ &= \left\|\lambda_n\left(T - T_n'\right) + \left(T_n'\right)^* - T^*\right\| \\ &\leq \left\|T - T_n'\right\| + \left\|\left(T_n'\right)^* - T^*\right\|. \end{aligned}$$

Therefore, $(\lambda_n T + \mu_n I)$ converges uniformly to T^*. Thus (λ_n) and (μ_n) are convergent and

$$T^* = (\lim \lambda_n) T + (\lim \mu_n) I.$$

□

Remark 2.2. *By cartisian decomposition, every operator $T \in B(H)$ is the sum of two sub-self-adjoint operators.*

Theorem 2.5. *Let $T \in \mathbb{S}(H)$ and δ, γ the endpoints of $W(T)$. Then,*

$$\|T\| = \max\{|\delta|, |\gamma|\}$$

and

$$\delta, \gamma \in \sigma(T).$$

Proof. (i) Since $w(T) = \max\{|\delta|, |\gamma|\}$ and T is normal, then $\|T\| = w(T) = \max\{|\delta|, |\gamma|\}$.

(ii) Since $\delta \in \overline{W(T)}$, there is a sequence of unit vectors (x_n) such that $\langle Tx_n, x_n\rangle \longrightarrow \delta$. Let $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$ such that $\alpha T + \beta I$ is self-adjoint. Then $(\alpha\delta + \beta), (\alpha\gamma + \beta) \in \mathbb{R}$. If $\alpha\delta + \beta \leq \alpha\gamma + \beta$, thus $\alpha(T - \delta I)$ and $\alpha(\gamma I - T)$ are positive operators. Hence

$$\left\|\left[\alpha(T - \delta I)\right]^{\frac{1}{2}} x_n\right\|^2 = \langle \alpha(T - \delta I) x_n, x_n\rangle \longrightarrow 0.$$

Also $\|(T - \delta I) x_n\| \longrightarrow 0$ and so $\delta \in \sigma_a(T) \subset \sigma(T)$. □

Proposition 2.1. *The class $\mathbb{S}(H)$ is invariant under unitary equivalence and under affine transformation.*

Proof. Let $T \in \mathbb{S}(H)$, U be a unitary operator in $B(H)$, $a, b \in \mathbb{C}$ with $a \neq 0$, and let $\lambda, \mu \in \mathbb{C}$ such that $T^* = \lambda T + \mu I$.

(i) We have

$$(U^*TU)^* = U^*T^*U = U^*(\lambda T + \mu I)U = \lambda(U^*TU) + \mu I.$$

Then $U^*TU \in \mathbb{S}(H)$.

(ii) It can be easly verified that

$$(aT + bI)^* = \frac{\lambda\overline{a}}{a}(aT + bI) + \left(\frac{|a|^2\mu + a\overline{b} - \lambda\overline{a}b}{a}\right) I.$$

Then $(aT + bI) \in \mathbb{S}(H)$. □

Proposition 2.2. *Let $T, T' \in \mathbb{S}(H)$, with $T^* = \lambda T + \mu I$ and $(T')^* = \lambda' T' + \mu' I$. Then*

(i) The straight support of $W(T')$ is parallel to the straight support of $W(T)$ if and only if $\lambda' = \lambda$.

(ii) The straight support of $W(T')$ is orthogonal to the straight support of $W(T)$ if and only if $\lambda' = -\lambda$.

Proof. If $W(T)$ is horizontal or vertical segment, then the two equivalents are clear from Theorem 2.3.

If $\lambda, \lambda' \notin \{1, -1\}$, then the direction vectors of $W(T)$ and $W(T')$ are

$$\overrightarrow{V} = \begin{pmatrix} 1 \\ \frac{-1+\Re\lambda}{\Im\lambda} \end{pmatrix} \text{ and } \overrightarrow{V'} = \begin{pmatrix} 1 \\ \frac{-1+\Re\lambda'}{\Im\lambda'} \end{pmatrix}$$

respectively. Therefore,

(i)

$$\overrightarrow{V} \parallel \overrightarrow{V'} \Longleftrightarrow \frac{-1+\Re\lambda}{\Im\lambda} = \frac{-1+\Re\lambda'}{\Im\lambda'} \Longleftrightarrow \lambda' = \lambda.$$

(ii)

$$\overrightarrow{V} \perp \overrightarrow{V'} \Longleftrightarrow \left(\frac{-1+\Re\lambda}{\Im\lambda}\right)\left(\frac{-1+\Re\lambda'}{\Im\lambda'}\right) = -1 \Longleftrightarrow \lambda' = -\lambda.$$

□

3. Conclusion

The geometrical properties of the numerical range often provide useful information about the analytic and algebraic properties of operators. In this paper, we give a necessary and sufficient condition for an operator so that his numerical range is a line segment and we determine their straight support in this case. We called the operator whose numerical range is a line segment by sub-self-adjoint operator and we initiated a study of this new class. In our future research, we will study the class of sub-self-adjoint operators and generalize some results of self-adjoint operators to this class. About the numerical range, we will try to show that the numerical range of a sub-self-adjoint operator is one of the two diagonals of the smallest rectangle containing the numerical range.

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