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EDGE-ZAGREB INDICES OF GRAPHS

C. YAMAC[1], M. S. OZ[2], I. N. CANGUL[1], §

Abstract. The algebraic study of graph matrices is an important area of Graph Theory giving information about the chemical and physical properties of the corresponding molecular structure. In this paper, we deal with the edge-Zagreb matrices defined by means of Zagreb indices which are the most frequently used graph indices.

Keywords: Graphs, adjacency, edge adjacency, energy.

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1. Introduction

Let $G = (V, E)$ be a graph with $| V(G) |= n$ vertices and $| E(G) |= m$ edges. For a vertex $v \in V(G)$, we denote the degree of v by d_v or $d_G(v)$. In particular, a vertex with degree one is called a pendant vertex. With slight abuse of language, one can use the term "pendant edge" for an edge having a pendant vertex. If u and v are two vertices of G connected by an edge e, then this situation is denoted by $e = uv$. In such a case, the vertices u and v are called adjacent vertices and the edge e is said to be incident with u and v. The study of adjacency and incidency with the help of corresponding matrices is a well known application of Graph Theory to Molecular Chemistry and the sub area of Graph Theory dealing with the energy of a graph is called Spectral Graph Theory which uses linear algebraic methods to calculate eigenvalues of a graph resulting in the molecular energy of that graph. In that sense, matrices are very helpful in the spectral study of graphs modelling some chemical structures. Apart from three most important kinds of matrices, that are Laplacian, adjacency and incidency matrices, there are nearly one hundred types of graph matrices, some giving important information about the molecules that are modelled by the corresponding graph. In [2], some formulae and recurrence relations on spectral polynomials of some graphs were calculated.

Topological graph indices are defined and used in many fields of science to study several properties of different objects such as atoms, molecules, people, countries, cities, firms, *etc.*

[1] Faculty of Arts and Science, Department of Mathematics, Bursa Uludag University, 16059 Bursa, Turkey.
e-mail: cilemyamac@hotmail.com, https://orcid.org/0000-0001-6044-1578.
e-mail: cangul@uludag.edu.tr, https://orcid.org/0000-0002-0700-5774, corresponding author.
[2] Department of Mathematics, Bursa Technical University, Bursa, Turkey.
e-mail: sinan.oz@btu.edu.tr, https://orcid.org/0000-0002-6206-0362.
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In particular, several topological graph indices have been defined and studied by many mathematicians and chemists as most graphs are generated from molecules by replacing atoms with vertices and bonds with edges. They are defined as topological graph invariants measuring several physical, chemical, pharmacological, pharmaceutical, biological, *etc.* properties of graphs modelling real life cases. They mainly can be grouped into three classes according to the way they are defined: by means of vertex degrees, matrices or distances.

Two of the well-known degree-based topological graph indices are called the first and second Zagreb indices denoted by $M_1(G)$ and $M_2(G)$, respectively:

$$M_1(G) = \sum_{u \in V(G)} d_G^2(u) \quad \text{and} \quad M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v).$$

These indices were defined in 1972 by Gutman and Trinajstic, [5]. In [3], some results on the first Zagreb index together with some other indices were obtained. In [4], the multiplicative versions of these indices were studied. Some relations between Zagreb indices and some other indices such as ABC, GA and Randic indices were obtained in [7]. Zagreb indices of subdivision graphs were studied in [9] and these were calculated for the line graphs of the subdivision graphs in [8]. A more generalized version of subdivision graphs is called r-subdivision graphs and Zagreb indices of r-subdivision graphs were calculated in [10]. These indices were calculated for several important graph classes in [11]. In this paper, we deal with the edge-Zagreb matrices defined by means of the second Zagreb index, see e.g. [1] and [6].

2. Edge-Zagreb matrices and polynomials

Let G be a connected simple graph having n vertices and m edges. The edge-Zagreb matrix $ZM(G)$ of G is a square matrix $[a_{ij}]_{n \times n}$ determined by the adjacency of vertices as follows:

$$a_{ij} = \begin{cases} d(v_i)d(v_j), & \text{if the vertices } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

We shall form the edge-Zagreb matrices $ZM(G)$ for several graphs and by means of them, we shall calculate the edge-Zagreb polynomials $P_G^{ez}(\lambda)$. These polynomials are the characteristic polynomials and their roots are the eigenvalues of the edge-Zagreb matrix. The sum of the absolute values of these eigenvalues gives the edge-Zagreb energy of the corresponding graph.

In our studies, we come up with a special square matrix given below. It does not correspond to a graph, but it appears frequently in our calculations:

$$\begin{bmatrix} 0 & 4 & 0 & \cdots & 0 & 0 & 0 \\ 4 & 0 & 4 & \cdots & 0 & 0 & 0 \\ 0 & 4 & 0 & \cdots & 0 & 0 & 0 \\ . & . & . & \cdots & . & . & . \\ 0 & 0 & 0 & \cdots & 0 & 4 & 0 \\ 0 & 0 & 0 & \cdots & 4 & 0 & 4 \\ 0 & 0 & 0 & \cdots & 0 & 4 & 0 \end{bmatrix}$$

We shall denote this matrix by Δ_n. The characteristic polynomial corresponding to this matrix will be denoted by $P_{\Delta_n}(\lambda)$ and is equal to

$$P_{\Delta_n}(\lambda) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k 2^{4k} \binom{n-k}{n-2k} \lambda^{n-2k}.$$

We now start the study of edge-Zagreb matrices. First we start with the path graph:

Theorem 2.1. *For $n \geq 4$, the formula for the edge-Zagreb polynomial of the path graph P_n obtained by means of the edge-Zagreb matrix is*

$$P^{ez}_{P_n}(\lambda) = \lambda^2 P_{\Delta_{n-2}}(\lambda) - 8\lambda P_{\Delta_{n-3}}(\lambda) + 16 P_{\Delta_{n-4}}(\lambda)$$

where $P^{ez}_{P_1}(\lambda) = \lambda$, $P^{ez}_{P_2}(\lambda) = \lambda^2 - 1$, and $P^{ez}_{P_3}(\lambda) = \lambda^3 - 8\lambda$.

Proof. The edge-Zagreb matrix of P_n is

$$\mathrm{ZM}(P_n) = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2 & 0 & 4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 4 & \cdots & 0 & 0 & 0 \\ . & . & . & . & . & \cdots & . & . & . \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2 & 0 \end{bmatrix}.$$

Hence

$$P^{ez}_{P_n}(\lambda) = |\lambda I - \mathrm{ZM}(P_n)| = \begin{vmatrix} \lambda & -2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -2 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -4 & \lambda & -4 & \cdots & 0 & 0 & 0 \\ . & . & . & . & . & \cdots & . & . & . \\ 0 & 0 & 0 & 0 & 0 & \cdots & \lambda & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & \lambda & -2 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -2 & \lambda \end{vmatrix}.$$

For $n = 1$, the one dimensional matrix whose entry is 0, so $P^{ez}_{P_1}(\lambda) = \lambda$. For $n = 2$ and 3, the proof is clear. So we may assume that $n \geq 4$. If we calculate $P^{ez}_{P_n}(\lambda)$ according to the first row, we get

$$|\lambda I - \mathrm{ZM}(P_n)| = \lambda \begin{vmatrix} \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ -4 & \lambda & -4 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & \cdots & 0 & 0 & 0 \\ . & . & . & & . & . & . \\ 0 & 0 & 0 & \cdots & \lambda & -4 & 0 \\ 0 & 0 & 0 & \cdots & -4 & \lambda & -2 \\ 0 & 0 & 0 & \cdots & 0 & -2 & \lambda \end{vmatrix} + 2 \begin{vmatrix} -2 & -4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda & -4 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & \cdots & 0 & 0 & 0 \\ . & . & . & & . & . & . \\ 0 & 0 & 0 & \cdots & \lambda & -4 & 0 \\ 0 & 0 & 0 & \cdots & -4 & \lambda & -2 \\ 0 & 0 & 0 & \cdots & 0 & -2 & \lambda \end{vmatrix}.$$

If we calculate the second determinant according to the first column, we get

$$|\lambda I - \mathrm{ZM}(P_n)| = \lambda \begin{vmatrix} \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ -4 & \lambda & -4 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & \cdots & 0 & 0 & 0 \\ . & . & . & & . & . & . \\ 0 & 0 & 0 & \cdots & \lambda & -4 & 0 \\ 0 & 0 & 0 & \cdots & -4 & \lambda & -2 \\ 0 & 0 & 0 & \cdots & 0 & -2 & \lambda \end{vmatrix} - 4 \begin{vmatrix} \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ -4 & \lambda & -4 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & \cdots & 0 & 0 & 0 \\ . & . & . & & . & . & . \\ 0 & 0 & 0 & \cdots & \lambda & -4 & 0 \\ 0 & 0 & 0 & \cdots & -4 & \lambda & -2 \\ 0 & 0 & 0 & \cdots & 0 & -2 & \lambda \end{vmatrix}.$$

The dimension of the first matrix is $(n-1)\times(n-1)$ and the dimension of the second matrix is $(n-2)\times(n-2)$. Let's calculate the determinant of the first and second matrices with respect to the last row. Then we get

$$\lambda(-1)^{2n-3}(-2)\begin{vmatrix} \lambda & -4 & 0 & \cdots & 0 & 0 & 0\\ -4 & \lambda & -4 & \cdots & 0 & 0 & 0\\ 0 & -4 & \lambda & \cdots & 0 & 0 & 0\\ . & . & . & & . & . & .\\ 0 & 0 & 0 & \cdots & \lambda & -4 & 0\\ 0 & 0 & 0 & \cdots & -4 & \lambda & 0\\ 0 & 0 & 0 & \cdots & 0 & -4 & -2 \end{vmatrix} + (-1)^{2n-2}\lambda^2\begin{vmatrix} \lambda & -4 & 0 & \cdots & 0 & 0 & 0\\ -4 & \lambda & -4 & \cdots & 0 & 0 & 0\\ 0 & -4 & \lambda & \cdots & 0 & 0 & 0\\ . & . & . & & . & . & .\\ 0 & 0 & 0 & \cdots & \lambda & -4 & 0\\ 0 & 0 & 0 & \cdots & -4 & \lambda & -4\\ 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{vmatrix}$$

$$+\ (4)(-1)^{2n-5}(2)\begin{vmatrix} \lambda & -4 & 0 & \cdots & 0 & 0 & 0\\ -4 & \lambda & -4 & \cdots & 0 & 0 & 0\\ 0 & -4 & \lambda & \cdots & 0 & 0 & 0\\ . & . & . & & . & . & .\\ 0 & 0 & 0 & \cdots & \lambda & -4 & 0\\ 0 & 0 & 0 & \cdots & -4 & \lambda & 0\\ 0 & 0 & 0 & \cdots & 0 & -4 & -2 \end{vmatrix} + (-1)^{2n-4}(-4)\lambda\begin{vmatrix} \lambda & -4 & 0 & \cdots & 0 & 0 & 0\\ -4 & \lambda & -4 & \cdots & 0 & 0 & 0\\ 0 & -4 & \lambda & \cdots & 0 & 0 & 0\\ . & . & . & & . & . & .\\ 0 & 0 & 0 & \cdots & \lambda & -4 & 0\\ 0 & 0 & 0 & \cdots & -4 & \lambda & -4\\ 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{vmatrix}.$$

Calculating the first and third determinants according to the last column gives

$$(-4)\lambda\begin{vmatrix} \lambda & -4 & 0 & \cdots & 0 & 0 & 0\\ -4 & \lambda & -4 & \cdots & 0 & 0 & 0\\ 0 & -4 & \lambda & \cdots & 0 & 0 & 0\\ . & . & . & & . & . & .\\ 0 & 0 & 0 & \cdots & \lambda & -4 & 0\\ 0 & 0 & 0 & \cdots & -4 & \lambda & -4\\ 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{vmatrix} + \lambda^2\begin{vmatrix} \lambda & -4 & 0 & \cdots & 0 & 0 & 0\\ -4 & \lambda & -4 & \cdots & 0 & 0 & 0\\ 0 & -4 & \lambda & \cdots & 0 & 0 & 0\\ . & . & . & & . & . & .\\ 0 & 0 & 0 & \cdots & \lambda & -4 & 0\\ 0 & 0 & 0 & \cdots & -4 & \lambda & -4\\ 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{vmatrix}$$

$$+\ 16\begin{vmatrix} \lambda & -4 & 0 & \cdots & 0 & 0 & 0\\ -4 & \lambda & -4 & \cdots & 0 & 0 & 0\\ 0 & -4 & \lambda & \cdots & 0 & 0 & 0\\ . & . & . & & . & . & .\\ 0 & 0 & 0 & \cdots & \lambda & -4 & 0\\ 0 & 0 & 0 & \cdots & -4 & \lambda & -4\\ 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{vmatrix} + (-4)\lambda\begin{vmatrix} \lambda & -4 & 0 & \cdots & 0 & 0 & 0\\ -4 & \lambda & -4 & \cdots & 0 & 0 & 0\\ 0 & -4 & \lambda & \cdots & 0 & 0 & 0\\ . & . & . & & . & . & .\\ 0 & 0 & 0 & \cdots & \lambda & -4 & 0\\ 0 & 0 & 0 & \cdots & -4 & \lambda & -4\\ 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{vmatrix}.$$

The dimensions of the first and last matrices are $(n-3)\times(n-3)$, the dimension of the second matrix is $(n-2)\times(n-2)$, and finally the dimension of the third matrix is $(n-4)\times(n-4)$. Consequently, we can write the formula for the edge-Zagreb polynomial of the path graph P_n as follows:

$$P^{ez}_{P_n}(\lambda) = \lambda^2 P_{\Delta_{n-2}}(\lambda) - 8\lambda P_{\Delta_{n-3}}(\lambda) + 16 P_{\Delta_{n-4}}(\lambda).$$

□

We now obtain the formula for the edge-Zagreb polynomial of the cycle graph C_n:

Theorem 2.2. *The formula for the edge-Zagreb polynomial of the cycle graph C_n obtained by means of the edge-Zagreb matrix is*

$$P^{ez}_{C_n}(\lambda) = -32\cdot 2^{2n-4} - 32P_{\Delta_{n-2}}(\lambda) + \lambda P_{\Delta_{n-1}}(\lambda).$$

Proof. The edge-Zagreb matrix of C_n is

$$\mathrm{ZM}(C_n) = \begin{bmatrix} 0 & 4 & 0 & 0 & 0 & \cdots & 0 & 0 & 4\\ 4 & 0 & 4 & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & 4 & 0 & 4 & 0 & \cdots & 0 & 0 & 0\\ 0 & 0 & 4 & 0 & 4 & \cdots & 0 & 0 & 0\\ . & . & . & . & . & \cdots & . & . & .\\ 0 & 0 & 0 & 0 & 0 & \cdots & 4 & 0 & 4\\ 4 & 0 & 0 & 0 & 0 & \cdots & 0 & 4 & 0 \end{bmatrix}.$$

Hence

$$|\lambda I - \mathrm{ZM}(C_n)| = \begin{bmatrix} \lambda & -4 & 0 & 0 & 0 & \cdots & 0 & 0 & -4 \\ -4 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -4 & \lambda & -4 & \cdots & 0 & 0 & 0 \\ . & . & . & . & . & \cdots & . & . & . \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & \lambda & -4 \\ -4 & 0 & 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{bmatrix}.$$

If we calculate this determinant according to the last row, we get

$$(-1)^{n+2}4 \begin{vmatrix} -4 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -4 \\ \lambda & -4 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -4 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ . & . & . & . & . & \cdots & . & . & . \\ 0 & 0 & 0 & 0 & 0 & \cdots & \lambda & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & \lambda & -4 \end{vmatrix} + (-1)^{2n}4 \begin{vmatrix} \lambda & -4 & 0 & 0 & 0 & \cdots & 0 & 0 & -4 \\ -4 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -4 & \lambda & -4 & \cdots & 0 & 0 & 0 \\ . & . & . & . & . & \cdots & . & . & . \\ 0 & 0 & 0 & 0 & 0 & \cdots & \lambda & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -4 & -4 \end{vmatrix}$$

$$+ \quad (-1)^{2n}\lambda \begin{vmatrix} \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ -4 & \lambda & -4 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & \cdots & 0 & 0 & 0 \\ . & . & . & & . & . & . \\ 0 & 0 & 0 & \cdots & \lambda & -4 & 0 \\ 0 & 0 & 0 & \cdots & -4 & \lambda & -4 \\ 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{vmatrix}.$$

If we calculate the determinant of the first matrix here with respect to the first row and calculate the determinant of the second matrix according to the last column, we get five determinants of dimensions $(n-2)\times(n-2)$, $(n-2)\times(n-2)$, $(n-2)\times(n-2)$, $(n-2)\times(n-2)$ and $(n-1)\times(n-1)$, respectively. Hence

$$\begin{aligned} P^{ez}_{C_n}(\lambda) &= (-1)^{n+5}16(-4)^{n-2} + (-1)^{2n+3}16P_{\Delta_{n-2}}(\lambda) + (-1)^{3n+1}16(-4)^{n-2} \\ &\quad +(-1)^{4n-1}16P_{\Delta_{n-2}}(\lambda) + \lambda P_{\Delta_{n-1}}(\lambda). \\ &= -32 \cdot 2^{2n-4} - 32P_{\Delta_{n-2}}(\lambda) + \lambda P_{\Delta_{n-1}}(\lambda), \end{aligned}$$

which is the required result.

□

The following result is very useful in the calculation of some determinants below:

Lemma 2.1. *If we divide a matrix A into four block matrices*

$$A = \left[\begin{array}{c|c} M & 0 \\ \hline K & N \end{array}\right] \text{ or } A = \left[\begin{array}{c|c} M & K \\ \hline 0 & N \end{array}\right]$$

where M and N are square matrices, then the determinant of A is equal to

$$|A| = |M|\,|N|\,.$$

It is now turn to calculate the edge-Zagreb polynomial of the tadpole graphs:

Theorem 2.3. *The formula for the edge-Zagreb polynomial of the tadpole graph* $T_{r,s}$ *obtained by means of the edge-Zagreb matrix is*

$$P^{ez}_{T_{r,s}}(\lambda)=\begin{cases}\left(\lambda^2-9\right)P_{\Delta_{r-1}}(\lambda)-72\lambda P_{\Delta_{r-2}}(\lambda)-9\cdot 2^{2r-1}\lambda\ , & s=1\\[2ex] \begin{array}{l}\lambda^2\left[\lambda P_{\Delta_{r-1}}(\lambda)-72P_{\Delta_{r-2}}(\lambda)-9\cdot 2^{2r-1}\right]\\ -40\lambda P_{\Delta_{r-1}}(\lambda)+288\left(P_{\Delta_{r-2}}(\lambda)+2^{2r-4}\right)\end{array}, & s=2\\[2ex] \begin{array}{l}\lambda P_{\Delta_{s-1}}(\lambda)\left[\lambda P_{\Delta_{r-1}}(\lambda)-72P_{\Delta_{r-2}}(\lambda)-9\cdot 2^{2r-1}\right]\\ +P_{\Delta_{s-2}}(\lambda)\left[-40\lambda P_{\Delta_{r-1}}(\lambda)+288\left(P_{\Delta_{r-2}}(\lambda)+2^{2r-4}\right)\right]\ ,\\ +P_{\Delta_{s-3}}(\lambda)\left[144P_{\Delta_{r-1}}(\lambda)\right].\end{array} & s>2.\end{cases}$$

Proof. Let $s=1$. First note that

$$P^{ez}_{T_{r,1}}(\lambda)=\begin{vmatrix}\lambda & -3 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\ -3 & \lambda & -6 & 0 & 0 & \cdots & 0 & 0 & -6\\ 0 & -6 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0\\ 0 & 0 & -4 & \lambda & -4 & \cdots & 0 & 0 & 0\\ . & . & . & . & . & \cdots & . & . & .\\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & \lambda & -4\\ 0 & -6 & 0 & 0 & 0 & \cdots & 0 & -4 & \lambda\end{vmatrix}.$$

If we calculate this determinant, we get the required result.

$$P^{ez}_{T_{r,1}}(\lambda)=\left(\lambda^2-9\right)P_{\Delta_{r-1}}(\lambda)-72\lambda P_{\Delta_{r-2}}(\lambda)-9\cdot 2^{2r-1}\lambda.$$

Let now $s=2$. Then $P^{ez}_{T_{r,2}}(\lambda)$ is

$$P^{ez}_{T_{r,2}}(\lambda)=\begin{vmatrix}\lambda & -2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\ -2 & \lambda & -6 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & -6 & \lambda & -6 & 0 & 0 & 0 & \cdots & 0 & 0 & -6\\ 0 & 0 & -6 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & 0 & 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0\\ . & . & . & . & \cdots & . & . & . & . & . & .\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -4 & \lambda & -4 & 0\\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -4 & \lambda & -4\\ 0 & 0 & -6 & 0 & \cdots & 0 & 0 & 0 & 0 & -4 & \lambda\end{vmatrix}.$$

If we calculate this determinant with respect to the third row, then we get

$$6\begin{vmatrix}\lambda & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\ -2 & -6 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & -6 & \lambda & -4 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & 0 & -4 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & 0 & 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0\\ . & . & . & \cdots & . & . & . & . & . & . & .\\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -4 & \lambda & -4 & 0\\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -4 & \lambda & -4\\ 0 & -6 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & -4 & \lambda\end{vmatrix}$$

$$
+\ \lambda \begin{vmatrix}
\lambda & -2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-2 & \lambda & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \lambda & -4 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -4 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\
. & . & . & \cdots & . & . & . & . & . & . & . \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & -4 & \lambda & -4 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -4 & \lambda & -4 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & -4 & \lambda
\end{vmatrix}
$$

$$
+\ 6 \begin{vmatrix}
\lambda & -2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-2 & \lambda & -6 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -6 & -4 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\
. & . & . & . & \cdots & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & -4 & \lambda & -4 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -4 & \lambda & -4 \\
0 & 0 & -6 & 0 & \cdots & 0 & 0 & 0 & 0 & -4 & \lambda
\end{vmatrix}
$$

$$
+\ (-1)^{r+s+3}(-6) \begin{vmatrix}
\lambda & -2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-2 & \lambda & -6 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & -6 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & -4 & \lambda & -4 & \cdots & 0 & 0 \\
. & . & . & . & \cdots & . & . & . & . & . \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & -4 & \lambda & -4 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -4 & \lambda \\
0 & 0 & -6 & 0 & \cdots & 0 & 0 & 0 & 0 & -4
\end{vmatrix}.
$$

If we calculate these determinants according to the second row, first row, third row and last row, respectively, and use Lemma 2.1 together with the properties of upper triangular determinants, we get

$$
\begin{aligned}
P^{ez}_{T_{r,2}}(\lambda) &= \lambda^2\left[\lambda P_{\Delta_{r-1}}(\lambda) - 72P_{\Delta_{r-2}}(\lambda) - 9\cdot 2^{2r-1}\right] - 40\lambda P_{\Delta_{r-1}}(\lambda) \\
&\quad +288\left(P_{\Delta_{r-2}}(\lambda) + 2^{2r-4}\right).
\end{aligned}
$$

Lastly, let $s \geq 3$. Then $P^{ez}_{T_{r,s}}(\lambda)$ is

$$
P^{ez}_{T_{r,s}}(\lambda) = \begin{vmatrix}
\lambda & -2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -4 & \lambda & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
. & \cdots & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & \cdots & 0 & 0 & -4 & \lambda & -4 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & -4 & \lambda & -6 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 & -6 & \lambda & -6 & 0 & 0 & \cdots & 0 & -6 \\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & -6 & \lambda & -4 & 0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & -4 & \lambda & -4 & \cdots & 0 & 0 \\
. & \cdots & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & \lambda & -4 \\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & 0 & -4 & \lambda
\end{vmatrix}.
$$

Proceeding as above, finally we get

$$
\begin{aligned}
P^{ez}_{T_{r,s}}(\lambda) &= \lambda P_{\Delta_{s-1}}(\lambda)\left[\lambda P_{\Delta_{r-1}}(\lambda) - 72P_{\Delta_{r-2}}(\lambda) - 9\cdot 2^{2r-1}\right] \\
&+ P_{\Delta_{s-2}}(\lambda)\left[-40\lambda P_{\Delta_{r-1}}(\lambda) + 288\left(P_{\Delta_{r-2}}(\lambda) + 2^{2r-4}\right)\right] \\
&+ P_{\Delta_{s-3}}(\lambda)\left[144P_{\Delta_{r-1}}(\lambda)\right].
\end{aligned}
$$

□

Next we calculate the edge-Zagreb polynomial of the star graph S_n:

Theorem 2.4. *The formula for the edge-Zagreb polynomial of the star graph S_n obtained by means of the edge-Zagreb matrix is*

$$P^{ez}_{S_n}(\lambda) = \lambda^{n-2}\left[\lambda^2 - (n-1)^3\right].$$

Proof. $P^{ez}_{S_n}(\lambda)$ is

$$P^{ez}_{S_n}(\lambda) = \begin{vmatrix} \lambda & -(n-1) & 0 & 0 & \cdots & 0 & 0 & 0 \\ -(n-1) & \lambda & -(n-1) & -(n-1) & \cdots & -(n-1) & -(n-1) & -(n-1) \\ 0 & -(n-1) & \lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & -(n-1) & 0 & \lambda & \cdots & 0 & 0 & 0 \\ . & . & . & . & \cdots & . & 0 & . \\ 0 & -(n-1) & 0 & 0 & \cdots & 0 & \lambda & 0 \\ 0 & -(n-1) & 0 & 0 & \cdots & 0 & 0 & \lambda \end{vmatrix}.$$

If we divide the matrix into block matrices so that the upper left one is 2×2 and the lower right one is $(n-2) \times (n-2)$, and if we use the following elementary row operations

$$\frac{n-1}{\lambda}R_3 + R_2 \longrightarrow R_2,$$
$$\frac{n-1}{\lambda}R_4 + R_2 \longrightarrow R_2,$$
$$\vdots$$
$$\frac{n-1}{\lambda}R_n + R_2 \longrightarrow R_2,$$

then we get

$$P^{ez}_{S_n}(\lambda) = \begin{vmatrix} \lambda & -(n-1) & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -(n-1) & \lambda - \frac{(n-2)(n-1)^2}{\lambda} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -(n-1) & \lambda & 0 & \cdots & 0 & 0 & 0 & 0 \\ . & . & . & . & \cdots & . & . & . & . \\ 0 & -(n-1) & 0 & 0 & \cdots & 0 & 0 & \lambda & 0 \\ 0 & -(n-1) & 0 & 0 & \cdots & 0 & 0 & 0 & \lambda \end{vmatrix}$$

which gives the required result. □

We similarly obtain the following result:

Theorem 2.5. *The formula for the edge-Zagreb polynomial of the complete graph K_n obtained by means of the edge-Zagreb matrix is*

$$P^{ez}_{K_n}(\lambda) = \left[\lambda - (n-1)^3\right]\left[\lambda + (n-1)^2\right]^{n-1}.$$

Proof. As

$$P^{ez}_{K_n}(\lambda) = \begin{vmatrix} \lambda & -(n-1)^2 & -(n-1)^2 & \cdots & -(n-1)^2 \\ -(n-1)^2 & \lambda & -(n-1)^2 & \cdots & -(n-1)^2 \\ . & . & . & . & . \\ -(n-1)^2 & \cdots & -(n-1)^2 & \lambda & -(n-1)^2 \\ -(n-1)^2 & \cdots & -(n-1)^2 & -(n-1)^2 & \lambda \end{vmatrix},$$

applying the elementary row operation $R_1 + R_2 + \cdots + R_n \longrightarrow R_1$, we get

$$\left[\lambda - (n-1)^3\right] \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ -(n-1)^2 & \lambda & -(n-1)^2 & \cdots & -(n-1)^2 \\ . & . & . & . & . \\ -(n-1)^2 & \cdots & -(n-1)^2 & \lambda & -(n-1)^2 \\ -(n-1)^2 & \cdots & -(n-1)^2 & -(n-1)^2 & \lambda \end{vmatrix}$$

and if we use the operations

$$\begin{gathered} (n-1)^2 R_1 + R_2 \longrightarrow R_2, \\ (n-1)^2 R_1 + R_3 \longrightarrow R_3, \\ \vdots \\ (n-1)^2 R_1 + R_n \longrightarrow R_n, \end{gathered}$$

we finally get

$$P^{ez}_{K_n}(\lambda) = \left[\lambda - (n-1)^3\right]\left[\lambda + (n-1)^2\right]^{n-1}.$$

□

The following result can be proven similarly:

Theorem 2.6. *The formula for the edge-Zagreb polynomial of the complete bipartite graph $K_{r,s}$ obtained by means of the edge-Zagreb matrix is*

$$P^{ez}_{K_{r,s}}(\lambda) = \lambda^{r+s-2}\left(\lambda^2 - r^3 s^3\right).$$

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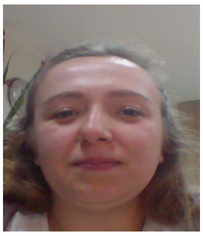
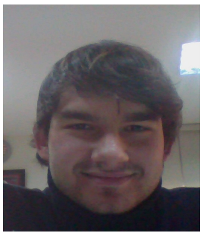


Cilem Yamac graduated and also had her MSc degree from the Mathematics Department of Bursa Uludag University.



Mert Sinan Oz graduated from Mathematics Department of Ege University is a PhD student at the Department of Mathematics of Bursa Uludag University and also working as a full time research assistant in Bursa Technical University.

Prof. Dr. Ismail Naci Cangul for photo and short biography see: Vol.6, No.2.