TWMS J. App. Eng. Math. V.10, No.1, 2010 , pp. ...

ON THE NUMERICAL SOLUTION TO OPTIMAL CONTROL PROBLEMS WITH NON-LOCAL CONDITIONS

K.R. AIDA-ZADE[1,2], V.M. ABDULLAYEV[3,4] , §

Abstract. Optimal control problems involving non-separated multipoint and integral conditions are investigated. For numerical solution to the problem, we propose to use first order optimization methods with application of the formulas for the gradient of the functional obtained in the work. To solve the adjoint boundary problems, we propose an approach. This approach makes it possible to reduce solving initial boundary problems to solving supplementary Cauchy problems and a linear algebraic system of equations. Results of numerical experiments are given.

Keywords: Optimal control, integral conditions, multipoint conditions, transfer method for intermediate conditions.

AMS Subject Classification: 49J15, 65L10

1. Introduction

Much research activity in the past years has been directed at solving boundary problems involving non-local multipoint and integral conditions, and the corresponding optimal control problems. This is connected with non-local character of information provided by measurement equipment. Namely, the measurements are not taken instantly, but during some time interval and the measurements at a separate point actually characterize the state of the object in some domain which contains the measurement point. Problems of this kind arise when controlling an object if it is otherwise impossible to affect the object instantly at time and locally at its separate points.

The investigations of boundary problems involving non-local conditions commenced at the beginning of the 20^{th} century [1, 2, 3], and they became more active by the efforts of many authors whose works were dedicated to both ordinary and partial differential

[1] Baku State University, Akad. Zahid Xalilov st. 23, Baku 1148, Azerbaijan.
[2] Institute of Mathematics and Mechanics of NAS of Azerbaijan, B.Vahabzade str. 9, Baku 1141, Azerbaijan
e-mail: kamil_aydazade@rambler.ru, ORCID: http://orcid.org/0000-0002-8439-5585;
[3] Azerbaijan State Oil and Industry University, 16/21 Azadliq Ave, Baku, Azerbaycan
[4] Institute of Control Systems of NAS of Azerbaijan, B.Vahabzade str. 9, Baku 1141, Azerbaijan
e-mail: vaqif_ab@rambler.ru, ORCID: https://orcid.org/0000-0001-7772-1226;
§ Manuscript received: January 29, 2018; accepted:
TWMS Journal of Applied and Engineering Mathematics, Vol.10, No.1;

equations [4, 5, 6, 7]. In [8, 9, 10, 11, 12, 13], optimal control problems for boundary problems involving non-local multipoint and integral conditions are investigated and necessary optimality conditions are obtained.

For linear boundary problems involving multipoint conditions, there exist efficient numerical methods of sweep and shift of conditions [14, 15, 16, 17]. For boundary problems involving integral conditions, the possibility of their reduction to problems involving multipoint conditions at the expense of introducing new variables and of increasing the number of differential equations has been implied. That is why any special numerical methods for problems of this kind have not been practically developed.

In the present work, we propose an approach to the numerical solution to boundary problems involving non-separated multipoint and integral conditions. This approach is similar to sweep method. We also describe how to use the approach when solving the corresponding optimal control problems. For optimal control problems, we obtain formulas for the gradient of the functional, as well as describe the numerical computation scheme based on first order optimization methods.

In the present work for optimal control problems, we obtain formulas for the gradient of the functional, as well as describe the numerical computation scheme based on first order optimization methods. Also, we propose an approach for numerical solution to the problem adjoint boundary problems.

2. Problem Statement and its investigation

Consider the following optimal control problem for the process described by an ordinary differential equations system, linear in the phase variable:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + C(t), \;\; t \in [t_0, T], \tag{1}$$

where $x(t) \in E^n$ is the phase variable; $u(t) \in U \subset E^r$ is the control function from the class of piecewise continuous functions, admissible values of $u(t)$ belong to a given compact set U; $A(t) \neq const$ is $(n \times n)$ matrix function, $B(t)$ is $(n \times r)$ matrix function, $C(t)$ is $n-$dimensional vector function, $A(t)$, $B(t)$, $C(t)$ are continuous with respect to t.

Non-separated multipoint and integral conditions are given in the following form:

$$\sum_{i=1}^{l_1} \int\limits_{\bar{t}_{2i-1}}^{\bar{t}_{2i}} \bar{D}_i(\tau)x(\tau)d\tau + \sum_{j=1}^{l_2} \tilde{D}_j x(\tilde{t}_j) = C_0, \tag{2}$$

where $\bar{D}_i\,(\tau)$ is the continuously differentiable $(n \times n)$ matrix function; $\tilde{D}_j$ is the $(n \times n)$ scalar matrix; C_0 is the n-dimensional vector; $\bar{t}_i$,$\tilde{t}_j$ time instances from $[t_0, T]$; $\bar{t}_{i+1} > \bar{t}_i$, $\tilde{t}_{j+1} > \tilde{t}_j$, $\quad i = 1, 2, ..., 2l_1 - 1$, $j = 1, 2, ..., l_2 - 1$, l_1, l_2 are given.

To be specific, without loss of generality, let us assume that

$$\min\left(\bar{t}_1,\; \tilde{t}_1\right) = t_0, \quad \max\left(\bar{t}_{2l_1},\; \tilde{t}_{l_2}\right) = T, \tag{3}$$

and for all $i = 1, 2, ..., 2l_1$, $j = 1, 2, ..., l_2$ and that the following natural condition holds

$$\tilde{t}_j \bar{\in} \left[\bar{t}_{2i-1}, \bar{t}_{2i}\right]. \tag{4}$$

The target functional is as follows:

$$J(u) = \Phi(\hat{x}) + \int\limits_{t_0}^{T} f^0(x, u)dt \to \min_{u(t) \in U}, \tag{5}$$

where the function Φ and its partial derivatives are continuous with respect to its arguments, and $f^0(x,u)$ is continuously differentiable with respect to (x,u) and continuous with respect to t; $\hat{t} = (\hat{t}_1, \hat{t}_2, ..., \hat{t}_{2l_1+l_2})$ is the ordered union of the sets $\tilde{t} = (\tilde{t}_1, \tilde{t}_2, ..., \tilde{t}_{l_2})$ and $\bar{t} = (\bar{t}_1, \bar{t}_2, ..., \bar{t}_{2l_1})$, i.e. $\hat{t}_j < \hat{t}_{j+1}$, $j = 1, 2, ..., 2l_1 + l_2 - 1$, $\hat{x}(\hat{t}) = (x(\hat{t}_1), x(\hat{t}_2), ..., x(\hat{t}_{2l_1+l_2})) \in R^{(2l_1+l_2)n}$.

The problem consists in determining the permissible control $u(t) \in U$ and the corresponding vector-function $x(t)$, which is a solution of the system of differential equations (1) satisfying the conditions (2) and minimizing the value of the objective functional (5).

Using the technique of the works [18, 19, 20, 21], we can obtain existence and uniqueness conditions for the solution to problem (1), (2) under every admissible control $u \in U$.

Theorem 2.1. *Let the matrix and vector functions $A(t)$, $B(t)$, $C(t)$ are continuous, and $D_i(t)$ are integrable for $t \in [t_0, T]$. For the solution of the system of differential equations (1), satisfying conditions (2), to exist and be unique under every permissible control $u(t) \in U$, it is necessary and sufficient that the following condition be satisfied:*

$$\det \left[\sum_{i=1}^{l_1} \int_{\bar{t}_{2i-1}}^{\bar{t}_{2i}} \bar{D}_i(\tau)\Phi(\tau, t_0)d\tau + \sum_{j=1}^{l_2} \tilde{D}_j \Phi(\tilde{t}_j, t_0) \right] \neq 0, \tag{6}$$

where $\Phi(t, \tau)$ is the fundamental matrix of solutions of the equation (1):

$$\dot{\Phi}_t(t, \tau) = A(t)\Phi(t), \ \Phi(t, t) = E, \ t \in [t_0, T],$$

E is the n-dimensional identity matrix.

To prove the theorem, we use the Cauchy formula for the solution of system (1):

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau) \left[B(\tau)u(\tau) + C(\tau)\right] d\tau. \tag{7}$$

Substituting (7) into ((2), we obtain a system of algebraic equations with respect to the initial condition $x_0 \in R^n$. For the solution of the algebraic system to exist and be unique, it is necessary that condition (6) be satisfied. After determining the initial condition x_0, it becomes clear that the solution of the Cauchy problem with respect to the system of differential equations (1) under the given conditions exists and is unique.

Theorem 2.2. *Assume that the admissible set of controls U is compact and convex, and functions $\Phi(\hat{x})$ and $f^0(x, u)$ are convex with respect to all their arguments. Then the solution of problem (1)-(5) exists. In case when one of the functions $\Phi(\hat{x}), f^0(x, u)$ is strictly convex, then the optimal solution of the problem is unique.*

Proof. We first prove the convexity of the functional in the problem (1)-(5). Let $u_1(t)$, $u_2(t)$ be arbitrary permissible controls from U, and $x_1(t)$, $x_2(t)$ are the corresponding solutions to the boundary-value problems (1) and (5), i.e.

$$\dot{x}_k(t) = A(t)x_k(t) + B(t)u_k(t) + C_k(t) \ , \tag{8}$$

$$\sum_{i=1}^{l_1} \int_{\bar{t}_{2i-1}}^{\bar{t}_{2i}} \bar{D}_i(\tau)x_k(\tau)d\tau + \sum_{j=1}^{l_2} \tilde{D}_j x_k(\tilde{t}_j) = C_0, \ k = 1, 2. \tag{9}$$

By the convexity of the permissible set U for an arbitrary $\lambda \in [0, 1]$, there takes place

$$u(t) = \lambda u_1(t) + (1 - \lambda)u_2(t) \in U.$$

Introduce the notation $x(t) = \lambda x_1(t) + (1-\lambda)x_2(t)$. Multiply both parts of (8) and (9) by λ when $k = 1$, and by $(1-\lambda)$ when $k = 2$; then, we obtain that the pair $(x(t), u(t))$ satisfies the system of differential equations (1) and the conditions (2), as well.

By the convexity of the functions $\Phi(\hat{x})$ and $f^0(x, u)$ with respect to all the arguments, we have:

$$J(u) = J(\lambda u_1(t) + (1-\lambda)u_2(t)) = \Phi(\lambda\hat{x}_1 + (1-\lambda)\hat{x}_2) +$$

$$+\int_{t_0}^{T} f^0(\lambda x_1 + (1-\lambda)x_2, \lambda u_1(t) + (1-\lambda)u_2(t))dt \leq \lambda\Phi(\hat{x}_1) + (1-\lambda)\Phi(\hat{x}_2) +$$

$$+\lambda\int_{t_0}^{T} f^0(x_1, u_1)dt + (1-\lambda)\int_{t_0}^{T} f^0(x_2, u_2)dt = \lambda J(u_1) + (1-\lambda)J(u_2). \quad (10)$$

It is clear that if one of the functions $\Phi(\hat{x})$, $f^0(x, u)$ is strictly convex, then the sign of the inequality in (10) is strict. Therefore, the functional of the problem (1)-(5) is strictly convex, and hence the minimum of the functional (5) is attained at a single point. The assertion of Theorem 2 follows from directly. □

3. Formula for the gradient of the functional of problem (1)-(5)

To solve optimal control problem (1)-(5) numerically with the application of first order optimization methods (see [22]), we obtain formulas for the gradient of the functional.

With respect to an arbitrary admissible process $(u(t), x(t;u))$, we define problem (1), (2) in increments, corresponding to an admissible control $\tilde{u} = u + \Delta u$:

$$\Delta\dot{x}(t) = A(t)\Delta x(t) + B(t)\Delta u(t), \ \ t \in [t_0, T] \ , \quad (11)$$

$$\sum_{i=1}^{l_1} \int_{\bar{t}_{2i-1}}^{\bar{t}_{2i}} \bar{D}_i(\tau)\Delta x(\tau)d\tau + \sum_{j=1}^{l_2} \tilde{D}_j \Delta x(\tilde{t}_j) = 0. \quad (12)$$

Here the following notations are used: $\Delta x(t) = x(t, \tilde{u}) - x(t, u)$, $\quad \Delta u(t) = \tilde{u}(t) - u(t)$.

Let $\psi(t)$ be an almost everywhere continuously differentiable vector function and let $\lambda \in E^n$ be as yet arbitrary numerical vector. Taking into account that $x(t)$ and $x(t)+\Delta x(t)$ are the solutions to problem (1)-(2) under corresponding values of the controls, for the increment of the functional (5), we obtain:

$$\Delta J(u) = J(u) - J(u+\Delta u) = \Phi(\hat{x} + \Delta\hat{x}) - \Phi(\hat{x}) + \int_{t_0}^{T} (f^0(x + \Delta x, u + \Delta u) - f^0(x))dt +$$

$$+\int_{t_0}^{T} \psi^*(t)\left[\Delta\dot{x}(t) - A(t)\Delta x(t) - B(t)\Delta u(t)\right]dt + \lambda^* \left[\sum_{i=1}^{l_1} \int_{\bar{t}_{2i-1}}^{\bar{t}_{2i}} \bar{D}_i(\tau)\Delta x(\tau)d\tau + \sum_{j=1}^{l_2} \tilde{D}_j \Delta x(\tilde{t}_j)\right],$$

where "*" is the transposition sign. Then for the increment of the functional, using the formula of partial integration, after grouping the corresponding terms, accurate within the terms of the first infinitesimal order, we obtain:

$$\Delta J(u) = \int_{t_0}^{T} \left[-\dot{\psi}^*(t) - \psi^*(t)A(t) + \lambda^* \sum_{i=1}^{l_1} \left[\chi(\bar{t}_{2i}) - \chi(\bar{t}_{2i-1})\right] \bar{D}_i(t) + \right.$$

$$+ f_x^0(x,u)\big]\,\Delta x(t)dt + \int_{t_0}^{T} \big\{f_u^0(x,u) - \psi^*(t)B(t)\big\}\Delta u(t)dt+$$

$$+ \sum_{k=2}^{2l_1+l_2-1} \left[\psi^{*-}(\hat{t}_k) - \psi^{*+}(\hat{t}_k) + \frac{\partial \Phi(\hat{x})}{\partial x(\hat{t}_k)}\right] \Delta x(\hat{t}_k)+$$

$$+ \sum_{j=1}^{l_2} \lambda^* \tilde{D}_j \Delta x(\tilde{t}_j) + \psi^*(T)\Delta x(T) - \psi^*(t_0)\Delta x(t_0)+$$

$$+ \int_{t_0}^{T} o_1(\|\Delta x(t)\|_{L_2^n[t_0,T]})dt + \int_{t_0}^{T} o_2(\|\Delta u(t)\|_{L_2^r[t_0,T]})dt + o_3(\left\|\Delta \hat{x}(\hat{t}_k)\right\|_{L_2^n[t_0,T]}), \tag{13}$$

where $\psi^+(\hat{t}_k) = \psi(\hat{t}_k + 0)$, $\psi^-(\hat{t}_k) = \psi(\hat{t}_k - 0)$, $k = 1, 2, ..., (2l_1 + l_2)$, $\chi(t)-$ is Heaviside function.

Let $\psi(t)$ be the solution to the following system of equations

$$\dot{\psi}(t) = -A^*(t)\psi(t) + \sum_{i=1}^{l_1} \left[\chi(\bar{t}_{2i}) - \chi(\bar{t}_{2i-1})\right] \bar{D}^*(t)\lambda + f_x^{0*}(x,u), \tag{14}$$

involving the following boundary conditions

$$\psi(t_0) = \begin{cases} \left(\frac{\partial \Phi(\hat{x})}{\partial x(\tilde{t}_1)}\right)^* + \tilde{D}_1^* \lambda, & if\ t_0 = \tilde{t}_1\,, \\ \left(\frac{\partial \Phi(\hat{x})}{\partial x(\bar{t}_1)}\right)^*, & if\ t_0 = \bar{t}_1\,, \end{cases} \tag{15}$$

$$\psi(T) = \begin{cases} -\left(\frac{\partial \Phi(\hat{x})}{\partial x(\tilde{t}_{l_2})}\right)^* - \tilde{D}_{l_2}^* \lambda, & if\ \tilde{t}_{l_2} = T\,, \\ -\left(\frac{\partial \Phi(\hat{x})}{\partial x(\bar{t}_{2l_1})}\right)^*, & if\ \bar{t}_{2l_1} = T, \end{cases} \tag{16}$$

as well as jump conditions at the intermediate $\tilde{t}_j$ such that $t_0 < \tilde{t}_j < T$,

$$\psi^+(\tilde{t}_j) - \psi^-(\tilde{t}_j) = \left(\frac{\partial \Phi(\hat{x})}{\partial x(\tilde{t}_j)}\right)^* + \tilde{D}_j^* \lambda, \qquad j = 1, 2, \dots l_2, \tag{17}$$

and at the points $\bar{t}_i$, $i = 1, 2, ..., 2\,l_1$ such that $t_0 < \bar{t}_i < T$,

$$\psi^+(\bar{t}_i) - \psi^-(\bar{t}_i) = \left(\frac{\partial \Phi(\hat{x})}{\partial x(\bar{t}_i)}\right)^*, \qquad i = 1, 2, \dots 2l_1. \tag{18}$$

Note the following about the estimate of the quantities $o_1(\|\Delta x(t)\|)$ and $o_3(\left\|\Delta x(\hat{t})\right\|)$. As it was noted above in the first paragraph, considered boundary problem (1)-(2) can be reduced to a non-local boundary problem involving both multipoint and two-point conditions. Problems of this kind are sufficiently well investigated in many works for the cases of different conditions imposed on the functions, on the parameters taking part in the statement, and for nonlinear problems as well. In works ([11, 12, 13]) for different variants of the conditions, the following estimate is obtained:

$$\|\Delta x(t)\| \le c\,\|\Delta u(t)\|\,, \tag{19}$$

where $c = const > 0$ does not depend on the choice of the admissible control.

It is clear that using the techniques of these works, we can obtain a similar estimate for boundary problem (1) and (2).

Thus the gradient of the target functional under the admissible control $u(t)$ in problem (1)-(3) is determined as follows:

$$\nabla J(u) = f_u^{0*}(x,u) - B^*(t)\psi(t), \tag{20}$$

where $x(t)$ and $\psi(t)$ are the solutions to direct system (1), (2) and to adjoint system (14)-(18), respectively, corresponding to this control.

On condition that there is a constructive algorithm for computing the value of the gradient of functional (25), it is not difficult to implement iterative techniques of first order minimization, particularly, of gradient projection method (see [22]):

$$u^{k+1}(t) = P_U(u^k(t) - \alpha_k \nabla J(u^k(t))), \quad k = 0,1,..., \tag{21}$$

$$\alpha_k = \arg\min_{\alpha \geq 0} J(P_U(u^k(t) - \alpha \nabla J(u^k(t)))) \,, \tag{22}$$

where $P_U(v)$ is the projection operator of the element $v \in E^r$ on the admissible set U; α_k is the one-dimensional minimization step.

On every iteration (21)-(22), the calculation of the gradient of the functional under given control confronts with two the most essential difficulties associated with the specific character of the problem, namely, with the problem of solution to non-autonomous differential equations system involving non-separated multipoint and integral conditions (1), (2), and with the problem of solution to adjoint boundary problem (14)-(18), the non-local conditions of which contain an unknown n-dimensional vector of parameters λ. As a whole, system of relations (1), (2), (14)–(18) for determining the gradient of the functional under given control $u(t)$ is closed: to determine unknown $2n$ functions $x(t), \psi(t)$, their $2n$ initial conditions, and n-dimensional vector λ, we have $2n$-dimensional differential equations system, n conditions in (2) and $2n$ conditions in (15), (16).

Below, we give an algorithm for overcoming these difficulties. It is based on using the operation of shift of conditions developed in [16] for solving systems of equations involving non-separated intermediate conditions and boundary conditions, including unknown parameters as well (see [23]). The concept of shift of intermediate conditions generalizes the known operation of shift of boundary conditions, and is based on developing the results of the [16] applied to this class of problems.

4. Numerical scheme of solution to the problem

We shall give below an algorithm of computing the gradient of the target functional under given control. To solve the problem (1), (2) under given admissible function $u(t)$, we can make use of, for example, the numerical method proposed in [16].

Solve the adjoint boundary problem (14)–(18) on the condition that the phase variable $x(t)$, the solution to problem (1) and (2), is already determined for given $u(t)$ by application of the procedure described above. To avoid cumbersome expressions in formulas and in description of numerical scheme of solution given below, we assume that $\tilde{t}_1 = t_0$ and $\tilde{t}_{l_2} = T$, and rewrite adjoint problem (14)–(18) in the following form:

$$\dot{\psi}(t) = A_1(t)\psi(t) + \sum_{i=1}^{l_1} \left[\chi(\bar{t}_{2i}) - \chi(\bar{t}_{2i-1})\right] \bar{D}_i(t)\lambda + C_1(t), \tag{23}$$

with the following boundary conditions:

$$\tilde{G}_1\psi(t_0) = \tilde{K}_1 + \tilde{D}_1\lambda, \tag{24}$$

$$\psi(T) = -\tilde{K}_{l_2} - \tilde{D}_{l_2}\lambda, \tag{25}$$

and jump conditions at the intermediate points $\tilde{t}_j$, for which $t_0 < \tilde{t}_j < T$:

$$\psi^+(\tilde{t}_j) - \psi^-(\tilde{t}_j) = \tilde{K}_j + \tilde{D}_j\lambda, \quad j = 2, 3, ..., l_2 - 1, \tag{26}$$

and at the points $\bar{t}_i$, for which $t_0 < \bar{t}_i < T$, $i = 1, 2, \ldots 2l_1$:

$$\psi^+(\bar{t}_i) - \psi^-(\bar{t}_i) = \bar{K}_i, \quad i = 1, 2, ...2\,l_1. \tag{27}$$

Here $\bar{G}_1 = I_n$ is the n-dimensional identity matrix, and the following notations are introduced for the matrices and vectors:

$$A_1(t) = -A^*(t),\ C_1(t) = \partial f^0(x,u)/\partial x,\ \tilde{K}_j = \partial\Phi(\hat{x})/\partial x(\tilde{t}_j), \tilde{D}_j^* = \tilde{D}_j,\ j = 1, 2, ..., l_2,$$

$$\bar{D}_i^*(t) = \bar{D}_i(t),\ i = 1, 2, ..., l_1, \quad \bar{K}_i = \partial\Phi(\hat{x})/\partial x(\bar{t}_i), \quad i = 1, 2..., 2\,l_1,$$

that were already calculated when solving the direct problem.

In problem (23)–(27), defined by system of n differential equations (28), in general case, we have $2n$ boundary conditions that include the unknown n-dimensional vector λ. Thus the conditions of problem (23)–(27) are closed, but there is a specific character which lies in the presence of discontinuities of the function $\psi(t)$ defined by jumps (26).

Condition (24) is called a condition shifted to the right in the semi-interval $t \in [\tilde{t}_1, \tilde{t}_2)$ by the matrix and vector functions $G_1(t),\ D_1(t) \in E^{n\times n}$, $K_1(t) \in E^n$ such that

$$G_1(t_0) = G_1(\tilde{t}_1) = \tilde{G}_1, \quad K_1(t_0) = K_1(\tilde{t}_1) = \tilde{K}_1, \quad \tilde{D}_1(t_0) = \tilde{D}_1(\tilde{t}_1) = \tilde{D}_1, \tag{28}$$

if for the solution $\psi(t)$ to (23), the following relation holds:

$$G_1(t)\psi(t) = K_1(t) + D_1(t)\lambda\,, \quad t \in [\tilde{t}_1, \tilde{t}_2). \tag{29}$$

Next, using the results of [23], we give the techniques to find the shifting functions $G_1(t),\ D_1(t), K_1(t)$. By using formula (29), we shift initial conditions (24) to the point $t = \tilde{t}_2 - 0$ and, taking shift condition (26) into account at the point $t = \tilde{t}_2 + 0$, we obtain

$$G_1(\tilde{t}_2)\psi(\tilde{t}_2 + 0) = \left[\ K_1(\tilde{t}_2) + G(\tilde{t}_2)\tilde{K}_2\right] + \left[\ D_1(\tilde{t}_2) + G_1(\tilde{t}_2)\tilde{D}_2\right]\ \lambda.$$

Introducing the notations

$$\tilde{t}_2 = \tilde{t}_2 + 0, \quad \tilde{G}_1^1 = G_1(\tilde{t}_2), \quad \tilde{K}_1^1 = K_1(\tilde{t}_2) + G(\tilde{t}_2)\tilde{K}_2, \quad \tilde{D}_1^1 = D_1(\tilde{t}_2) + G_1(\tilde{t}_2)\tilde{D}_2,$$

we obtain the conditions similar to (23) and defined at the point $\tilde{t}_2$:

$$\tilde{G}_1^1\psi(\tilde{t}_2) = \tilde{K}_1^1 + \tilde{D}_1^1\lambda.$$

By shifting condition (24) $l_2 - 1$ times and taking (25) into account, we obtain a linear system of $2n$ algebraic equations with respect to $\psi(\tilde{t}_{l_2}) = \psi(T)$ and λ. After solving this system, we determine the vector function $\psi(t)$ from right to left from Cauchy problem with respect to (23).

Illustrate the stages of the shift process applied to condition (24). To be specific we assume that $[\bar{t}_1, \bar{t}_2]\ \subset [\tilde{t}_1, \tilde{t}_2)$ and $\tilde{t}_1 = t_0$. Shift of condition is carried out successively in the intervals $[\tilde{t}_1, \bar{t}_1)$, $[\bar{t}_1, \bar{t}_2)$, $[\bar{t}_2, \tilde{t}_2)$, by using formulas (29).

1) For $t \in [\tilde{t}_1, \bar{t}_1)$, we shift initial conditions (24) to the point $t = \bar{t}_1 - 0$ and, taking shift condition (27) into account at the point $t = \bar{t}_1$, we obtain

$$G_1(\bar{t}_1)\psi(\bar{t}_1 + 0) = \left[\ K_1(\bar{t}_1) + G_1(\bar{t}_1)\bar{K}_1\,\right] + D_1(\bar{t}_1)\,\lambda.$$

Assuming $\bar{t}_1 = \bar{t}_1 + 0$, introduce the notations

$$\tilde{G}_1^1 = G_1(\bar{t}_1), \quad \tilde{K}_1^1 = K_1(\bar{t}_1) + G_1(\bar{t}_1)\bar{K}_1, \quad \tilde{D}_1^1 = D_1(\bar{t}_1),$$

following which, we obtain the initial conditions similar to (24) and defined at the point $\bar{t}_1$, we obtain

$$\tilde{G}_1^1\psi(\bar{t}_1) = \tilde{K}_1^1 + \tilde{D}_1^1\lambda. \tag{30}$$

2) For $t \in [\bar{t}_1, \bar{t}_2)$, we shift conditions (30) to the point $t = \bar{t}_2 - 0$ and, taking shift condition (27) into account at the point $t = \bar{t}_2$, we obtain

$$G_1(\bar{t}_2)\psi(\bar{t}_2 + 0) = \left[\, K_1(\bar{t}_2) + G_1(\bar{t}_2)\bar{K}_2 \,\right] + D_1(\bar{t}_2)\,\lambda.$$

Assuming $\bar{t}_2 = \bar{t}_2 + 0$, introduce the notations

$$\tilde{G}_1^2 = G_1(\bar{t}_2), \quad \tilde{K}_1^2 = K_1(\bar{t}_2) + G_1(\bar{t}_2)\bar{K}_2\,, \quad \tilde{D}_1^2 = D_1(\bar{t}_2)\,,$$

and obtain initial conditions equivalent to (30) and defined at the point $\bar{t}_2$

$$\tilde{G}_1^2\psi(\bar{t}_2) = \tilde{K}_1^2 + \tilde{D}_1^2\lambda\,. \tag{31}$$

3) For $t \in [\bar{t}_2, \tilde{t}_2)$ we shift conditions (31) to the point $t = \tilde{t}_2 - 0$ and, taking shift condition (26) into account at the point $t = \tilde{t}_2$, we obtain

$$G_1(\tilde{t}_2)\psi(\tilde{t}_2 + 0) = \left[\, K_1(\tilde{t}_2) + G_1(\tilde{t}_2)\bar{K}_2 \,\right] + \left[D_1(\tilde{t}_2) + G_1(\tilde{t}_2)\tilde{D}_2\right]\,\lambda\,.$$

Assuming $\tilde{t}_2 = \tilde{t}_2 + 0$, introduce the notations

$$\tilde{G}_1^3 = G_1(\tilde{t}_2)\,, \quad \tilde{K}_1^3 = K_1(\tilde{t}_2) + G_1(\tilde{t}_2)\bar{K}_2\,, \quad \tilde{D}_1^3 = D_1(\tilde{t}_2) + G_1(\tilde{t}_2)\tilde{D}_2\,,$$

and obtain the conditions equivalent to (31) and defined at the point $\tilde{t}_2$

$$\tilde{G}_1^3\psi(\tilde{t}_2) = \tilde{K}_1^3 + \tilde{D}_1^3\lambda\,. \tag{32}$$

The functions $G_j(t),\ K_j(t),\ D_j(t),\ \ j = 1, 2, ..., l_2$ that shift conditions (24) successively to the right (i.e. the functions $G_j(t),\ K_j(t),\ D_j(t),\ \ j = 1, 2, ..., l_2$ must satisfy (28), (29)), are not determined uniquely. For example, it is possible to use functions proposed in the following theorem.

Theorem 4.1. *Let the functions $G_1(t),\ K_1(t),\ D_1(t)$ be the solution to the following Cauchy problems for $t \in (\tilde{t}_1, \tilde{t}_2]$:*

$$\dot{G}_1(t) = Q^0(t)G_1(t) - G_1(t)A_1(t), \qquad G_1(\tilde{t}_1) = \tilde{G}_1,$$

$$\dot{D}_1(t) = Q^0(t)D_1(t) + G_1(t)\sum_{i=1}^{l_1}\left[\chi(\bar{t}_{2i}) - \chi(\bar{t}_{2i-1})\right]\bar{D}_i(t), \quad D_1(\tilde{t}_1) = \tilde{D}_1,$$

$$\dot{K}_1(t) = Q^0(t)K_1(t) + G_1(t)C_1(t), \qquad K_1(\tilde{t}_1) = \tilde{K}_1,$$

$$\dot{Q}(t) = Q^0(t)Q(t), \qquad Q(\tilde{t}_1) = I_{n\times n}, \tag{33}$$

$$Q^0(t) = \Bigg[G_1(t)A_1(t)G_1^*(t) - G_1(t)\sum_{i=1}^{l_1}\left[\chi(\bar{t}_{2i}) - \chi(\bar{t}_{2i-1})\right]\bar{D}_i(t)D_1^*(t) -$$

$$-G_1(t)C_1(t)K_1^*(t)] \times \left[G_1(t)G_1^*(t) + D_1(t)D_1^*(t) + K_1(t)K_1^*(t)\right]^{-1}.$$

Then these functions shift condition (24) to the right on the semi-interval $t \in [\tilde{t}_1, \tilde{t}_2)$, and relation (29) holds true for them. The following condition

$$\|G_1(t)\|^2_{R^{n\times n}} + \|D_1(t)\|^2_{R^{n\times n}} + \|K_1(t)\|^2_{R^n} =$$

$$= \left\|\tilde{G}_1\right\|^2_{R^{n\times n}} + \left\|\tilde{D}_1\right\|^2_{R^{n\times n}} + \left\|\tilde{K}_1\right\|^2_{R^n} = const, \quad t \in \left(\tilde{t}_1, \tilde{t}_2\right), \tag{34}$$

also holds. Condition (34) provides stability for the solution to Cauchy problem (33).

It is not difficult to carry on similar considerations and to obtain formulas for shifting condition (25) successively to the left.

Thus to implement iterative procedure (22), it is necessary to go through the following steps on every iteration for given $u(t) = u^k(t), t \in [t_0, T], k = 0, 1, ...$:

1) to solve the problem (1), (2) by using the numerical scheme proposed in [16], and to determine the phase trajectory $x(t),\ t \in [t_0, T]$;

2) to solve problem (23)–(27) with the use of shift procedure (29) applied to the boundary conditions, and to determine the adjoint vector-function $\psi(t), t \in [t_0, T]$ and the vector of dual variables λ;

3) to substitute the obtained values of $x(t)$, $\psi(t)$, $t \in [t_0, T]$ into formula (20) and to determine the value of the gradient of the functional.

Instead of gradient projection method (21), other efficient first order numerical optimization methods can be used (see [22]).

5. Numerical experiments

Problem. Consider the following illustrative test optimal control problem for $t \in [0; 1]$, $n = 2$, $r = 1$, $U \equiv E^1$:

$$\begin{cases} \dot{x}_1(t) = 4tx_1(t) - x_2(t) + tu - 5t^2 + 5t + 3, \\ \dot{x}_2(t) = 3x_1(t) + 2tx_2(t) - 2t^3 - 6t + 3. \end{cases} \tag{35}$$

$$\int\limits_0^{0.25} \begin{pmatrix} \tau & -2 \\ 0 & 3 \end{pmatrix} x(\tau)d\tau + \begin{pmatrix} 5 & 1 \\ 2 & 3 \end{pmatrix} x(0.5) +$$

$$+ \int\limits_{0.8}^{1} \begin{pmatrix} \tau - 1 & 2 \\ 1 & 0 \end{pmatrix} x(\tau)d\tau = \begin{pmatrix} -0.3025 \\ 4.6756 \end{pmatrix}. \tag{36}$$

$$\bar{D}_1(t) = \begin{pmatrix} t & -2 \\ 0 & 3 \end{pmatrix}, \quad \bar{D}_2(t) = \begin{pmatrix} t-1 & 2 \\ 1 & 0 \end{pmatrix}, \quad \tilde{D}_1 = \begin{pmatrix} 5 & 1 \\ 2 & 3 \end{pmatrix}.$$

$$J(u) = \int\limits_0^1 [x_1(t) - u(t) + 2]^2\, dt + x_1^2(0,5) +$$

$$+ [x_2(0,5) - 1.25]^2 + [x_1(1) - 1]^2 + [x_2(1) - 2]^2. \tag{37}$$

In the problem, it is assumed that the control $u(t)$ can take on arbitrary values, and therefore, in the iteration procedure (21)-(26), the projection operation is not carried out. It is easy to see that the exact solution of the problem is $u^*(t) = 2t + 1$, $\quad x_1^*(t) = 2t - 1$, $x_2^*(t) = t^2 + 1$, To which corresponds the minimal value of the functional $J(u^*) = 0$.

According to formulas (14)-(18), the adjoint problem is as follows:

$$\dot{\psi}_1(t) = -4t\psi_1(t) - 3\psi_2(t) + (\chi(0.25) - \chi(0))\,(t\lambda_1) +$$
$$+ (\chi(1) - \chi(0.8))((t-1)\lambda_1 + \lambda_2) + 2(x_1(t) - u(t) + 2),$$
$$\dot{\psi}_2(t) = \psi_1(t) - 2t\psi_2(t) + (\chi(0.25) - \chi(0))\,(-2\lambda_1 + 3\lambda_2) + (\chi(1) - \chi(0.8))(2\lambda_1),$$

$$\psi_1(0) = 0 \ , \ \psi_2(0) = 0 \ , \ \psi_1(1) = -2\,[x_1(1) - 1], \ \psi_2(1) = -2\,[x_2(1) - 2]\,,$$

$$\psi_1^+(0.5) - \psi_1^-(0.5) = 2x_1(0.5) + 5\lambda_1 + 2\lambda_2 \ ,$$

$$\psi_2^+(0.5) - \psi_2^-(0.5) = 2\,[x_2(0.5) - 1.25] + \lambda_1 + 3\lambda_2 \ .$$

The gradient of the functional is determined as follows:

$$\nabla J(u) = -\,[\,x_1(t) - u(t) + 2\,] - t\psi_1(t).$$

Numerical experiments were carried out for different initial controls $u^0(t)$ and for different numbers N of partition of time interval. Fourth order Runge-Kutta method and conjugate gradient method were used. In figure 1, we give the results of solution to system (35), (36) and to the corresponding adjoint system. We also show the values of the components of the normalized gradients ($\nabla^{nor.}_{an.}J$) calculated by the proposed formulas (20) and the values of the components of the normalized gradients ($\nabla^{nor.}_{app.}J$) obtained by finite difference approximation:

$$\partial J(u)/\partial u_j \approx (J(u + \delta e_j) - J(u))/\delta, \tag{38}$$

where u_j is the value of the control $u = (u_1, u_2, ..., u_N)$ at thej^{th} discretization point; e_j is the N- dimensional unit vector consisting of zeros except for the j^{th} component. The value of δ equals 0.01 and 0.001.

Notice that instead of formula (38) it is possible to use the formulas proposed in the author's papers [24, 25]. These formulas are most effective at points with small values of the components of the gradient (in particular, in the neighborhood of the extremum).

Figure 1. **Initial values of the controls, of the phase variables, and of the normalized gradients calculated using both the proposed formulas and (38)**

t	$u^{(0)}(t)$	$x_1^{(0)}(t)$	$x_2^{(0)}(t)$	$\psi_1^{(0)}(t)$	$\psi_2^{(0)}(t)$	$\nabla^{nor.}_{an.}J$	$\nabla^{nor.}_{app.}J$	
							$\delta = 10^{-2}$	$\delta = 10^{-3}$
0	1.000	1.5886	1.2034	-9.2836	-6.3653	-0.0161	-0.0156	-0.0161
20	2.000	1.5641	2.4702	-0.5626	3.2122	-0.0106	-0.0102	-0.0107
40	3.000	1.2382	3.5937	2.9294	14.8639	-0.0053	-0.0049	-0.0052
60	4.000	0.6657	4.4538	-4.5081	14.2492	0.0147	0.0147	0.0145
80	5.000	-0.0781	4.9528	-10.9008	11.4433	0.0411	0.0413	0.0410
100	6.000	-0.8973	5.0218	-15.1806	7.0519	0.0668	0.0688	0.0667
120	7.000	-1.6767	4.6295	-21.2792	9.1844	0.0996	0.0990	0.0994
140	8.000	-2.2835	3.7919	-22.7128	2.1275	0.1169	0.1164	0.1168
160	9.000	-2.5710	2.5845	-14.7216	-0.0383	0.1039	0.1034	0.1038
180	10.000	-2.3853	1.1544	-6.8798	-0.6530	0.0827	0.0823	0.0826
200	11.000	-1.5759	-0.2660	0.0000	-0.0000	0.0800	0.0795	0.0800

The initial value of the functional is $J(u^0) = 56.28717$, $\lambda_1 = 0.2387$, $\lambda_2 = 0.1781$. The values of the functional obtained in the course of the iterations are as follows:

$$J(u^1) = 1.93187, \quad J(u^2) = 0.10445, \quad J(u^3) = 0.00868,$$

$$J(u^4) = 0.00023, \quad J(u^5) = 0.00004.$$

On the sixth iteration of conjugate gradient method, we obtain the results given in figure 2 with the minimal value of the functional $J(u^6)$ equal to 10^{-6}.

Figure 2. **The exact solution to the problem and the solution obtained after the sixth iteration**

t	Solution obtained					Exact solution		
	$u^{(6)}(t)$	$x_1^{(6)}(t)$	$x_2^{(6)}(t)$	$\psi_1^{(6)}(t)$	$\psi_2^{(6)}(t)$	$u^*(t)$	$x_1^*(t)$	$x_2^*(t)$
0	0.9999	-1.0000	1.0001	0.0059	-0.0040	1.0000	-1.0000	1.0000
20	1.2000	-0.8001	1.0101	0.0095	0.0027	1.2000	-0.8000	1.0100
40	1.4001	-0.6001	1.0401	0.0089	0.0098	1.4000	-0.6000	1.0400
60	1.6001	-0.4001	1.0902	0.0038	0.0114	1.6000	-0.4000	1.0900
80	1.7999	-0.2001	1.1602	-0.0012	0.0114	1.8000	-0.2000	1.1600
100	1.9999	-0.0001	1.2500	-0.0049	0.0099	2.0000	0.0000	1.2500
120	2.1999	0.1999	1.3599	-0.0055	0.0044	2.2000	0.2000	1.3600
140	2.3998	0.4000	1.4899	-0.0053	0.0025	2.4000	0.4000	1.4900
160	2.6001	0.6001	1.6399	-0.0026	0.0014	2.6000	0.6000	1.6400
180	2.8001	0.8001	1.8101	-0.0008	0.0006	2.8000	0.8000	1.8100
200	3.0001	1.0001	2.0001	0.0000	-0.0000	3.0000	1.0000	2.0000

6. Conslusion

In the work, we propose the technique for numerical solution to optimal control problems for ordinary differential equations systems involving non-separated multipoint and integral conditions. Note that a mere numerical solution to the differential systems presents certain difficulties. The adjoint problem also has a specific character which lies both in the equation itself and in the presence of an unknown vector of Lagrange coefficients in the conditions.

The formulas proposed in the work, as well as the computational schemes make it possible to take into account all the specific characters which occur when calculating the gradient of the functional. Overall, the proposed approach allows us to use a rich arsenal of first order optimization methods and the corresponding standard software to solve the considered optimal control problems.

References

[1] Nicoletti, O., (1897), Sulle condizioni iniziali che determinano gli integrali delle equazioni differenziali ordinarie, Atti R. Sci. Torino, 33, pp. 746–748.

[2] Tamarkin, Ya. D., (1917), On some general problems of ordinary differential equations theory and on series expansion of arbitrary functions,Petrograd.

[3] Vallee-Poussin, Ch. J., (1929), Sur l'quation diffrentielle linaire du second ordre. Dtermination d'une intgrale par deux valeurs assignes. Extension aux quations d'ordre n., J. Math. Pures Appl., 8, pp.125–144.

[4] Aida-zade, K. R., Abdullaev, V. M., (2012) On an approach to designing control of the distributed-parameter processes, Autom. Remote Control, 73 (9), pp. 1443–1455.

[5] Abdullayev, V. M., Aida-zade, K. R., (2018) Numerical solution of the problem of determining the number and locations of state observation points in feedback control of a heating process, Comput. Math. Math. Phys., 58 (1), pp.78–89.

[6] Bouziani, A., (2002), On the solvability of parabolic and hyperbolic problems with a boundary integral condition, Intern.J. Math. Sci., 31 (4), pp. 202–213.

[7] Pulkina, L. S., (2004), Non-local problem with integral conditions for a hypergolic equation, Differential Equations, 40 (7), pp. 887–891.

[8] Abdullayev, V. M., (2017), Identification of the functions of response to loading for stationary systems, Cybern. Syst. Analysis, 53 (3), pp. 417–425.

[9] Abdullaev, V. M., Aida-zade, K. R., (2006), Numerical solution of optimal control problems for loaded lumped parameter systems, Comput. Math. Math. Phys., 46 (9), pp.1487–1502.

[10] Abdullayev, V. M., Aida-zade, K. R., (2017), Optimization of loading places and load response functions for stationary systems, Comput. Math. Math. Phys., 57 (4), pp. 634–644.

[11] Aschepkov, L. T., (1981), Optimal control of system with intermediate conditions, Journal of Applied Mathematics and Mechanics, 45 (2), pp. 215–222.
[12] Vasileva, O. O., Mizukami, K., (2000), Dynamical processes described by boundary problem: necessary optimality conditions and methods of solution, Journal of Computer and System Sciences International (A Journal of Optimization and Control), 1, pp.95–100.
[13] Vasilev, O. V., Terleckij, V. A., (1995), Optimal control of a boundary problem, Proceedings of the Steklov Institute of Mathematics, (211), pp. 221–130.
[14] Abdullaev, V. M., Aida-zade, K. R., (2014), Numerical method of solution to loaded nonlocal boundary-value problems for ordinary differential equations, Comput. Math. Math. Phys., 54 (7), pp. 1096–1109.
[15] Abramov, A. A., (1961), On the transfer of boundary conditions for systems of ordinary linear differential equations (a variant of the dispersive method), Zh. Vychisl. Mat. Mat. Fiz., 3 (1), pp.542–545.
[16] Aida-zade, K. R., Abdullaev, V. M., (2013), On the solution of boundary-value problems with nonseparated multipoint and integral conditions, Differential Equations, 49 (9), pp. 1114–1125.
[17] Moszynski, K., (1964), A method of solving the boundary value problem for a system of linear ordinary differential equation, Algorytmy. Varshava, 11 (3), pp. 25–43.
[18] Bondarev, A. N., Laptinskii, V. N., (2011), A multipoint boundary value problem for the Lyapunov equation in the case of strong degeneration of the boundary conditions, Differential Equations, 47 (6), pp. 776–784.
[19] Dzhumabaev, D. S., Imanchiev, A. E., (2005), Well-posed solvability of linear multipoint boundary problem, Matematicheskij zhurnal. Almaaty, 5 (1(15)), pp. 30–38.
[20] Kiguradze, I. T., (1987), Boundary value problems for systems of ordinary differential equations, Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat. Nov. Dostizh., 30, pp. 3–103.
[21] Samoilenko, A. M., Laptinskii, V. N., Kenzhebaev, K. K., (1999), Constructive Methods of Investigating Periodic and Multipoint Boundary - Value Problems, Kiev.
[22] Vasilyev, F. P., (2002), Optimization Methods, M: Faktorial.
[23] Aida-zade, K. R., Abdullayev, V. M., (2016), Solution to a class of inverse problems for system of loaded ordinary differential equations with integral conditions, J. of Inverse and Ill-posed Problems, 24 (5), pp. 543–558.
[24] Karchevsky, A.L., (2000), Numerical solution of the one-dimensional inverse problem for the elasticity system, Doklady RAS. 375 (2), pp. 235-238.
[25] Karchevsky, A.L., (2010), Reconstruction of pressure velocities and boundaries of thin layers in thinly-stratified layers, J. Inv. Ill-Posed Problems. 18 (4), pp. 371-388.

Kamil Aida-zade was born in 1950 in Qusar region of Azerbaijan. He graduated from Baku State University in 1972 and received a diploma in mathematics. He received degree in Candidate Physic-Mathematical Sciences (1979) from the Computing Center of the Georgia Academy of Sciences and degree in Doctor Physic-Mathematical Sciences from the (1989) Institute of Cybernetic of the Ukraine Academy of Sciences. He was elected Corresponding Member of the Azerbaijan National Academy of Sciences (2014). Now he is the head of "Informatics" department at the Baku State University. His research interests include: computing mathematics; optimization and optimal control; inverse problems; mathematical modeling.

Vagif Abdullayevwas born in 1973 in Lerik region of Azerbaijan. He graduated from Applied Mathematics and Cybernetics Faculty of Baku State University in 1995, got his Ph.D. degree in 2005, Doctor of Physical-Mathematical Sciences since degree 2016. He was elected Professor of the Azerbaijan National Academy of Sciences (2017). Now he is the Professor in Azerbaijan State Oil and Industry University. His research interests include: computing mathematics; optimization and optimal control; inverse problems; mathematical modeling.