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# ON FUZZY QUOTIENT $BCK$-ALGEBRAS

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Abstract. In this paper, by considering the concept of fuzzy congruence in some algebraic structures, we specially study fuzzy congruence in $BCK$-algebras. We prove that there is a bijection between the set of fuzzy ideals and the set of fuzzy congruences in $BCK$-algebras. Then we show that for each fuzzy ideal $\mu$, there is an associated algebra $X/\mu$ that is a $BCK$-algebra. Also, we obtain a congruence relation on a $BCK$-algebra by fuzzy ideals.
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## 1. Introduction

The notion of $BCK$-algebra was formulated first in 1966 by Imai and Iséki [6]. This notion is originated from two different ways. One of the motivations is based on set theory. Another motivation is from classical and non-classical propositional clacului. As is well known, there is close relationship between the notion of the set difference in set theory and the implication functor in logical systems. Then the following problems arise from this relationship. What is the most essential and fundamental common properties? Can we establish a good theory of general algebra? To give an answer this problems, Y. Imai and K. Iséki introduced a notion of a new class of general algebras, which is called a $BCK$-algebra. This name is taken from $BCK$-system of C. A. Meredith. $BCK$-algebras have been applied to many branches of mathematics, such as group theory, functional analysis, probability theory and topology. The concept of fuzzy subset was introduced by Zadeh for the first time [14]. At present these ideas have been applied to other algebraic structures such as groups, rings, modules and since then many studies were performed about this subject on fuzzy new algebraic structures. In 1993 the consept of fuzzy sets was applied to $BCI$-algebras [1, 7]. The concept of a fuzzy relation on a set was introduced by Zadeh [14]. In [8], Kondo defined the quotient $BCI$-algebras induced by fuzzy ideals. Note that each congruence class in quotient $BCI$-algebras induced by fuzzy ideals is not a fuzzy set but it is a crisp set. In [11], A. Rezaei and A. Borumand Saeid studied and introduced fuzzy congruence relations on $CI$-algebras. Recently, some reasearchers worked on $MV$-algebras

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and ideals in them (see [10, 12, 13]). In this paper, specially, we present the definitions of fuzzy congruence, fuzzy congruence classes and fuzzy quotient algebras in $BCK$-algebras. We will show that the elements in fuzzy quotient algebras induced by fuzzy ideals are fuzzy sets in $BCK$-algebras. Hence we prove that there is a bijection between the set of fuzzy ideals and the set of fuzzy congruence. For each fuzzy ideal $\mu$, there is an associated algebra $X/\mu$. We prove that $X/\mu$ is a $BCK$-algebra and it is isomorphic to the $BCK$-algebra $X/\mu_{\mu(0)}$. Finally, we obtain a congruence relation on a $BCK$-algebra by fuzzy ideals.

## 2. Preliminaries

In this section, we review related lemmas and theorems that we use in the next sections.

**Definition 2.1.** [9] *A $BCK$-algebra is a structure $X = (X, *, 0)$ of type $(2, 0)$ such that:*
$(BCK1)$ $((x * y) * (x * z)) * (z * y) = 0$,
$(BCK2)$ $(x * (x * y)) * y = 0$,
$(BCK3)$ $x * x = 0$,
$(BCK4)$ $0 * x = 0$,
$(BCK5)$ $x * y = y * x = 0$ *implies that* $x = y$, *for all* $x, y, z \in X$.
*The relation $x \leq y$ which is defined by $x * y = 0$ is a partial order on $X$ with $0$ as least element. In $BCK$-algebra $X$, for any $x, y, z \in X$, we have*
$(BCK6)$ $(x * y) * z = (x * z) * y$,
$(BCK7)$ $x \leq y$ *implies* $z * y \leq z * x$,
$(BCK8)$ $x \leq y$ *implies* $x * z \leq y * z$.

*Let $(X, *, 0)$ be a $BCK$-algebra. Then $\emptyset \neq X_0 \subseteq X$ is called to be a subalgebra of $X$, if for any $x, y \in X_0$, $x * y \in X_0$, i.e., $X_0$ is closed under the binary operation "$*$" of $X$. $X$ is called bounded, if there exists $1 \in X$ such that $x \leq 1$, for any $x \in X$ and in this case, we let $Nx = 1 * x$. $X$ is said to be commutative, if $y * (y * x) = x * (x * y)$, for all $x, y \in X$. Subset $\emptyset \neq I \subseteq X$ is called an ideal of $X$, if $0 \in I$ and for any $x, y \in X$, $x * y \in I$ and $y \in I$, implies that $x \in I$. In a $BCK$-algebra $X$, we let $x \wedge y = y * (y * x)$ and in a bounded $BCK$-algebra $X$, we let $x \vee y = N(Nx \wedge Ny)$, for all $x, y \in X$. In bounded commutative $BCK$-algebra $X$, for any $x, y \in X$, $x \vee y$ is the least upper bound and $x \wedge y$ is the grate lower bound of $x, y$ and so $(L, \vee, \wedge)$ is a bounded lattice.*

**Definition 2.2.** [14] *Let $X$ be a set. A fuzzy set in $X$ is a mapping $\mu : X \to [0, 1]$. The notations $1_X$ and $0_X$ represent two special fuzzy sets in $X$ satisfying $1_X = 1$ and $0_X = 0$, for every $x \in X$, respectively. For every sequence $\{a_1, \cdots, a_n\}$ of real numbers, $a_1 \wedge \cdots \wedge a_n = min\{a_1, \cdots, a_n\}$ and $a_1 \vee \cdots \vee a_n = max\{a_1, \cdots, a_n\}$. For any fuzzy sets $f, g$ in $X$, $f \leq g$ means that $f(x) \leq g(x)$, for every $x \in X$. Let $\mu$ be a fuzzy set in $X$, $t \in [0, 1]$, the set $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called a level subset of $\mu$.*

**Definition 2.3.** [5] *A fuzzy set $\mu$ in $BCK$-algebra $X$ is a fuzzy ideal of $X$, if it satisfies*
$(F1)$ $\mu(0) \geq \mu(x)$, *for all* $x \in X$,
$(F2)$ $\mu(y) \geq \mu(x) \wedge \mu(y * x)$, *for all* $x, y \in X$.

**Definition 2.4.** [2] *A fuzzy relation $\mu$ in a set $X$ is a fuzzy subset of $X \times X$. $\mu$ is $\varepsilon$-reflexive in $X$ if $\mu(x, x) \geq \varepsilon > 0$, for all $x \in X$. $\mu$ is symmetric in $X$ if $\mu(x, y) = \mu(y, x)$, for all $x, y \in X$. $\mu$ is transitive in $X$ if $\mu \circ \mu \subseteq \mu$.*

**Proposition 2.1.** [5] *Let $\mu$ be a fuzzy ideal in $BCK$-algebra $X$. Then*
$(i)$ *if* $x \leq y$, *then* $\mu(x) \geq \mu(y)$,

*(ii)* $\mu(x * y) \geq \mu(x * z) \wedge \mu(z * y)$,
*(iii)* *if* $\mu(x * y) = \mu(0)$, *then* $\mu(x) \geq \mu(y)$, *for all* $x, y, z \in X$.

**Definition 2.5.** [3] *An MV-algebra is a structure* $M = (M, \oplus, ', 0)$ *of type* $(2, 1, 0)$ *such that:*
$(MV1)$ $(M, \oplus, 0)$ *is an Abelian monoid,*
$(MV2)$ $(a')' = a$,
$(MV3)$ $0' \oplus a = 0'$,
$(MV4)$ $(a' \oplus b)' \oplus b = (b' \oplus a)' \oplus a$,
*If we define the constant* $1 = 0'$ *and operations* $\odot$ *and* $\ominus$ *by* $a \odot b = (a' \oplus b')'$, $a \ominus b = a \odot b'$, *then*
$(MV5)$ $(a \oplus b) = (a' \odot b')'$,
$(MV6)$ $x \oplus 1 = 1$,
$(MV7)$ $(a \ominus b) \oplus b = (b \ominus a) \oplus a$,
$(MV8)$ $a \oplus a' = 1$,
*for every* $a, b \in A$. *It is clear that* $(M, \odot, 1)$ *is an abelian monoid. Now, if we define auxiliary operations* $\vee$ *and* $\wedge$ *on* $M$ *by* $a \vee b = (a \odot b') \oplus b$ *and* $a \wedge b = a \odot (a' \oplus b)$, *for every* $a, b \in M$, *then* $(M, \vee, \wedge, 0)$ *is a bounded distributive lattice. An ideal of MV-algebra* $M$ *is a subset* $I$ *of* $M$, *satisfying the following condition:* $(I1)$ $0 \in I$, $(I2)$ $x \leq y$ *and* $y \in I$ *implies that* $x \in I$, $(I3)$ $x \oplus y \in I$, *for every* $x, y \in I$.. *Let* $M$ *and* $K$ *be two MV-algebras. A mapping* $f : M \to K$ *is called an MV-homomorphism if* $(H1)$ $f(0) = 0$, $(H2)$ $f(x \oplus y) = f(x) \oplus f(y)$ *and* $(H3)$ $f(x') = (f(x))'$, *for every* $x, y \in M$. *If* $f$ *is one to one (onto), then* $f$ *is called an MV-monomorphism (epimorphism) and if* $f$ *is onto and one to one, then* $f$ *is called an MV-isomorphism.*

The following results were proved in $MV$-algebras. Then this results are proved in $BCK$-algebras, easily.

**Lemma 2.1.** [4] *Let* $A$ *be an MV-algebra and* $\mu : A \to [0, 1]$ *be a fuzzy set on* $A$. *Then* $\mu$ *is a fuzzy ideal on* $A$ *if and only if*
(1) $\mu(x) \leq \mu(0)$, (2) $x \leq y$ *implies that* $\mu(x) \leq \mu(y)$, *for all* $x, y \in A$ *and* .

**Theorem 2.1.** [4] *Let* $\mu$ *be a fuzzy set in MV-algebra* $A$. $\mu$ *is a fuzzy ideal if and only if for all* $t \in [0, 1]$, $\mu_t$ *is either empty or an ideal of* $A$.

**Corollary 2.1.** [4] $I$ *is an ideal of MV-algebra* $A$ *if and only if* $\chi_I$ *is a fuzzy ideal of* $A$, *where* $\chi_I$ *is characteristic function of* $I$.

**Definition 2.6.** [14] *Let* $X, Y$ *be two MV-algebras,* $\mu$ *be a fuzzy subset of* $X$, $\mu'$ *be a fuzzy subset of* $Y$ *and* $f : X \to Y$ *be a homomorphism. The image of* $\mu$ *under* $f$ *denoted by* $f(\mu)$ *is a fuzzy set of* $Y$ *defined by*

$$f(\mu)(y) = \begin{cases} sup_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

*for all* $y \in Y$. *The preimage of* $\mu'$ *under* $f$ *denoted by* $f^{-1}(\mu')$ *is a fuzzy set of* $X$ *defined by: for all* $x \in X$, $f^{-1}(\mu')(x) = \mu'(f(x))$.

**Theorem 2.2.** [5] *Let* $\mu$ *be a fuzzy ideal in MV-algebra* $A$. *For any* $x, y, z \in A$, $\mu(x \vee y) = \mu(x) \wedge \mu(y)$.

**Note**: From now on, in this paper, we let $X$ be a $BCK$-algebra.

## 3. On congruence relations induced by fuzzy ideals

In [8], Kondo defined the quotient $BCI$-algebras induced by fuzzy ideals. Note that each congruence class in quotient $BCI$-algebras induced by fuzzy ideals is not a fuzzy set but it is a crisp set. In the following, we present the notions of fuzzy congruences, fuzzy congruence classes and fuzzy quotient algebras in $BCK$-algebras. Note that the definition of fuzzy relation in most of algebraic structures is the same. We will show that the elements in fuzzy quotient algebras induced by fuzzy ideals are fuzzy sets in $BCK$-algebras.

**Definition 3.1.** *A fuzzy relation $\theta$ from $X \times X$ to $[0,1]$ is called a fuzzy congruence in $X$ if it satisfies the following:*
$(C1)$ $\theta(0,0) = \theta(x,x)$, *for all* $x \in X$,
$(C2)$ $\theta(x,y) = \theta(y,x)$, *for all* $x, y \in X$,
$(C3)$ $\theta(x,z) \geq \theta(x,y) \wedge \theta(y,z)$, *for all* $x, y, z \in X$,
$(C4)$ $\theta(x*z, y*z) \geq \theta(x,y)$ *(right compatible) and* $\theta(z*x, z*y) \geq \theta(x,y)$ *(left compatible).*

**Example 3.1.** *Let $X = \{0, a, b, 1\}$ and $*$ be defined as follows:*

| $*$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $a$ | $a$ | 0 | $a$ | 0 |
| $b$ | $b$ | $b$ | 0 | 0 |
| 1 | 1 | $b$ | $a$ | 0 |

*Then $(X, *, 0)$ is a $BCK$-algebra. Consider fuzzy relation $\theta$ from $X \times X$ to $[0,1]$ with* $\theta(0,0) = \theta(a,a) = \theta(b,b) = \theta(1,1) = 0.8$, $\theta(a,1) = \theta(b,0) = \theta(1,a) = \theta(0,b) = 0.5$ *and* $\theta(0,1) = \theta(1,0) = \theta(a,b) = \theta(b,a) = \theta(a,0) = \theta(b,1) = \theta(0,a) = \theta(1,b) = 0.3$. *It is easily checked that $\theta$ is a fuzzy congruence in $X$.*

**Proposition 3.1.** *If $\theta$ is a fuzzy congruence in $X$, then*
$(i)$ $\theta(Nx, Ny) \geq \theta(x,y)$,
$(ii)$ $\theta(x \wedge z, y \wedge z) \geq \theta(x,y)$,
$(iii)$ $\theta(x \vee z, y \vee z) \geq \theta(x,y)$
$(iv)$ $\theta(0,0) \geq \theta(x,y)$,
$(v)$ $\theta(x,y) = \theta(x*y, 0)$,
$(vi)$ *if $\theta$ satisfies the conditions $(C_2)$, $(C_3)$ and $(C_4)$, then $(C_1)$ is equivalet to $\theta(0,0) \geq \theta(x,y)$, for all $x, y \in X$.*

*Proof.* $(i), (ii), (iii)$: By $C_4$, the proof is clear.
$(iv)$ We have $\theta(0,0) = \theta(x,x)$ and $\theta(x,x) \geq \theta(x,y) \wedge \theta(y,x) = \theta(x,y)$. Then $\theta(0,0) \geq \theta(x,y)$.
$(v)$ By $(C_4)$, $\theta(x,y) \leq \theta(x*y, y*y) = \theta(x*y, 0)$. On the other hand, $\theta(x*y, 0) = \theta(x*y, x*x) \geq \theta(y,x) = \theta(x,y)$. Hence $\theta(x,y) = \theta(x*y, 0)$.
$(vi)$ Let $\theta(0,0) = \theta(x,x)$. By $(C_2)$ and $(C_3)$, $\theta(0,0) = \theta(x,x) \geq \theta(x,y) \wedge \theta(y,x) = \theta(x,y)$. Converesly, by $(C_4)$, $\theta(0,0) \leq \theta(x*0, x*0) = \theta(x,x)$. $\square$

Let $\theta$ be a fuzzy relation on $X$. Consider $U_t(\theta) = \{(x,y) \in X \times X | \theta(x,y) \geq t\}$ and $U_{t>}(\theta) = \{(x,y) \in X \times X | \theta(x,y) > t\}$, where $t \in [0,1]$.

**Theorem 3.1.** *If $\theta$ is a fuzzy congruence relation and $U_t(\theta) \neq \emptyset$, for $t \in [0,1]$, then $U_t(\theta)$ is a congruence relation on $X$.*

*Proof.* Since $U_t(\theta)$ is not empty, there is an element $(u,v) \in X \times X$ such that $(u,v) \in U_t(\theta)$. This means that $t \leq \theta(u,v)$. Since $\theta$ is the congruence, by Proposition 3.1 $(iv)$, we have

$t \leq \theta(u, v) \leq \theta(0, 0) = \theta(x, x)$. That is, $(x, x) \in U_t(\theta)$. Also, the relation is clearly symmetric.

Let $(x, y), (y, z) \in U_t(\theta)$. Since $t \leq \theta(x, y)$ and $t \leq \theta(y, z)$, we have $t \leq \theta(x, y) \wedge \theta(y, z) \leq \theta(x, z)$. Hence $(x, z) \in U_t(\theta)$.

Now, we assume that $(x, y) \in U_t(\theta)$. Since $t \leq \theta(x, y) \leq \theta(x * u, y * u)$, for every $u \in X$, we have $(x * u, y * u) \in U_t(\theta)$ and similarly $(u * x, u * y) \in U_t(\theta)$. Therefore, $U_t(\theta)$ is the congruence on $X$. □

Since the definition of $U_t(\theta)$ is a general definition and it is not rerated to properties of algebraic structures, we can easily prove that the following theorem in all algebraic structures as $BCK$-algebras:

**Theorem 3.2.** *Let $\theta$, $\theta_1$ and $\theta_2$ be fuzzy congruence relations in $X$. Then*
*(i) $U_t(\theta) = \bigcap_{0 \leq s < t} U_{s>}(\theta)$ and $U_{t>}(\theta) = \bigcup_{t < s \leq 1} U_s(\theta)$.*
*(ii) $\theta$ is a fuzzy left(right) compatible relation if and only if $U_t(\theta)$ ($U_{t>}(\theta)$) is a left(right) compatible relation on $X$.*
*(iii) Let the composition $\theta_1 \circ \theta_2$ is defined by $\theta_1 \circ \theta_2 = sup_{z \in X} min(\theta_1(x, z), \theta_2(z, y))$. Then for every $t \in [0, 1]$,*
*$\theta_1 = \theta_2$ if and only if $U_{t>}(\theta_1) = U_{t>}(\theta_2)$ and $U_{t>}(\theta_1 \circ \theta_2) = U_{t>}(\theta_1) \circ U_{t>}(\theta_2)$.*
*(iv) $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$ if and only if $U_{t>}(\theta_1) \circ U_{t>}(\theta_2) = U_{t>}(\theta_2) \circ U_{t>}(\theta_1)$, for every $t \in [0, 1]$, where $\theta_1 \neq \emptyset$ and $\theta_2 \neq \emptyset$.*

*Proof.* $(i)$ Let $t \in [0, 1]$. By definition $U_t(\theta)$ and $U_{t>}(\theta)$, we have

$$U_{t>}(\theta) = \{(x, y) \in X \times X | \theta(x, y) > t\} = \bigcup_{t < s \leq 1} \{(x, y) \in X \times X | \theta(x, y) \geq s\} = \bigcup_{t < s \leq 1} U_s(\theta)$$

and

$$U_t(\theta) = \{(x, y) \in X \times X | \theta(x, y) \geq t\} = \bigcap_{0 \leq s < t} \{(x, y) \in X \times X | \theta(x, y) > s\} = \bigcup_{0 \leq s < t} U_{s>}(\theta).$$

$(ii)$ The proof is easy.
$(iii)$ Let $\theta_1 = \theta_2$ and $(x, y) \in U_{t>}(\theta_1)$. Then $\theta_1(x, y) = \theta_2(x, y) > t$ and so $(x, y) \in U_{t>}(\theta_2)$. Hence $U_{t>}(\theta_1) \subseteq U_{t>}(\theta_2)$. Similarly, $U_{t>}(\theta_2) \subseteq U_{t>}(\theta_1)$ and so $U_{t>}(\theta_1) = U_{t>}(\theta_2)$.
Conversely, let $U_{t>}(\theta_1) = U_{t>}(\theta_2)$, but $\theta_1 \neq \theta_2$ . Then there exists $(x, y) \in X \times X$ such that $t_1 = \theta_1(x, y) \neq \theta_2(x, y) = t_2$. Without loss of generality, let $t_1 > t_2$. Then $\theta_1(x, y) > t_2$ and so $(x, y) \in U_{t>}(\theta_2) = U_{t>}(\theta_1)$. Hence $\theta_2(x, y) > t_1$ and so $t_2 > t_1$, which is a contradiction.
Also, let $(x, y) \in X \times X$ and $t \in [0, 1]$. Then

$$\begin{aligned}
(x, y) \in U_{t>}(\theta_1 \circ \theta_2) &\leftrightarrow \theta_1 \circ \theta_2 > t \leftrightarrow sup_{z \in X} min(\theta_1(x, z), \theta_2(z, y)) > t \\
&\leftrightarrow \exists z_0 \in X, min(\theta_1(x, z_0), \theta_2(z_0, y)) > t \leftrightarrow \theta_1(x, z_0) > t \text{ and } \theta_2(z_0, y) > t \\
&\leftrightarrow \exists z_0 \in X, (x, z_0) \in U_{t>}(\theta_1) \text{ and } (z_0, y) \in U_{t>}(\theta_2) \\
&\leftrightarrow (x, y) \in U_{t>}(\theta_1) \circ U_{t>}(\theta_2).
\end{aligned}$$

Therefore, $U_{t>}(\theta_1 \circ \theta_2) = U_{t>}(\theta_1) \circ U_{t>}(\theta_2)$.
$(iv)$ $(\Rightarrow)$ Let $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$. Then by Theorem $(iii)$, the proof is clear.
$(\Leftarrow)$ Let $U_{t>}(\theta_1) \circ U_{t>}(\theta_2) = U_{t>}(\theta_2) \circ U_{t>}(\theta_1)$, for every $t \in [0, 1]$. Then by $(i)$, $(ii)$ and

$(iii)$,

$$\begin{aligned} U_t(\theta_1 \circ \theta_2) = \bigcap_{0 \le s < t} U_{s>}(\theta_1 \circ \theta_2) &= \bigcap_{0 \le s < t} U_{s>}(\theta_1) \circ U_{s>}(\theta_2) = \bigcap_{0 \le s < t} U_{s>}(\theta_2) \circ U_{s>}(\theta_1) \\ &= \bigcap_{0 \le s < t} U_{s>}(\theta_2 \circ \theta_1) = U_t(\theta_2 \circ \theta_1). \end{aligned}$$

Now, let $x, y \in X$ such that $(\theta_1 \circ \theta_2)(x, y) = t$. Then $(x, y) \in U_t(\theta_1 \circ \theta_2) = U_t(\theta_2 \circ \theta_1)$ and so $(\theta_1 \circ \theta_2)(x, y) \ge t = (\theta_1 \circ \theta_2)$. Similarly, $(\theta_2 \circ \theta_1)(x, y) \ge (\theta_1 \circ \theta_2)(x, y)$. Therefore, $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$.

□

**Definition 3.2.** *Let $\theta$ be a fuzzy congruence in $X$ and $x \in X$. Define the fuzzy set $\theta^x$ in $X$ by $\theta^x(y) = \theta(x, y)$, for all $y \in X$. The fuzzy set $\theta^x$ is called a fuzzy congruence class of $x$ by $\theta$ in $X$. The set $X/\theta = \{\theta^x | x \in X\}$ is called a fuzzy quotient set by $\theta$.*

**Example 3.2.** *Consider $BCK$-algebra $X = \{0, a, b, 1\}$ with fuzzy congruence relation $\theta$ in Example 3.1. A fuzzy quotient set by $\theta$ is $X/\theta = \{\theta^0, \theta^a, \theta^b, \theta^1\}$.*

**Lemma 3.1.** *Let $\theta$ be a fuzzy congruence in $X$. Then $\theta^0$ is a fuzzy ideal in $X$.*

*Proof.* Since $\theta$ is a fuzzy congruence in $X$, by Proposition 3.1 $(iv)$, we have $\theta^0(0) = \theta(0, 0) \ge \theta(0, x) = \theta^0(x)$, for every $x \in X$.

Also, since $\theta$ is a fuzzy congruence in $X$, we have $\theta(0, y) \ge \theta(0, y * x) \wedge \theta(y * x, y)$ and $\theta(y * x, y) = \theta(y * x, y * 0) \ge \theta(x, 0)$. Hence $\theta(0, y) \ge \theta(0, y * x) \wedge \theta(0, x)$. Thus $\theta^0(y) \ge \theta^0(y * x) \wedge \theta^0(x)$, for all $x, y \in X$. Therefore, $\theta^0$ is a fuzzy ideal in $X$. □

**Lemma 3.2.** *Let $\mu$ be a fuzzy ideal in $X$. Then $\theta_\mu(x, y) = \mu(x * y) \wedge \mu(y * x)$ is a fuzzy congruence in $X$.*

*Proof.* $(C1)$ and $(C2)$ are clear.
$(C3)$ Let $x, y, z \in X$. By Proposition 2.1 $(ii)$,

$$\begin{aligned} \theta_\mu(x, z) &= \mu(x * z) \wedge \mu(z * x) \ge (\mu(x * y) \wedge \mu(y * z)) \wedge (\mu(z * y) \wedge \mu(y * x)) \\ &= (\mu(x * y) \wedge \mu(y * x)) \wedge (\mu(y * z) \wedge \mu(z * y)) \\ &= \theta_\mu(x, y) \wedge \theta_\mu(y, z). \end{aligned}$$

$(C4)$ Let $x, y, z \in X$. By $(BCK1)$, we have $(x*z)*(y*z) \le x*y$ and $(y*z)*(x*z) \le y*x$. Then by Proposition 2.1 $(i)$,

$$\theta_\mu((x * z), (y * z)) = \mu((x * z) * (y * z)) \wedge \mu((y * z) * (x * z)) \ge \mu(x * y) \wedge \mu(y * x) = \theta_\mu(x, y).$$

Similarly, $\theta_\mu(z * x, z * y) \ge \theta_\mu(x, y)$. □

**Theorem 3.3.** *There is a bijection between the set of fuzzy ideals and the set of fuzzy congruences in $X$.*

*Proof.* By Lemmas 3.1 and 3.2, it is easily checked that $\mu = (\theta_\mu)^0$ and $\theta = \theta_{\theta^0}$ for each fuzzy ideal $\mu$ and fuzzy congruence $\theta$ in $X$. Hence there is a bijection between the set of fuzzy ideals and the set of fuzzy congruences in $X$. □

Let $\mu$ be a fuzzy ideal in $X$, $\mu^x$ denote the fuzzy congruence class of $x$ by $\theta_\mu$ in $X$, for every $x \in X$ and $X/\mu$ be the fuzzy quotient set by $\theta_\mu$. In following, we introduce congruence relations induced by fuzzy ideals.

**Proposition 3.2.** *Let $\mu$ be a fuzzy ideal in $X$. Then $\mu^x = \mu^y$ if and only if $\mu(x * y) = \mu(y * x) = \mu(0)$, for all $x, y \in X$.*

*Proof.* Let $\mu^x = \mu^y$, for $x, y \in X$. We have $\mu^u(v) = \theta^u_\mu(v) = \theta_\mu(u, v) = \mu(u * v) \wedge \mu(v * u)$, for any $u, v \in X$. Since $\mu^x = \mu^y$, $\mu^x(x) = \mu^y(x)$, for all $x \in X$. It follows that $\mu(x * x) \wedge \mu(x * x) = \mu(y * x) \wedge \mu(x * y)$ and so by $(F1)$, $\mu(x * y) = \mu(y * x) = \mu(0)$.

Conversely, let $\mu(x * y) = \mu(y * x) = \mu(0)$. By Proposition 2.1 $(ii)$, we have $\mu(x * z) \geq \mu(x*y) \wedge \mu(y*z)$ and $\mu(y*z) \geq \mu(y*x) \wedge \mu(x*z)$, for all $z \in X$. Since $\mu(x*y) = \mu(y*x) = \mu(0)$, we have $\mu(x * z) \geq \mu(y * z)$ and $\mu(y * z) \geq \mu(x * z)$ and so $\mu(x * z) = \mu(y * z)$.

Similarly, we have $\mu(z * x) = \mu(z * y)$. This implies that

$$\mu^x(z) = \mu(x * z) \wedge \mu(z * x) = \mu(y * z) \wedge \mu(z * y) = \mu^y(z), \ for\ all\ z \in X.$$

Hence $\mu^x = \mu^y$ □

By Proposition 3.2, consider the binary relation $\sim_\mu$ on $X$ by $x \sim_\mu y \Longleftrightarrow \mu(y * x) = \mu(x * y) = \mu(0)$ where $\mu$ is a fuzzy ideal in $X$.

**Lemma 3.3.** *Let $\mu$ be a fuzzy ideal in $X$. Then $\sim_\mu$ is an equivalent relation on $X$.*

*Proof.* It is clear that $\sim_\mu$ is reflexive and symmetric. Let $x \sim_\mu y$ and $y \sim_\mu z$, for any $x, y, z \in X$. Then $\mu(y * x) = \mu(x * y) = \mu(z * y) = \mu(y * z) = \mu(0)$. We have $z * y \leq (y * x) * (z * x)$ and $x * y \leq (y * z) * (x * z)$. Then

$$\mu(z * x) \geq \mu(y * x) \wedge \mu((y * x) * (z * x)) \geq \mu(y * x) \wedge \mu(z * y) = \mu(0)$$

and

$$\mu(x * z) \geq \mu(y * z) \wedge \mu((y * z) * (x * z)) \geq \mu(y * z) \wedge \mu(x * y) = \mu(0)$$

and so $\mu(z * x) = \mu(x * z) = \mu(0)$. It results that $x \sim_\mu z$. Therefore, $\sim_\mu$ is an equivalent relation on $X$. □

**Theorem 3.4.** *Let $\mu$ be a fuzzy ideal of $X$. Then $\sim_\mu$ is a congruence relation on $X$.*

*Proof.* By Lemma 3.3, it is cofitient to prove that $x \sim_\mu y$ implies $z * x \sim_\mu z * y$, for any $x, y, z \in X$. Let $x \sim_\mu y$. Then $\mu(y*x) = \mu(x*y) = \mu(0)$. Since $y*x \leq (z*y)*(z*x)$ and $x*y \leq (z*x)*(z*y)$, we have $\mu(0) = \mu(x*y) \leq \mu((z*x)*(z*y))$ and $\mu(0) = \mu(y*x) \leq \mu((z * y) * (z * x))$ and so $\mu((z * x) * (z * y)) = \mu((z * y) * (z * x)) = \mu(0)$. Therefore, $z * x \sim_\mu z * y$.

□

**Theorem 3.5.** *Let $\mu$ be a fuzzy ideal in $X$. Then $X/\mu$ is a $BCK$-algebra ($BCK$-quotient algebra induced by fuzzy ideal $\mu$).*

*Proof.* For every $\mu^x, \mu^y \in A/\mu$, we define $\mu^x * \mu^y = \mu^{x*y}$. We prove that the operation on $X/\mu$ is well defined. Let $\mu^x = \mu^s$, $\mu^y = \mu^t$. Then $x \sim_\mu s$ and $y \sim_\mu t$. By Theorem 3.4, we have $x * y \sim_\mu s * t$ and so $\mu^{x*y} = \mu^{s*t}$. It is routine to prove that $X/\mu = (X/\mu, *, \mu^0)$ is a $BCK$-algebra. □

**Example 3.3.** *Let $\Omega = \{1, 2\}$ and $X = \mathcal{P}(\Omega)$. Then $(X, *, \emptyset)$ is a $BCK$-algebra. If $\mu$ is a non-constant fuzzy ideal such that $\mu(X) \neq \mu(\emptyset)$, for $X \neq \emptyset$, then $\mu^{\{1\}} = \{\{1\}\}$, $\mu^{\{2\}} = \{\{2\}\}$, $\mu^{\emptyset} = \{\emptyset\}$ and $\mu^{\{1,2\}} = \{\{1, 2\}\}$ and so $X/\mu = (X/\mu, *, \mu^{\emptyset})$ is a $BCK$-algebra.*

**Remark 3.1.** *By Theorem 3.5, we conclude that $\mu^x \vee \mu^y = \mu^{x \vee y}$ and $\mu^x \wedge \mu^y = \mu^{x \wedge y}$, where $\mu$ is a fuzzy ideal in $X$ and $x, y \in X$.*

**Note.** Let $I$ be an ideal of $X$. Then a congruence relation $\sim_I$ induced by $I$ will obtained by $x \sim_I y$ if an only if $x * y, y * x \in I$.

**Theorem 3.6.** *Let $I$ be an ideal of $X$ and $\chi_I$ be the characteristic function of $I$. Then $x \sim_I y$ if and only if $x \sim_{\chi_I} y$.*

*Proof.* We have

$$x \sim_I y \iff y * x, x * y \in I \iff \chi_I(y * x) = 1 \text{ and } \chi_I(x * y) = 1$$
$$\iff \chi_I(y * x) = \chi_I(I) \text{ and } \chi_I(x * y) = \chi_I(I) \iff x \sim_{\chi_I} y.$$

□

**Theorem 3.7.** *Let $X$ and $Y$ be $BCK$-algebras, $f : X \longrightarrow Y$ be a $BCK$-epimorphism and $\mu$ be a fuzzy ideal of $Y$. Then $X/f^{-1}(\mu) \cong Y/\mu$.*

*Proof.* We know $f^{-1}(\mu)$ is a fuzzy ideal of $X$. Then by Theorem 3.5, $X/f^{-1}(\mu)$ and $Y/\mu$ are $BCK$-algebras. Define $g : X/f^{-1}(\mu) \longrightarrow Y/\mu$ by $g((f^{-1}(\mu))^x) = \mu^{f(x)}$. Let $(f^{-1}(\mu))^x = (f^{-1}(\mu))^y$, for any $x, y \in X$. Then $f^{-1}(\mu)(x*y) = f^{-1}(\mu)(y*x) = f^{-1}(\mu)(0)$ and hence $\mu(f(x)*f(y)) = \mu(f(y)*f(x)) = \mu(f(0)) = \mu(0)$. This means that $f(x) \sim_\mu f(y)$, that is, $\mu^{f(x)} = \mu^{f(y)}$. Hence $g$ is well-defined. For injectiveness of $g$, we suppose that $g((f^{-1}(\mu))^x) = g((f^{-1}(\mu))^y)$, that is, $\mu^{f(x)} = \mu^{f(y)}$, for any $x, y \in X$. Since $f(x) \sim_\mu f(y)$, we have $\mu(f(x) * f(y)) = \mu(f(y) * f(x)) = \mu(0)$. It follows that $f^{-1}(\mu)(x*y) = f^{-1}(\mu)(y * x) = f^{-1}(\mu)(0)$ and hence $(f^{-1}(\mu))^x = (f^{-1}(\mu))^y$. It is easy to show that $g$ is a surjective $BCK$-homomorphism. Therefore, $X/f^{-1}(\mu) \cong Y/\mu$. □

**Corollary 3.1.** *Let $X$ and $Y$ be $BCK$-algebras, $f : X \longrightarrow Y$ be a $BCK$-epimorphism and $I$ be an ideal of $Y$. Then*
*(i) $X/f^{-1}(I) \cong Y/I$.*
*(ii) $X/(Ker(f)) \cong Y$*

*Proof.* $(i)$ We know that $f^{-1}(I)$ is an ideal of $X$. Consider

$$\chi_{f^{-1}(I)} = \begin{cases} 1, & \text{if } x \in f^{-1}(I) \\ 0, & \text{otherwise} \end{cases}$$

Hence $\chi_{f^{-1}(I)} = f^{-1}(\chi_I)(x)$ and so $X/f^{-1}(I) = X/\chi_{f^{-1}(I)}$. Furthermore, we have $X/f^{-1}(I) = X/f^{-1}(\chi_I)$ and $Y/I = Y/\chi_I$. Now, by Theorem 3.7, we have $X/f^{-1}(\chi_I) \cong Y/\chi_I$. Therefore, $X/f^{-1}(I) \cong Y/I$.
$(ii)$ By $(i)$, the proof is clear. □

**Theorem 3.8.** *For two fuzzy quotient algebras $\xi$ and $\eta$ which are defined by $\xi : X/f^{-1}(\mu) \to [0,1]$, $\xi(x/f^{-1}(\mu)) = f^{-1}(\mu)(x)$ and $\eta : f(X)/\mu \to [0,1]$, $\eta(f(x)/\mu) = \mu(f(x))$, respectively, there exists a bijective map $h$ from $X/f^{-1}(\mu)$ to $f(X)/\mu$ such that $\eta \circ h = \xi$.*

*Proof.* The proof is routine.

□

**Theorem 3.9.** *Let $\mu$ be a fuzzy ideal in $X$. Define a mapping $f : X \to X/\mu$ by $f(x) = \mu^x$. Then*
*(1) $f$ is a surjective homomorphism,*
*(2) $Ker(f) = \mu_{\mu(0)}$,*
*(3) $X/\mu$ is isomorphic to the $BCK$-algebra $X/\mu_{\mu(0)}$.*

*Proof.* (1) Clearly, $f$ is surjective. We have $f(x * y) = \mu^{x*y} = \mu^x * \mu^y = f(x) * f(y)$ and $f(0) = \mu^0$. Hence $f$ is a surjective homomorphism.

(2) $x \in Ker(f)$ if and only if $f(x) = \mu^0$ if and only if $\mu^x = \mu^0$ if and only if $x \sim_{\mu_{\mu(0)}} 0$ if and only if $x \in \mu_{\mu(0)}$. Hence $Ker(f) = \mu_{\mu(0)}$.

(3) By (1) and (2) we have $X/\mu$ is isomorphic to the $BCK$-algebra $X/\mu_{\mu(0)}$. □

**Example 3.4.** *Consider $BCK$-algebra $X$ and $X/\mu$ in Example 3.3. We have* $\mu_{\mu(\emptyset)} = \{x \in X | \mu(x) = \mu(\emptyset)\} = \{\emptyset\}$. *Hence*

$$X/\mu_{\mu(\emptyset)} = X/\{\emptyset\} \cong X \cong \{\{\emptyset\}, \{\{1\}\}, \{\{2\}\}, \{\{1,2\}\}\} = \{\mu^{\emptyset}, \mu^{\{1\}}, \mu^{\{2\}}, \mu^{\{1,2\}}\} = X/\mu.$$

In following, we present the generalization of the congruence relation induced by a fuzzy idela of $X$. It shows that more congruence relations on $X$ can be induced by a fuzzy ideal of $X$. Moreover, the above congruence relation is an special case of the following congruence relations to be defined.
Let $\mu$ be a fuzzy subset of $X$ and $\alpha \in [0,1]$. A binary relation $\bar{\mu}^{\alpha}$ on $X$ is defined as follows:

$$\bar{\mu}^{\alpha} = \{(x,y) : x, y \in X,\ \mu(x * y) > \alpha \text{ and } \mu(y * x) > \alpha\}$$

**Lemma 3.4.** *Let $\mu$ be a fuzzy ideal of $X$, $\alpha \in [0,1]$ and $\bar{\mu}^{\alpha} \neq \emptyset$. Then for every $(x,y) \in \bar{\mu}^{\alpha}$ and $z \in X$,*

$$\mu((z * y) * (z * x)) > \alpha,\ \mu((z * x) * (z * y)) > \alpha,$$

$$\mu((y * z) * (x * z)) > \alpha,\ \mu((x * z) * (y * z)) > \alpha.$$

*Proof.* By $(BCK1)$ and definition of fuzzy ideal, we have $\mu(((z * y) * (z * x)) * (x * y)) = \mu(0) \geq \mu(y * x) > \alpha$. Hence $\mu((z * y) * (z * x)) \geq \mu(((z * y) * (z * x)) * (x * y)) \wedge \mu(x * y) > \alpha$. Similarly, we can prove other cases. □

**Theorem 3.10.** *Let $\mu$ be a fuzzy ideal of $X$, $\alpha \in [0,1]$ and $\bar{\mu}^{\alpha} \neq \emptyset$. Then $\bar{\mu}^{\alpha}$ is an equivalent relation on $X$.*

*Proof.* Since $\bar{\mu}^{\alpha} \neq \emptyset$, there exists $x, y \in X$ such that $\mu(y * x) > \alpha$. By definition of fuzzy ideal, $\mu(0) \geq \mu(y * x) > \alpha$. We have $\mu(x * x) = \mu(0) > \alpha$, for every $x \in X$. It results the reflexitivity of $\bar{\mu}^{\alpha}$. It is clear that $\bar{\mu}^{\alpha}$ is symmetric. For proving the trasitivity of $\bar{\mu}^{\alpha}$, let $(x,y), (y,z) \in \bar{\mu}^{\alpha}$. By Lemma 3.4, we have $\mu((z * x) * (z * y)) > \alpha$. Then $\mu(z * x) \geq \mu((z * x) * (z * y)) \wedge \mu(z * y) > \alpha$. Similarly, we can prove $\mu(x * z) > \alpha$. Therefore, $(x,z) \in \bar{\mu}^{\alpha}$. □

**Theorem 3.11.** *Let $\mu$ be a fuzzy ideal of $X$ and $\alpha \in [0,1]$ such that $\mu(0) > \alpha$. Then $\bar{\mu}^{\alpha}$ is a congruence relation on $X$.*

*Proof.* Since $\mu(0) > \alpha$, we have $\mu(x,x) = \mu(0) > \alpha$ and so $\bar{\mu}^{\alpha} \neq \emptyset$. Let $(x,y), (z,t) \in \bar{\mu}^{\alpha}$. Then by Lemma 3.4, we get $(z * x, z * y), (z * y, t * y) \in \bar{\mu}^{\alpha}$ and so by Theorem 3.10, we have $(z * x, t * y) \in \bar{\mu}^{\alpha}$. Hence $\bar{\mu}^{\alpha}$ is a congruence relation on $X$. □

## 4. Conclusions

We tried to improve the studying of fuzzy congruence in algebraic structures and proved some results on $BCK$-algebras. Since congruence relations are interesting and important subjects in fuzzy logic, we hope that we helped to open new fields to anyone that is interested to studying of these concepts in $BCK$-algebras.

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