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ON PROGRESSIVE WAVE SOLUTION FOR NON-PLANAR KDV EQUATION IN A PLASMA WITH q-NONEXTENSIVE ELECTRONS AND TWO OPPOSITELY CHARGED IONS

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ABSTRACT. In this paper, the ion-acoustic wave is investigated in a plasma with q-nonextensive electrons and two oppositely charged ions with varying masses. These parameters are found to modify the linear dispersion relation and nonlinear solitary structures. The reductive perturbation method is employed to derive modified Korteweg-de Vries (KdV) equation. To solve the obtained governing evolution equation, the exact solution in the planar geometry is obtained and used to obtain an analytical approximate progressive wave solution for the nonplanar evolution equation. The analytical approximate solution so obtained is compared with the numerical solution of the same nonplanar evolution equation and the results are presented in 2D and 3D figures. The results revealed that both solutions are in good agreement. A parametric study is carried out to investigate the effect of different physical parameters on the nonlinear evolution solution behavior. The obtained solution allows us to study the impact of various plasma parameters on the behavior of the nonplanar ion-acoustic solitons. The suitable application of the present investigations can be found in laboratory plasmas, where oppositely charged ions and nonthermal electrons dwell.

Keywords: Nonplanar solitons, Modified KdV, Cylindrical and spherical solitons, Electronegative plasmas , Analytical approximate solutions
AMS Subject Classification: 35Q53, 65Z05

1. Introduction

The plasmas holding negative ions as well as electrons and positive ions, are classified as electronegative plasmas. Such plasmas have been investigated intensively, not only because of their many industrial applications but also owing to the reality that their basic

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properties differ predominantly from the usual electropositive plasmas. Spherically convergent positive and negative ion-acoustic pulses are also investigated experimentally][1]. Both experimental and theoretical explorations unveiled the reality that massive negative ions addition modify the plasma characteristics and dynamics predominantly [2,3]. By reasons of their vital roles in astrophysical scenarios, neutral beam sources [4], plasma processing reactors [5], synthesis of nano-materials [6], processing/manufacturing of semiconductors[7], thorough investigations are necessary. The occurrence of negative ions in Earth's ionosphere [8] and cometary comae [9] are confirmed through experiments. These negative ions are comparatively more useful than positive ions in silicon etching plasma devices [10] when alternating irradiations of negative and positive ions are utilized. Recently, the investigations related to the propagation characteristics of ion-acoustic waves (IAWs) in nonplanar geometries have been also focused in Refs [11-15]. The evolution equation of IAWs in nonplanar geometries (spherically and cylindrically symmetric) has been found by Maxon and Viecelli [11]. They have found that the cylindrical solitons travel slower than the spherical ones however faster than the one-dimensional solitons of the same amplitude. Cylindrical and spherical soliton-like waves have been investigated experimentally [16-18]. The experimental verification of theoretically earlier proposed cylindrical and spherical solitons in electronegative plasma by Das and Singh [19] was done by Williams *et al*[20]. The multiple scale expansion technique was used to derive and analyze the effect of the nonthermal electrons on nonplanar IAWs in an unmagnetized plasma with warm adiabatic ions [21]. Hershkowitz and Romesser [22] have shown experimentally that the cylindrical soliton evolves from the compressive cylindrical pulses in a collisionless plasma and while the converging soliton passes through the center the shape of the soliton is maintained. The spherical symmetric solitons along with cylindrical solitons have been observed in plasma as in Refs.[23,24]. The experimental data obtained from the spacecraft missions communicate the existence of nonthermal/superthermal particles in the space plasma environments. Such species are interpreted through non-Maxwellian distributions that obey power-law feature [25]. During the previous decades, it is proved that systems with long-range interactions and long-time memory can not be fully explained with the conventional Boltzmann-Gibbs (BG), and therefore a need was recognized to develop new statistics. Consequently, non-extensive statistics was materialized for studying systems with long-range interactions and long-time memory [26-28]. One of the possible one-dimensional q-distribution function exhibiting electron nonextensive behavior is proposed as [29]

$$f_e(v) = C_q \left[1 - (q-1)\left(\frac{m_e v^2}{2K_B T_e} - \frac{e\varphi}{K_B T_e}\right)\right]^{\frac{1}{q-1}}$$

where $C_q = n_{e0}\frac{\Gamma(\frac{1}{1-q})}{\Gamma(\frac{1}{1-q}-\frac{1}{2})}\sqrt{\frac{m_e(1-q)}{2\pi K_B T_e}}$ is a normalization constant for $-1 < q < 1$ and $C_q = n_{e0}(\frac{1+q}{2})\frac{\Gamma(\frac{1}{q-1}+\frac{1}{2})}{\Gamma(\frac{1}{q-1})}\sqrt{\frac{m_e(q-1)}{2\pi K_B T_e}}$ for $q > 1$.

The significance, application and development of such distribution function is given in Ref.[30].

In this work, the reductive perturbation method [31] is utilized to get modified evolution equation. To solve the obtained governing evolution equation, the exact solution in simple plane geometry is obtained, afterwards which is utilized to derive an analytical approximate progressive wave solution for the equation in nonplanar geometry. The analytical approximate solution so obtained is compared with the numerical solution of the same nonplanar evolution equation. With the aid of appropriate parameters, the results are presented in 2D and 3D figures. These results revealed that both analytican and exact numerical solutions are very much similar same. One of the advantages of the present analytical solution is that it allows readers to carry out parametric studies. A parametric investigation is carried out to study different physical parameters impact on the behavior of the analytical solution. Moreover, the impact of various plasma parameters on nonplanar ion-acoustic solitons can also be elaborated. The suitable application of the present investigations can be found in laboratory plasmas, where oppositely charged ions and nonthermal electrons dwell.

2. Derivation of evolution equation

A homogeneous, unmagnetized and collisionless plasma having oppositely charged ions with varying masses and q-nonextensive electrons is considered. The positive ions having temperature T_i, while negative ions with temperature T_n are present in the plasma system and at the same time the electrons T_e (such as $T_e \gg T_{i,n}$) are assumed obeying q-nonextensive distribution. Normalized equations describing the plasma model under consideration are written, in axially symmetric non-planar case, as

$$\frac{\partial n_i}{\partial t} + \frac{1}{r^p}\frac{\partial\left(r^p n_i v_i\right)}{\partial r} = 0, \tag{1}$$

$$\frac{\partial v_i}{\partial t} + v_i\frac{\partial v_i}{\partial r} = -\alpha\frac{\partial \Phi}{\partial r} - \frac{\alpha\theta_i}{n_i}\frac{\partial p_i}{\partial r}, \tag{2}$$

$$\frac{\partial p_i}{\partial t} + v_i\frac{\partial p_i}{\partial r} + 3p_i\frac{1}{r^p}\frac{\partial(r^p v_i)}{\partial r} = 0, \tag{3}$$

$$\frac{\partial n_n}{\partial t} + \frac{1}{r^p}\frac{\partial\left(r^p n_n v_n\right)}{\partial r} = 0, \tag{4}$$

$$\frac{\partial v_n}{\partial t} + v_n\frac{\partial v_n}{\partial r} = \frac{\partial \Phi}{\partial r} - \frac{\theta_n}{n_n}\frac{\partial p_n}{\partial r}, \tag{5}$$

$$\frac{\partial p_n}{\partial t} + v_n\frac{\partial p_n}{\partial r} + 3p_n\frac{1}{r^p}\frac{\partial(r^p v_n)}{\partial r} = 0 \tag{6}$$

where $n_{i,n}$, $v_{i,n}$ and $p_{i,n}$ are the number densities, normalized fluid velocity in the radial direction and pressures of positive and negative ions, respectively. For q-nonextensive electrons, we can write the number density as

$$n_e = \left[1 + (q-1)\,\Phi\right]^{(q+1)/2(q-1)} \tag{7}$$

Assuming weak nonlinearity limit $\Phi \ll 1$ and approximation $(q-1)\Phi \ll 1$, Taylor series expansion of electron density Eq.(7), results in $n_e = \left[1 + \alpha_1\Phi + \alpha_2\Phi^2\right]$ where $\alpha_1 = (q+1)/2$, and $\alpha_2 = (q+1)(3-q)/4$.

Poisson equation in rescaled form is as follows

$$\frac{1}{r^p}\frac{\partial}{\partial r}(r^p\frac{\partial \Phi}{\partial r}) = \delta\mu\{1 + \alpha_1\Phi + \alpha_2\Phi^2\} + \delta\nu n_n - \delta n_i \tag{8}$$

where $p = 0, 1, 2$ for planar, cylindrical and spherical geometries, respectively. Here Φ is the electrostatic potential. The symbols m_i and m_n show the masses of positive and negative ions respectively and $\alpha = m_n/m_i$.Various ratios appear as: $\nu = n_{n0}/n_{i0}$, $\mu = n_{e0}/n_{i0}$, $\delta = n_{i0}/n_{n0}$, $\theta_i = T_i/T_e$ and $\theta_n = T_n/T_e$. For $m_n > m_i$ and $n_{i0} \sim n_{n0}$, the plasma quantities are rescaled using $t \longrightarrow T\omega_{pn}$, $\nabla \longrightarrow \lambda_D\nabla$, $n_j \longrightarrow N_j/N_{j0}$, $v_j \longrightarrow V_j/c_{sn}$, $\Phi \longrightarrow e\varphi/K_BT_e$, where K_B is the Boltzmann constant and $c_{sn} = \sqrt{K_BT_e/m_n}$ is the speed of sound. Here $\omega_{pn} = \sqrt{4\pi n_{n0}e^2/m_n}$ is the ion plasma frequency and $\lambda_D = \sqrt{K_BT_e/4\pi n_{n0}e^2}$ is the Debye length as a characteristic scaling length.

2.1. **Reductive perturbation analysis.** In this sub-section we shall employ the reductive perturbation method to obtain the evolution equation for the plasma in nonplanar geometry. The stretched variables in this technique are as follows [32, 33]

$$\xi = \epsilon^{\frac{1}{2}}(r - \lambda t), \qquad \tau = \epsilon^{\frac{3}{2}} t \tag{9}$$

Here λ shows rescaled phase speed by c_{sn} and ϵ $(0 < \epsilon \leq 1)$ is a negligibly small factor that determines nonlinearity scale. The plasma parameters n_j, v_j and Φ can be expanded in a power series of ϵ as follows,

$$\begin{aligned} n_j &= 1 + \epsilon n_j^{(1)} + \epsilon^2 n_j^{(2)} + \epsilon^3 n_j^{(3)} + ... \\ v_j &= \epsilon v_j^{(1)} + \epsilon^2 v_j^{(2)} + \epsilon^3 v_j^{(3)} + ... \\ p_j &= 1 + \epsilon p_j^{(1)} + \epsilon^2 p_j^{(2)} + \epsilon^3 p_j^{(3)} + ... \\ \Phi &= \epsilon \Phi^{(1)} + \epsilon^2 \Phi^{(2)} + \epsilon^3 \Phi^{(3)} + ... \end{aligned} \tag{10}$$

where $j = i$ (for positive ions), n (for negative ions), and e (for electron). Employing Eq. (9) and expansions Eq. (10) into the basic set of model equations (1)- (8). Afterwards, comparing and setting the coefficients of like powers of ϵ equal to zero , one obtains a set of differential equations. Simplification of the set of differential equations obtained in this way, yields cylindrical (spherical) KdV equation as

$$\frac{\partial \Phi^{(1)}}{\partial \tau} + \frac{p}{2\tau}\Phi^{(1)} + A\Phi^{(1)}\frac{\partial \Phi^{(1)}}{\partial \xi} + B\frac{\partial^3 \Phi^{(1)}}{\partial \xi^3} = 0, \tag{11}$$

where the coefficients A and B are defined by

$$A = \frac{1}{\Delta}\left(\frac{3\alpha^2\delta(\lambda^2 - 3\alpha\theta_i) + 12\alpha^3\delta\theta_i}{(\lambda^2 - 3\alpha\theta_i)^3} + \frac{3\delta\nu(\lambda^2 - 3\theta_n) + 12\delta\nu\theta_n}{(\lambda^2 - 3\theta_n)^3} - \delta\mu\alpha_2\right),$$

$$\Delta = \frac{2\alpha\delta\lambda}{(\lambda^2 - 3\alpha\theta_i)^2} + \frac{2\delta\nu\lambda}{(\lambda^2 - 3\theta_n)^2}, \; B = \frac{1}{\Delta}. \tag{12}$$

Here, in order to save the space, we did not give the details of the derivation of equations (11) and (12). For the details of the derivation, the reader are refereed to the reference [34]. In Eq.(11), the term $\frac{p}{2\tau}\Phi^{(1)}$ arises when the plasma is contained in non-planar geometries. For $p = 0$, Eq.(11) is a KdV equation in a planar geometry. An instructive overview of nonextensive statistics including various plasma scenarios and its applications can be found in Refs.[35-38]. Here, the findings of our research work will be elaborated with specific laboratory parameters that appear in Ref. [39].

3. Approximate solution of the modified KdV equation

Nonlinear partial differential equations governing the propagation of dust acoustic waves in plasma can be solved using suitable methods directly or after suitable transformation depending on the nature of the equation's coefficients and the non-linearity; see for example (tanh function method[42,43], inverse scattering method [40], Jacobi elliptic function expansion method [44,45], G'/G-expansion method [46-49], generalized expansion method [50], weighted residual method [51-56], sine-cosine method [41]). In this section, an analytical approximate solution in integral sense for the nonplanar KdV equation (11) is presented using weighted residual method. For sufficiently large τ, the term $\Phi^{(1)}/\tau$ may be negligibly small and the equation (11) reduces to the KdV equation in planar case [11,30],

$$\frac{\partial \phi_0}{\partial \tau} + A\phi_0 \frac{\partial \phi_0}{\partial \xi} + B\frac{\partial^3 \phi_0}{\partial \xi^3} = 0, \tag{13}$$

which has a steady state (solitary wave) solution given by

$$\phi_0(\xi,\tau) = a_0 \,\mathrm{sech}^2\,\eta \ , \ \eta = \omega_0(\xi - u_0\tau) \tag{14}$$

$$\omega_0^2 = \frac{Aa_0}{12B} \ , \ u_0 = \frac{Aa_0}{3} \ ,$$

where a_0 is the wave amplitude, ω_0 is the wave width and u_0 is the constant velocity of propagation. An approximate solution to Eq.(11) is proposed as:

$$\Phi^{(1)}(\xi,\tau) = a(\tau)\ \mathrm{sech}^2\ \eta \ , \ \eta = \omega(\tau)(\xi - u(\tau)) \ , \tag{15}$$

with

$$\omega^2(\tau) = \frac{Aa(\tau)}{12B}, \quad u' = \frac{Aa(\tau)}{3}, \tag{16}$$

where the prime shows differentiation of the some quantity w.r.t τ. As a matter of fact, the expressions (16) are formally the same with those of (14), except that in (16) $a(\tau)$ is still undetermined.

When (15) along with (16) is substituted into the equation (11), it will not be satisfied identically, i. e., there will a residue term $R(\eta,\tau)$, which reads

$$R(\eta,\tau) = \left[a' + \frac{p\,a}{2\tau} - \frac{2a\,\omega'}{\omega}\eta \tanh\eta\right] \mathrm{sech}^2\eta \tag{17}$$

In order to determine $a(\tau)$ we need an additional differential equation in terms of $a(\tau)$. To obtain such an equation, we shall utilize the weighted residual method, as explained in [55-57]. For that purpose, we multiply the equation (17) by a weighing function of the form $\mathrm{sech}^2\eta$. Then integrating the result over η in the limits $-\infty$ to ∞ and setting them equal to zero, we obtain

$$\int_{-\infty}^{\infty} R(\eta,\tau)\mathrm{sech}^2\eta d\eta = [(a' + \frac{pa}{2\tau})\int_{-\infty}^{\infty} \mathrm{sech}^4\eta d\eta$$

$$-2a\frac{\omega'}{\omega}\int_{-\infty}^{\infty} \eta\ \mathrm{sech}^4\eta\ \tanh\eta d\eta = 0. \tag{18}$$

Carrying out the last integral in equation (18), one obtains

$$(\frac{p}{2}\frac{a}{\tau} + a' - \frac{\omega' a}{2\omega}) < \mathrm{sech}^4\eta >= 0, \quad < \mathrm{sech}^4\eta >= \int_{-\infty}^{\infty} \mathrm{sech}^4\eta d\eta. \tag{19}$$

Here $\mathrm{sech}^2\eta$ is a square integrable function, thus from Eq. (20) we have

$$\frac{p}{2}\frac{a}{\tau} + a' + \frac{\omega' a}{2\omega} = 0, \quad \text{or} \quad \frac{a'}{a} + \frac{\omega'}{2\omega} + \frac{p}{2\tau} = 0. \tag{21}$$

Integrating Eq.(21) from τ_∞ to τ, where τ_∞ is the time at which the solution of Eq. (11) can be approximated by the solution given in Eq. (14), along with Eq.(19) results in

$$a(\tau) = a_0(\tau_\infty/\tau)^{\frac{2p}{3}} \ , \ \omega(\tau) = \omega_0(\tau_\infty/\tau)^{\frac{p}{3}},$$

$$u(\tau) = u_0\tau_\infty\left(1 + \frac{3}{3-2p}\left((\tau_\infty/\tau)^{2p/3-1} - 1\right)\right). \tag{22}$$

Thus, the solution $\Phi^{(1)}$ may be expressed by

$$\Phi^{(1)} = \left(\frac{3u_0}{A}\right)\left(\frac{\tau_\infty}{\tau}\right)^{\frac{2p}{3}} \text{sech}^2\eta,$$
$$\eta = \left(\frac{u_0}{4B}\right)^{1/2}\left(\frac{\tau_\infty}{\tau}\right)^{\frac{p}{3}}\left(\xi - u_0\tau_\infty\left(1 + \frac{3}{3-2p}\left((\tau_\infty/\tau)^{2p/3-1} - 1\right)\right)\right). \tag{23}$$

The basic advantage of the analytical solution (22)-(23) is that a parametric study can be carried out for solitons excited in nonplanar geometry. A parametric study is carried out to investigate the effect of different physical parameters on the nonlinear evolution solution behavior, and results in:

i. For $p = 0$ or $\tau = \tau_\infty$ the evolution solution in Eq. (23) reduces to the planar solution in Eq. (14).

ii. Solution in Eq. (23) shows that the wave amplitude, wave width and wave velocity of the KdV equation in planar geometry ($p = 0$), cylindrical geometry ($p = 1$), and spherical geometry ($p = 2$) are different from each others.

iii. As the value of the initial velocity u_0 increases the initial amplitude a_0 increases and consequently the wave amplitude $a(\tau)$ while the wave width $1/\omega(\tau)$ decreases.

iv. For fixed values of the initial amplitude a_0 the spherical solitary amplitude is taller than the cylindrical as is evident from the amplitude component $(\tau_\infty/\tau)^{\frac{2p}{3}}$.

v. As τ increases the solution amplitude component $(\tau_\infty/\tau)^{\frac{2p}{3}}$ decreases and consequentially the wave amplitude $a(\tau)$ while the wave width $1/\omega(\tau)$ increases.

vi. According to the values of the plasma parameters the sign of the nonlinearity coefficient A can be changed and consequently the polarity of the wave amplitude $a(\tau) = \left(\frac{3u_0}{A}\right)\left(\frac{\tau_\infty}{\tau}\right)^{\frac{2p}{3}}$. In other words, it leads to the formation of both rarefactive and compressive solitons as shown in Figures 9 and 10.

4. Numerical results and discussion

In order to test the effectiveness of the present analytical solution of equation Eq. (13), the same evolution equation is solved numerically by the spectral numerical scheme discussed and used in Refs.[55-57] with a 4^{th} order Runge-Kutta time integrator with a time step $\Delta\tau = 0.005$ and a space step $\Delta\xi = 0.01$ from τ_∞ = -10 to $\tau = -2.5$. In doing so the planar solution is utilized as our initial solution. The results are illustrated in Figures 1-10. Figure 1 displays analytical and numerical solutions of nonplanar geometry ($p = 1$, for cylindrical and $p = 2$, for spherical) in 3D, by giving variations in entropic index ($q = 0.6$, 0.7, 0.8) with $\theta_i = 0.5$, $\theta_n = 0.1$, $\alpha = 146/40$ and $\mu = 0.7$. Figure 2 shows the same solutions at different values of time i.e., $\tau = -10, -7.5, -5, -2.5$. Figure 3 shows the amplitude of the solitons with time for different $q = 0.6, 0.7, 0.8$. By looking at this figure, it can be concluded that the nonlinear solitary waves amplitude diminish with the passage of time τ which is physically true because no plasma system is ideally dissipation free. Furthermore, Figures (1) to (3) show that the amplitude of wave decreases faster as one departs away from center of the sphere. Such decaying trend continues with the passage of time as expected from Eq. (22), where the cylindrical (spherical) amplitude

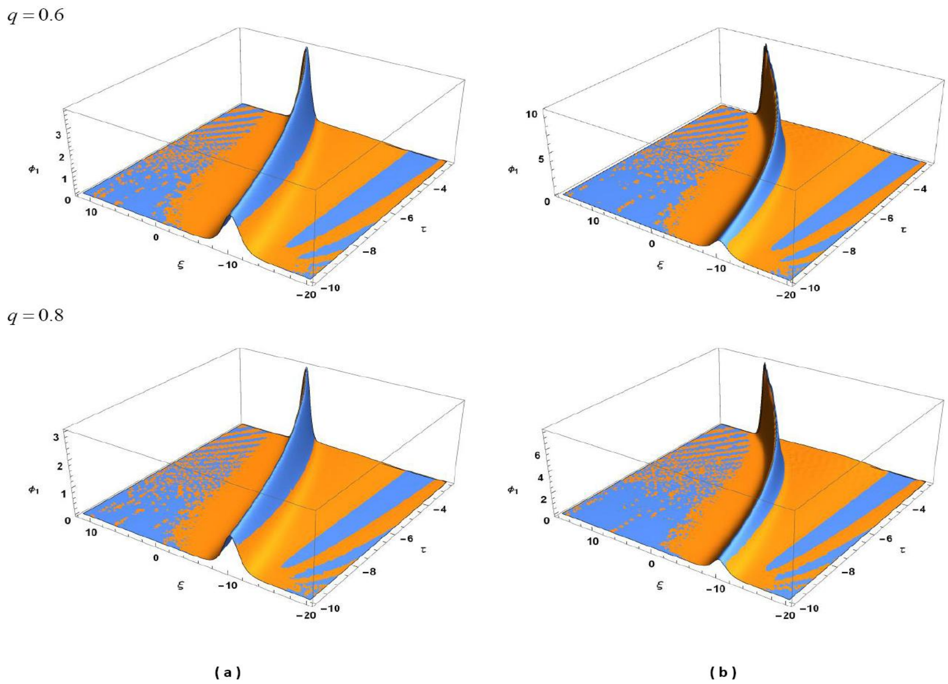

FIGURE 1. Numerical (orange colour) and analytical (blue colour) solutions of the cylindrical (a) and the spherical (b) KdV equation for various values of the entropic index q.

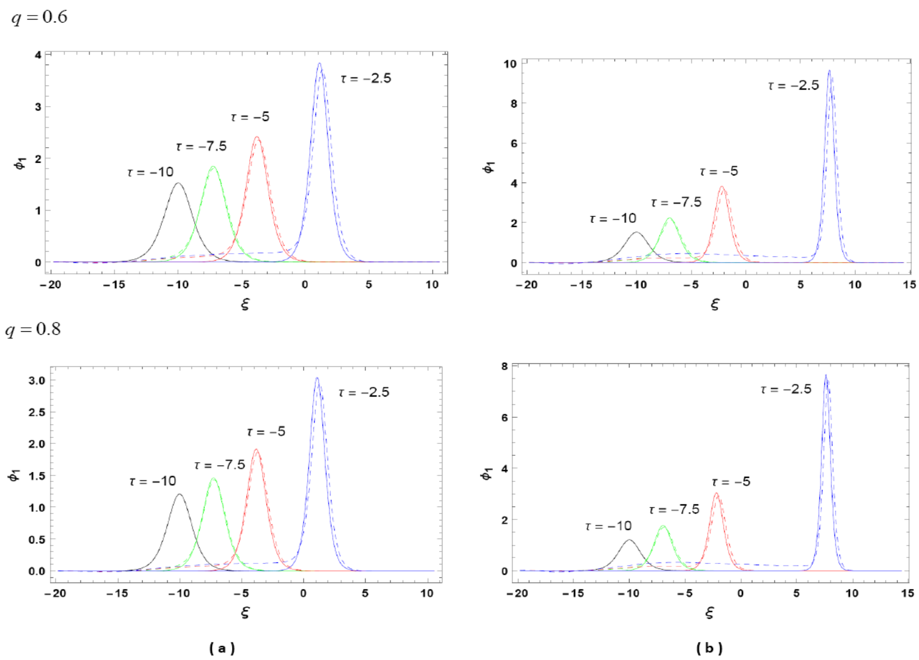

FIGURE 2. The numerical solution (dashed line) and the analytical solution (solid line) of the cylindrical (a) and the spherical (b) KdV equation for various values of the entropic index q and time τ.

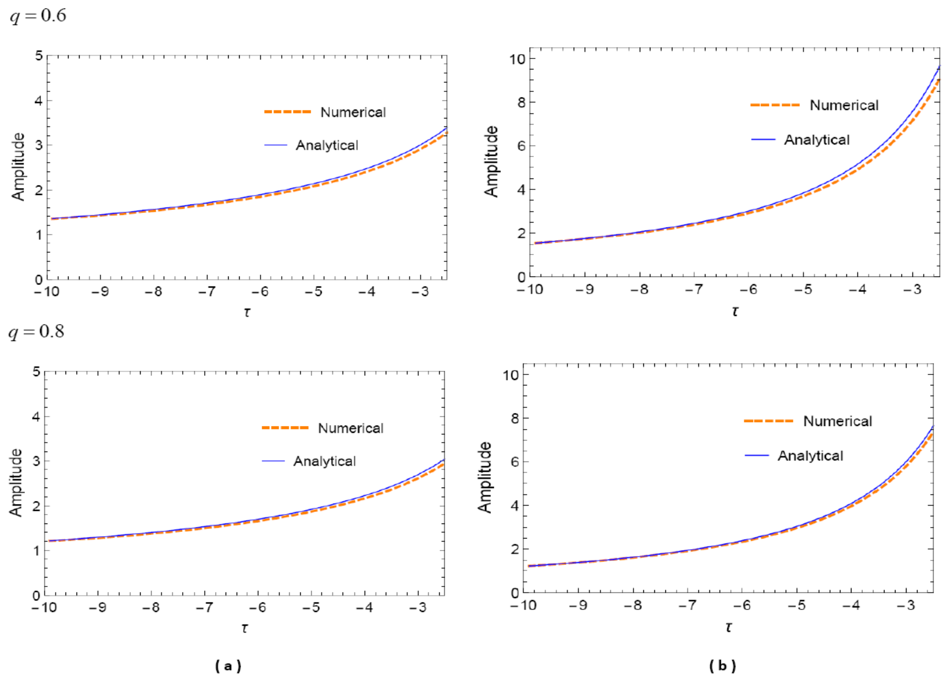

FIGURE 3. Variation of the cylindrical (a) and spherical (b) wave amplitude against time for various values of the entropic index q.

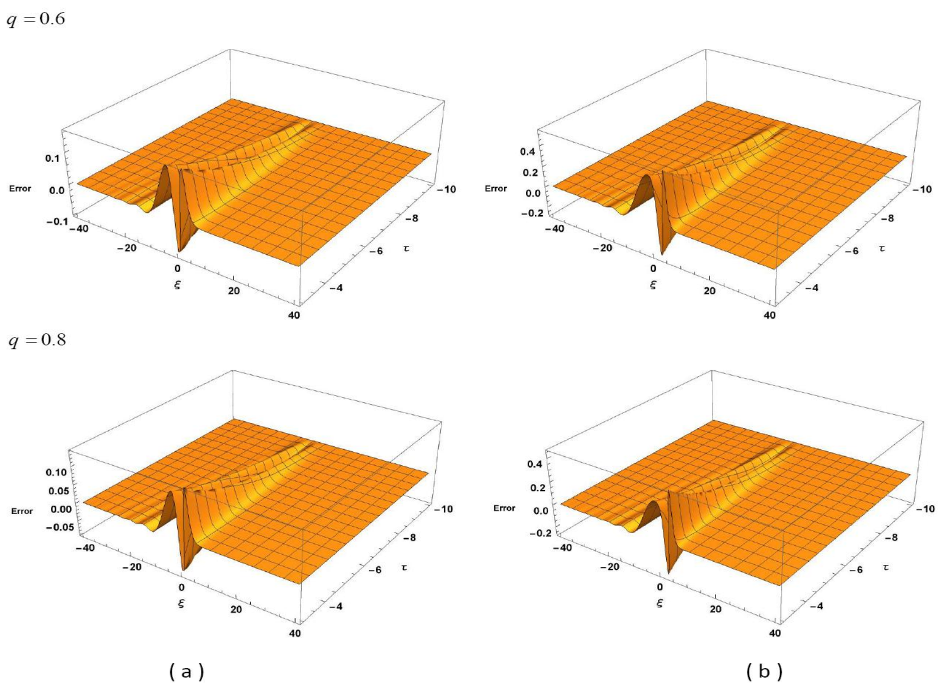

FIGURE 4. 3D differences between analytical and numerical solutions for (a) cylindrical and (b) spherical geometries for various values of the entropic index q.

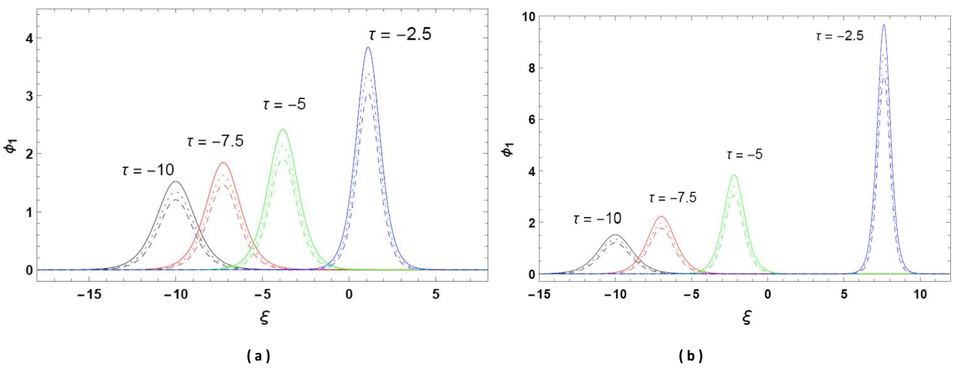

FIGURE 5. Time evolution of compressive IA solitons in (a) cylindrical geometry and (b) spherical geometry for changing the values of $\mu = 0.6$ (solid curves), $\mu = 0.7$ (tiny dashed curves) and $\mu = 0.8$ (small dashed curves) with $\theta_i = 0.5$, $\theta_n = 0.1$, $\alpha = 146/40$ and $q = 0.8$.

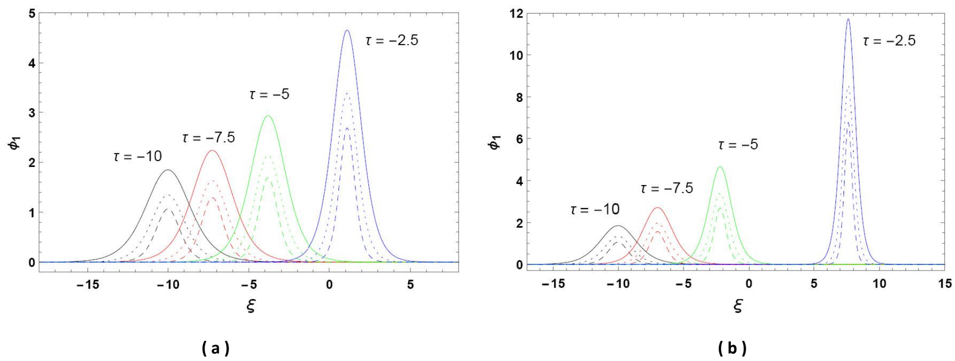

FIGURE 6. Time evolution of compressive IA solitons in (a) cylindrical geometry and (b) spherical geometry for different values of $\theta_i = 0.3$ (solid curves), $\theta_i = 0.4$ (tiny dashed curves) and $\theta_i = 0.5$ (small dashed curves) with $\theta_n = 0.1$, $\mu = 0.6$, $\alpha = 146/40$ and $q = 0.8$.

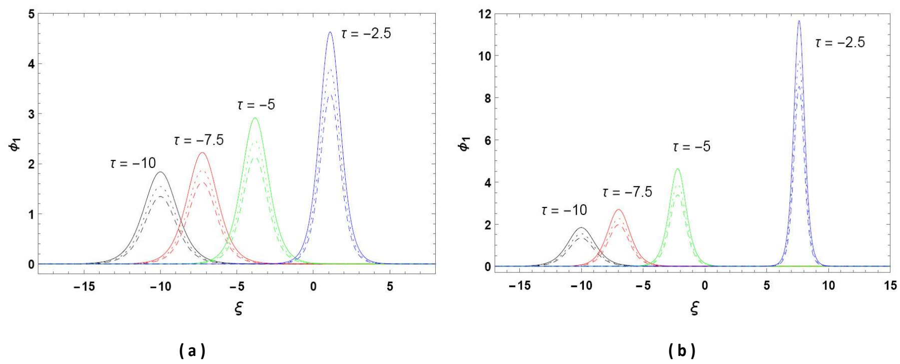

FIGURE 7. Time evolution of compressive IA solitons in (a) cylindrical geometry and (b) spherical geometry for $\alpha = 140/40$ (solid curves), and $\alpha = 140/131$ (small dashed curves) with $\theta_i = 0.5$, $\theta_n = 0.1$, $\mu = 0.6$ and $q = 0.8$.

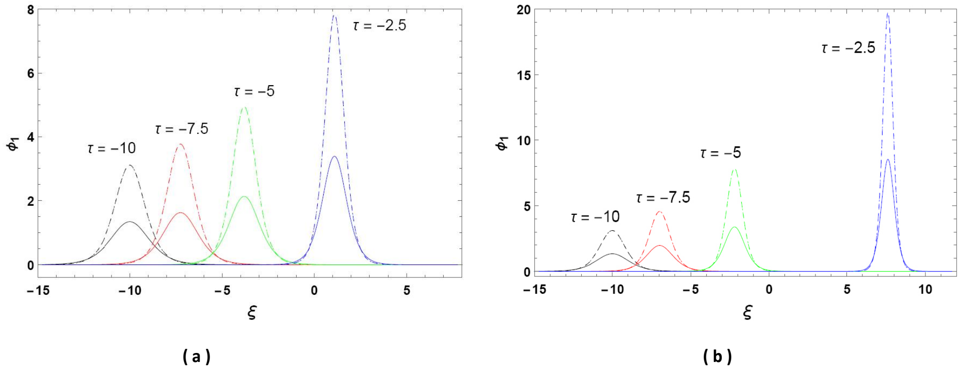

FIGURE 8. Time evolution of compressive and rarefactive IA solitons in (a) cylindrical geometry and (b) spherical geometry for varying the electron nonextensivity parameter $q = 0.2$ (solid curves), $q = 0.3$ (tiny dashed curves) and $q = 0.9$ (small dashed curves) with $\theta_i = 0.1$, $\theta_n = 0.1$, $\mu = 0.6$ and $\alpha = 146/40$.

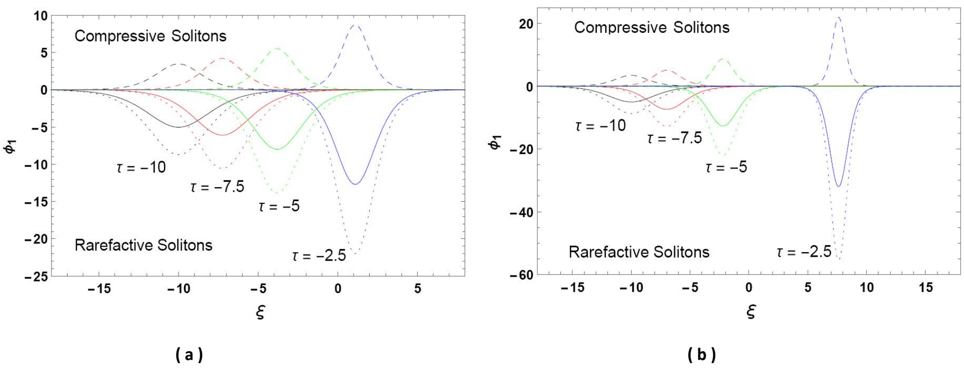

FIGURE 9. Time evolution of compressive and rarefactive IA solitons in (a) cylindrical geometry and (b) spherical geometry for varying the electron-to-positive ion density ratio $\mu = 0.3$ (solid curves), $\mu = 0.4$ (tiny dashed curves) and $\mu = 0.7$ (small dashed curves) with $\theta_i = 0.1$, $\theta_n = 0.1$, $q = 0.7$ and $\alpha = 146/40$.

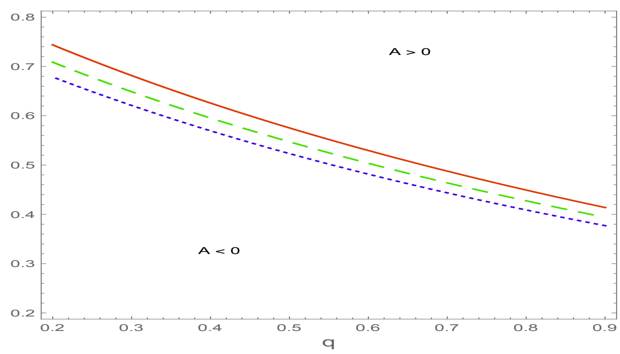

FIGURE 10. Contour plot for the non-linearity coefficient A against the electron nonextensivity parameter q and the electron-to-positive ion density ratio μ for $\theta_i = 0.1$ [solid (red)], $\theta_i = 0.15$ [dashed (green)], $\theta_i = 0.2$ [dotted (blue)], $\theta_n = 0.1$ and $\alpha = 146/40$.

component $(\tau_\infty/\tau)^{\frac{2p}{3}}$ decreases. Moreover, Figures (1) to (3) show that the spherical solitary waves amplitude is greater and more significant than the cylindrical waves for the same time. All these figures, indicate that both analytical and numerical solutions are very much similar. Figure 4 shows the 3D relative differences between the analytical and numerical solutions in the nonplanar geometry ($p = 1$, and $p = 2$) for varying q and illustrates how much analytical solutions satisfy the equation Eq. (11) in integral form. Figure 5 shows that the amplitude and the width of fast IA nonplanar hump solitons is reduced for higher μ. This figure demonstrates the impact of μ on the nonplanar KdV solitons. The decrease in the profile of the fast IA nonplanar compressive solitons is obvious for enhanced μ. Figure 6 has been included in the illustrations to show the effect of positive ion-to-electron temperature ratio (θ_i) on nonplanar solitons. The parameter θ_i is found to impact the nonplanar solitons in a manner that amplitudes of the pulses diminishes with higher θ_i. It is important to mention here that the temperature impacts the Debye length directly. It means that whenever temperature is varied, Debye length is also modified. This is the reason why amplitude and width of the nonplanar solitons is reduced with enhanced positive ion-to-electron temperature ratio (θ_i). In order to make a comparative analysis, Figure 7 shows solitons which has been formed considering two different types of electronegative plasmas (i.e., Xe^+-SF_6^- and Ar^+-SF_6^-), by varying α (= 146/40, 146/131). This illustration exhibits that nonplanar solitons in Xe^+-SF_6^- plasma have higher amplitude and smaller width compared to those formed in case of Ar^+-SF_6^-. As the plasma under investigation contains two massive ions with opposite charges, both hump (compressive) and dip (rarefactive) solitons should be formed.
As the present experimental plasma contains both positive and negative ions, the electrons follow q-nonextensive distribution. Therefore, both of compressive and rarefactive nonplanar solitons can be sustained depending upon the choice of plasma parameters. Figures 8 and 9 show that each of the variation in nonextensivity (determinable through q) and the electron-to-positive ion density ratio parameter μ affects significantly the formation of hump and dip solitons. The nonlinearity coefficient A of Kortewede de Vries is a very crucial factor that can be utilized to determine the nature of the solitary waves. Whenever this coefficient is positive i.e., $A > 0$ it means compressive solitons exist, whereas $A < 0$ means the rarefactive solitons will be formed. Figure 10 has been included to point out that both hump and dip solitons can be formed in electronegative plasmas but formation of such structures is highly crucial to the choice of plasma parameters.
It has been noted that variation in parameter q, plays a crucial role in deciding amplitude and widths of the solitons (either compressive and rarefactive solitons). Also such solitons are highly influenced by parameters like α μ θ_i and θ_nThe significant conclusions of the present studies are:

- IA solitons (fast mode) sustained in spherical geometry ($p = 2$) are higher as compared to those formed in cylindrical geometry ($p = 1$).
- the spherical solitons move faster than those formed in cylindrical geometry. In other words, the nonplanar solitary waves are modified largely as compared to those of the solitary waves propagating in planar geometry.

5. Conclusions

We have derived the nonplanar evolution equation for ion acoustic IA waves in an electronegative plasma along with nonextensive q-distributed electrons using a multiple scale expansion method. The exact solution of this evolution equation in the planar geometry is obtained and it has been used to obtain an analytical approximate progressive wave

solution for the nonplanar evolution equation. To check the accuracy of the obtained analytical approximate solution, the illustrations are compared with the numerical spectral solution of the same nonplanar evolution equation. Applying suitable relevant plasma parameters the results are prepared/presented in the form of 2D and 3D figures. These figures indicate suitability of the applied technique and approximate analytical solutions for the progressive wave. A parametric study can be performed using this technique/solution which has no CPU time-consuming or round off error. A parametric study is presented to investigate influence of different plasma parameters on the nonlinear evolution solution behavior. A parametric study is presented to investigate the effect of different physical parameters on the nonlinear evolution solution behavior. Debye length is modified owing to the variations in important plasma parameters and, hence consequent modifications occur in the amplitude and width of the solitons.
Such electronegative plasmas like (H^+, O_{-2}) and (H^+, H^-) exist in D and F regions of the Earth's ionosphere [8,59,60]. Present studies can be helpful in understanding plasma waves that can be sustained both in space [59], and laboratory experiments holding different polarity ions (varying in masses) and electrons [60,61].

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