TWMS J. App. Eng. Math. V.10, N.1, 2020, pp. 232-240

FURTHER RESULTS OF NEUTROSOPHIC SUBALGEBRAS IN BCK/BCI-ALGEBRAS BASED ON NEUTROSOPHIC POINTS

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Abstract. In this paper, we investigate several properties of $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra, $(\in, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra, $(q_{(k_T,k_I,k_F)}, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra, and $(q_{(k_T,k_I,k_F)}, \in)$-neutrosophic subalgebra.

Keywords: $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra; $(\in, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra; $(q_{(k_T,k_I,k_F)}, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra.

AMS Subject Classification: 06F35, 03G25, 03B52

1. Introduction

Smarandache [10, 11] introduced the concept of neutrosophic sets which is more general platform to extend the notions of the classical set and (intuitionistic, interval valued) fuzzy set. Neutrosophic set theory is applied to several parts which is referred to the site http://fs.gallup.unm.edu/neutrosophy.htm. Jun [3] introduced the notion of neutrosophic subalgebras in BCK/BCI-algebras based on neutrosophic points. Borumand and Jun [1] studied several properties of $(\in, \in \vee q)$-neutrosophic subalgebras and $(q, \in \vee q)$-neutrosophic subalgebras in BCK/BCI-algebras. Muhiuddin et al. [9] studied further results on $(\in, \in)$-neutrosophic subalgebras and ideals in BCK/BCI-algebras. Also, Kim et al. [4] considered a general form of neutrosophic points, and then they discussed generalizations of the papers [3] and [1]. As a generalization of $(\in, \in \vee q)$-neutrosophic subalgebras, they introduced the notions of $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra, and $(\in, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra in BCK/BCI-algebras, and investigated several properties. They discussed characterizations of $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra, and considered relations between $(\in, \in)$-neutrosophic subalgebra, $(\in, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra and $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra. Recently, Muhiuddin et al. applied the neutrosophic set theory to the BCK/BCI-algebras on various aspects (see for e.g., [6], [7], [8], [9]).

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§ Manuscript received: May 17, 2018; accepted: April 10, 2019.

In this paper, we investigate further properties of $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra, $(q_{(k_T,k_I,k_F)}, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra, $(\in, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra and $(q_{(k_T,k_I,k_F)}, \in)$-neutrosophic subalgebra in BCK/BCI-algebras.

2. Preliminaries

By a *BCI-algebra* we mean a set X with a binary operation $*$ and the special element 0 satisfying the axioms:

(a1) $((x * y) * (x * z)) * (z * y) = 0$,
(a2) $(x * (x * y)) * y = 0$,
(a3) $x * x = 0$,
(a4) $x * y = y * x = 0 \Rightarrow x = y$,

for all $x, y, z \in X$. If a BCI-algebra X satisfies the axiom

(a5) $0 * x = 0$ for all $x \in X$,

then we say that X is a *BCK-algebra*. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

The collection of all BCK-algebras and all BCI-algebras are denoted by $\mathcal{B}_K(X)$ and $\mathcal{B}_I(X)$, respectively. Also $\mathcal{B}(X) := \mathcal{B}_K(X) \cup \mathcal{B}_I(X)$.

We refer the reader to the books [2] and [5] for further information regarding BCK/BCI-algebras.

Let X be a non-empty set. A neutrosophic set (NS) in X (see [10]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we consider the following sets (see [3]):

$$\begin{aligned}
&T_\in(A; \alpha) := \{x \in X \mid A_T(x) \geq \alpha\},\\
&I_\in(A; \beta) := \{x \in X \mid A_I(x) \geq \beta\},\\
&F_\in(A; \gamma) := \{x \in X \mid A_F(x) \leq \gamma\},\\
&T_q(A; \alpha) := \{x \in X \mid A_T(x) + \alpha > 1\},\\
&I_q(A; \beta) := \{x \in X \mid A_I(x) + \beta > 1\},\\
&F_q(A; \gamma) := \{x \in X \mid A_F(x) + \gamma < 1\},\\
&T_{\in\vee q}(A; \alpha) := \{x \in X \mid A_T(x) \geq \alpha \text{ or } A_T(x) + \alpha > 1\},\\
&I_{\in\vee q}(A; \beta) := \{x \in X \mid A_I(x) \geq \beta \text{ or } A_I(x) + \beta > 1\},\\
&F_{\in\vee q}(A; \gamma) := \{x \in X \mid A_F(x) \leq \gamma \text{ or } A_F(x) + \gamma < 1\}.
\end{aligned}$$

We say $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are *neutrosophic $\in$-subsets*; $T_q(A; \alpha)$, $I_q(A; \beta)$ and $F_q(A; \gamma)$ are *neutrosophic q-subsets*; and $T_{\in\vee q}(A; \alpha)$, $I_{\in\vee q}(A; \beta)$ and $F_{\in\vee q}(A; \gamma)$ are *neutrosophic* $\in \vee\, q$*-subsets*. It is clear that

$$T_{\in\vee\, q}(A;\alpha) = T_{\in}(A;\alpha) \cup T_q(A;\alpha), \tag{1}$$

$$I_{\in\vee\, q}(A;\beta) = I_{\in}(A;\beta) \cup I_q(A;\beta), \tag{2}$$

$$F_{\in\vee\, q}(A;\gamma) = F_{\in}(A;\gamma) \cup F_q(A;\gamma). \tag{3}$$

Given $\Phi, \Psi \in \{\in, q, \in\vee\, q\}$, a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is called a $(\Phi,\ \Psi)$-*neutrosophic subalgebra* of X (see [3]) if the following assertions are valid.

$$\begin{aligned} &x \in T_\Phi(A;\alpha_x),\ y \in T_\Phi(A;\alpha_y) \ \Rightarrow\ x * y \in T_\Psi(A;\alpha_x \wedge \alpha_y),\\ &x \in I_\Phi(A;\beta_x),\ y \in I_\Phi(A;\beta_y) \ \Rightarrow\ x * y \in I_\Psi(A;\beta_x \wedge \beta_y),\\ &x \in F_\Phi(A;\gamma_x),\ y \in F_\Phi(A;\gamma_y) \ \Rightarrow\ x * y \in F_\Psi(A;\gamma_x \vee \gamma_y)\end{aligned} \tag{4}$$

for all $x, y \in X$, α_x, α_y, $\beta_x, \beta_y, \in (0,1]$ and $\gamma_x, \gamma_y \in [0,1)$.

In what follows, let k_T, k_I and k_F denote arbitrary elements of $[0,1)$ unless otherwise specified. If k_T, k_I and k_F are the same number in $[0,1)$, then it is denoted by k, i.e., $k = k_T = k_I = k_F$.

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$, we consider the following sets (see [4]):

$$\begin{aligned} &T_{q_{k_T}}(A;\alpha) := \{x \in X \mid A_T(x) + \alpha + k_T > 1\},\\ &I_{q_{k_I}}(A;\beta) := \{x \in X \mid A_I(x) + \beta + k_I > 1\},\\ &F_{q_{k_F}}(A;\gamma) := \{x \in X \mid A_F(x) + \gamma + k_F < 1\},\\ &T_{\in\vee\, q_{k_T}}(A;\alpha) := \{x \in X \mid A_T(x) \geq \alpha \text{ or } A_T(x) + \alpha + k_T > 1\},\\ &I_{\in\vee\, q_{k_I}}(A;\beta) := \{x \in X \mid A_I(x) \geq \beta \text{ or } A_I(x) + \beta + k_I > 1\},\\ &F_{\in\vee\, q_{k_F}}(A;\gamma) := \{x \in X \mid A_F(x) \leq \gamma \text{ or } A_F(x) + \gamma + k_F < 1\}.\end{aligned}$$

We say $T_{q_{k_T}}(A;\alpha)$, $I_{q_{k_I}}(A;\beta)$ and $F_{q_{k_F}}(A;\gamma)$ are *neutrosophic* q_k*-subsets*; and $T_{\in\vee\, q_{k_T}}(A;\alpha)$, $I_{\in\vee\, q_{k_I}}(A;\beta)$ and $F_{\in\vee\, q_{k_F}}(A;\gamma)$ are *neutrosophic* $\in\vee\, q_k$*-subsets*. For $\Phi \in \{\in, q, q_k, q_{k_T}, q_{k_I}, q_{k_F}, \in\vee\, q, \in\vee\, q_k, \in\vee\, q_{k_T}, \in\vee\, q_{k_I}, \in\vee\, q_{k_F}\}$, the element of $T_\Phi(A;\alpha)$ (resp., $I_\Phi(A;\beta)$ and $F_\Phi(A;\gamma)$) is called a *neutrosophic* T_Φ*-point* (resp., *neutrosophic* I_Φ*-point* and *neutrosophic* F_Φ*-point*) with value α (resp., β and γ).

It is clear that

$$T_{\in\vee\, q_{k_T}}(A;\alpha) = T_{\in}(A;\alpha) \cup T_{q_{k_T}}(A;\alpha), \tag{5}$$

$$I_{\in\vee\, q_{k_I}}(A;\beta) = I_{\in}(A;\beta) \cup I_{q_{k_I}}(A;\beta), \tag{6}$$

$$F_{\in\vee\, q_{k_F}}(A;\gamma) = F_{\in}(A;\gamma) \cup F_{q_{k_F}}(A;\gamma). \tag{7}$$

3. Generalizations of neutrosophic subalgebras

Definition 3.1 ([4])**.** *A neutrosophic set* $A = (A_T, A_I, A_F)$ *in* $X \in \mathcal{B}(X)$ *is called an* $(\in, \in\vee q_{(k_T,k_I,k_F)})$*-neutrosophic subalgebra of* X *if*

$$\begin{aligned} &x \in T_{\in}(A;\alpha_x),\ y \in T_{\in}(A;\alpha_y) \ \Rightarrow\ x * y \in T_{\in\vee q_{k_T}}(A;\alpha_x \wedge \alpha_y),\\ &x \in I_{\in}(A;\beta_x),\ y \in I_{\in}(A;\beta_y) \ \Rightarrow\ x * y \in I_{\in\vee\, q_{k_I}}(A;\beta_x \wedge \beta_y),\\ &x \in F_{\in}(A;\gamma_x),\ y \in F_{\in}(A;\gamma_y) \ \Rightarrow\ x * y \in F_{\in\vee\, q_{k_F}}(A;\gamma_x \vee \gamma_y)\end{aligned} \tag{8}$$

for all $x, y \in X$*,* α_x, α_y*,* $\beta_x, \beta_y, \in (0,1]$ *and* $\gamma_x, \gamma_y \in [0,1)$*.*

An $(\in, \in\vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra with $k_T = k_I = k_F = k$ is called an $(\in, \in\vee q_k)$*-neutrosophic subalgebra*.

Lemma 3.1 ([4]). *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, the following are equivalent.*

(1) *$A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X.*
(2) *$A = (A_T, A_I, A_F)$ satisfies the following assertion.*

$$(\forall x, y \in X) \left(\begin{array}{l} A_T(x * y) \geq \bigwedge\{A_T(x), A_T(y), \frac{1-k_T}{2}\} \\ A_I(x * y) \geq \bigwedge\{A_I(x), A_I(y), \frac{1-k_I}{2}\} \\ A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), \frac{1-k_F}{2}\} \end{array} \right). \quad (9)$$

Theorem 3.1. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of $X \in \mathcal{B}(X)$, then neutrosophic q_k-subsets $T_{q_{k_T}}(A; \alpha)$, $I_{q_{k_I}}(A; \beta)$ and $F_{q_{k_F}}(A; \gamma)$ are subalgebras of X for all $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2})$ whenever they are nonempty.*

Proof. Assume that $T_{q_{k_T}}(A; \alpha)$, $I_{q_{k_I}}(A; \beta)$ and $F_{q_{k_F}}(A; \gamma)$ are nonempty neutrosophic q_k-subsets for all $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2})$. Let $x, y \in T_{q_{k_T}}(A; \alpha)$. Then $A_T(x) + \alpha + k_T > 1$ and $A_T(y) + \alpha + k_T > 1$. Using Lemma 3.1 implies that

$$\begin{aligned} A_T(x * y) + \alpha + k_T &\geq \bigwedge\{A_T(x), A_T(y), \tfrac{1-k_T}{2}\} + \alpha + k_T \\ &= \bigwedge\{A_T(x) + \alpha + k_T, A_T(y) + \alpha + k_T, \tfrac{1-k_T}{2} + \alpha + k_T\} \\ &> 1 \end{aligned}$$

and so that $x * y \in T_{q_{k_T}}(A; \alpha)$. Hence $T_{q_{k_T}}(A; \alpha)$ is a subalgebra of X. Similarly, we can induce that $I_{q_{k_I}}(A; \beta)$ is a subalgebra of X. Now, let $x, y \in F_{q_{k_F}}(A; \gamma)$. Then $A_F(x) + \gamma + k_F < 1$ and $A_F(y) + \gamma + k_F < 1$. It follows from Lemma 3.1 that

$$\begin{aligned} A_F(x * y) + \gamma + k_F &\leq \bigvee\{A_F(x), A_F(y), \tfrac{1-k_F}{2}\} + \gamma + k_F \\ &= \bigvee\{A_F(x) + \gamma + k_F, A_F(y) + \gamma + k_F, \tfrac{1-k_F}{2} + \gamma + k_F\} \\ &< 1. \end{aligned}$$

Thus $x * y \in F_{q_{k_F}}(A; \gamma)$. Therefore $T_{q_{k_T}}(A; \alpha)$, $I_{q_{k_I}}(A; \beta)$ and $F_{q_{k_F}}(A; \gamma)$ are subalgebras of X. □

Corollary 3.1 ([3]). *If $A = (A_T, A_I, A_F)$ is an $(\in, \in \vee q)$-neutrosophic subalgebra of $X \in \mathcal{B}(X)$, then neutrosophic q-subsets $T_q(A; \alpha)$, $I_q(A; \beta)$ and $F_q(A; \gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$ whenever they are nonempty.*

Proof. It follows from taking $k_T = k_I = k_F = 0$ in Theorem 3.1. □

Definition 3.2 ([4]). *A neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$ is called a $(q_{(k_T,k_I,k_F)}, \in\vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of X if*

$$\begin{array}{l} x \in T_{q_{k_T}}(A; \alpha_x),\ y \in T_{q_{k_T}}(A; \alpha_y) \ \Rightarrow\ x * y \in T_{\in\vee\, q_{k_T}}(A; \alpha_x \wedge \alpha_y), \\ x \in I_{q_{k_I}}(A; \beta_x),\ y \in I_{q_{k_I}}(A; \beta_y) \ \Rightarrow\ x * y \in I_{\in\vee\, q_{k_I}}(A; \beta_x \wedge \beta_y), \\ x \in F_{q_{k_F}}(A; \gamma_x),\ y \in F_{q_{k_F}}(A; \gamma_y) \ \Rightarrow\ x * y \in F_{\in\vee\, q_{k_F}}(A; \gamma_x \vee \gamma_y) \end{array} \quad (10)$$

for all $x, y \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y, \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

Theorem 3.2. *If $A = (A_T, A_I, A_F)$ is a $(q_{(k_T,k_I,k_F)}, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of $X \in \mathcal{B}(X)$, then the nonempty neutrosophic $\in \vee\, q_k$-subsets $T_{\in\vee\, q_{k_T}}(A; \alpha)$, $I_{\in\vee\, q_{k_I}}(A; \beta)$ and $F_{\in\vee\, q_{k_F}}(A; \gamma)$ are subalgebras of X for all $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2})$.*

Proof. Assume that $T_{\in\vee\, q_{k_T}}(A;\alpha)$, $I_{\in\vee\, q_{k_I}}(A;\beta)$ and $F_{\in\vee\, q_{k_F}}(A;\gamma)$ are nonempty neutrosophic $\in\vee\, q_k$-subsets for all $\alpha\in(\frac{1-k_T}{2},1]$, $\beta\in(\frac{1-k_I}{2},1]$ and $\gamma\in[0,\frac{1-k_F}{2})$. Let $x,y\in I_{\in\vee\, q_{k_I}}(A;\beta)$. Then

$$A_I(x)\geq\beta \text{ or } A_I(x)+\beta+k_I>1$$

and

$$A_I(y)\geq\beta \text{ or } A_I(y)+\beta+k_I>1.$$

If $A_I(x)+\beta+k_I>1$ and $A_I(y)+\beta+k_I>1$, then obviously $x*y\in I_{\in\vee\, q_{k_I}}(A;\beta)$. Assume that $A_I(x)\geq\beta$ and $A_I(y)+\beta+k_I>1$. Then $A_I(x)+\beta+k_I\geq 2\beta+k_I>1$. Hence $x*y\in I_{\in\vee\, q_{k_I}}(A;\beta)$. By the similar way, if $A_I(y)\geq\beta$ and $A_I(x)+\beta+k_I>1$, then $x*y\in I_{\in\vee\, q_{k_I}}(A;\beta)$. Suppose that $A_I(x)\geq\beta$ and $A_I(y)\geq\beta$. Then $A_I(x)+\beta+k_I\geq 2\beta+k_I>1$ and $A_I(y)+\beta+k_I\geq 2\beta+k_I>1$. It follows that $x*y\in I_{\in\vee\, q_{k_I}}(A;\beta)$. Hence $I_{\in\vee\, q_{k_I}}(A;\beta)$ is a subalgebra of X. Similarly, we can verify that $T_{\in\vee\, q_{k_T}}(A;\alpha)$ is a subalgebra of X. Now, let $x,y\in F_{\in\vee\, q_{k_F}}(A;\gamma)$. Then

$$x\in F_{\in}(A;\gamma) \text{ or } x\in F_{q_{k_F}}(A;\gamma)$$

and

$$y\in F_{\in}(A;\gamma) \text{ or } y\in F_{q_{k_F}}(A;\gamma).$$

If $x\in F_{q_{k_F}}(A;\gamma)$ and $y\in F_{q_{k_F}}(A;\gamma)$, then clearly $x*y\in F_{\in\vee\, q_{k_F}}(A;\gamma)$. If $x\in F_{\in}(A;\gamma)$ and $y\in F_{\in}(A;\gamma)$, then $A_F(x)+\gamma+k_F\leq 2\gamma+k_F<1$ and $A_F(y)+\gamma+k_F\leq 2\gamma+k_F<1$, that is, $x,y\in F_{q_{k_F}}(A;\gamma)$ which implies that $x*y\in F_{\in\vee\, q_{k_F}}(A;\gamma)$. Suppose that $x\in F_{\in}(A;\gamma)$ and $y\in F_{q_{k_F}}(A;\gamma)$. Then $A_F(x)+\gamma+k_F\leq 2\gamma+k_F<1$, i.e., $x\in F_{q_{k_F}}(A;\gamma)$. It follows that $x*y\in F_{\in\vee\, q_{k_F}}(A;\gamma)$. Similarly, if $x\in F_{q_{k_F}}(A;\gamma)$ and $y\in F_{\in}(A;\gamma)$, then $x*y\in F_{\in\vee\, q_{k_F}}(A;\gamma)$. Therefore $F_{\in\vee\, q_{k_F}}(A;\gamma)$ is a subalgebra of X. □

Corollary 3.2 ([3])**.** *If $A=(A_T,A_I,A_F)$ is a $(q,\in\vee\, q)$-neutrosophic subalgebra of $X\in\mathcal{B}(X)$, then the nonempty neutrosophic $\in\vee\, q$-subsets $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are subalgebras of X for all $\alpha,\beta\in(0.5,1]$ and $\gamma\in[0,0.5)$.*

Given a neutrosophic set $A=(A_T,A_I,A_F)$ in a set X, consider the following sets:

$$X_{k_T}:=\{x\in X\mid A_T(x)>k_T\},$$

$$X_{k_I}:=\{x\in X\mid A_I(x)>k_I\},$$

and

$$X_{k_F}:=\{x\in X\mid A_F(x)<k_F\}.$$

Theorem 3.3. *Let $A=(A_T,A_I,A_F)$ be an $(\in,\, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra of $X\in\mathcal{B}(X)$. If $k_T\in[0,\frac{1-A_T(x)\wedge A_T(y)}{2}]$, $k_I\in[0,\frac{1-A_I(x)\wedge A_I(y)}{2}]$ and $k_F\in(\frac{1-A_F(x)\vee A_F(y)}{2},1)$, then the sets X_{k_T}, X_{k_I} and X_{k_F} are subalgebras of X.*

Proof. Let $x,y\in X_{k_T}$. Then $A_T(x)>k_T$ and $A_T(y)>k_T$. If $A_T(x*y)\leq k_T$, then

$$A_T(x*y)+\alpha+k_T\leq 2k_T+\alpha\leq 1$$

where $\alpha=A_T(x)\wedge A_T(y)$. Hence $x*y\notin T_{q_{k_T}}(A;\alpha)$, a contradiction since $x\in T_{\in}(A;A_T(x))$ and $y\in T_{\in}(A;A_T(y))$. Thus $A_T(x*y)>k_T$, that is, $x*y\in X_{k_T}$. Similarly, if $x,y\in X_{k_I}$, then $x * y \in X_{k_I}$. Let $x, y \in X_{k_F}$. Then $A_F(x) < k_F$ and $A_F(y) < k_F$. If $A_F(x * y) \geq k_F$, then

$$A_F(x * y) + \gamma + k_F \geq 2k_F + \gamma \geq 1$$

where $\gamma = A_F(x) \vee A_F(y)$, and so $x * y \notin F_{q_{k_F}}(A; \gamma)$. This is a contradiction, and thus $A_F(x * y) < k_F$, i.e., $x * y \in X_{k_F}$. Therefore X_{k_T}, X_{k_I} and X_{k_F} are subalgebras of X. □

Corollary 3.3. *Let $A = (A_T, A_I, A_F)$ be an $(\in, q_{(k_T, k_I, k_F)})$-neutrosophic subalgebra of $X \in \mathcal{B}(X)$. If $k_T \in [0, \frac{1 - A_T(x) \wedge A_T(y)}{2}]$, $k_I \in [0, \frac{1 - A_I(x) \wedge A_I(y)}{2}]$ and $k_F \in (\frac{1 - A_F(x) \vee A_F(y)}{2}, 1)$, then $X_{k_T} \cap X_{k_I} \cap X_{k_F}$ is a subalgebra of X.*

Theorem 3.4. *If $A = (A_T, A_I, A_F)$ is a $(q_{(k_T, k_I, k_F)}, \in)$-neutrosophic subalgebra of $X \in \mathcal{B}(X)$ with $k_F \in (0, \frac{1}{2}]$, then the sets X_{k_T}, X_{k_I} and X_{k_F} are subalgebras of X.*

Proof. Let $x, y \in X_{k_I}$. Then $A_I(x) > k_I$ and $A_I(y) > k_I$, which imply that $A_I(x) + k_I + 1 > 1$ and $A_I(y) + k_I + 1 > 1$. Hence $x, y \in I_{q_{k_I}}(A; 1)$, and so $x * y \in I_\in(A; 1)$. If $x * y \notin X_{k_I}$, then $A_I(x * y) \leq k_I < 1 = 1 \wedge 1$, that is, $x * y \notin I_\in(A; 1 \wedge 1) = I_\in(A; 1)$. This is a contradiction, and thus $x * y \in X_{k_I}$. By the similar way, we can verify that if $x, y \in X_{k_T}$, then $x * y \in X_{k_T}$. Now, let $x, y \in X_{k_F}$. Then $A_F(x) < k_F$ and $A_F(y) < k_F$. Since $k_F \leq \frac{1}{2}$, it follows that $A_F(x) + k_F + 0 < 1$ and $A_F(y) + k_F + 0 < 1$, that is, $x, y \in F_{q_{k_F}}(A; 0)$. Thus $x * y \in F_\in(A; 0)$, and so $A_F(x * y) = 0 < k_F$, i.e., $x * y \in X_{k_F}$. Therefore X_{k_T}, X_{k_I} and X_{k_F} are subalgebras of X. □

Theorem 3.5. *Given a a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, the nonempty neutrosophic $\in$-subsets $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are subalgebras of X for all $\alpha \in (\frac{1 - k_T}{2}, 1]$, $\beta \in (\frac{1 - k_I}{2}, 1]$ and $\gamma \in [0, \frac{1 - k_F}{2})$ if and only if the following assertion is valid.*

$$(\forall x, y \in X) \begin{pmatrix} A_T(x * y) \vee \frac{1 - k_T}{2} \geq A_T(x) \wedge A_T(y) \\ A_I(x * y) \vee \frac{1 - k_I}{2} \geq A_I(x) \wedge A_I(y) \\ A_F(x * y) \wedge \frac{1 - k_F}{2} \leq A_F(x) \vee A_F(y) \end{pmatrix}. \tag{11}$$

Proof. Suppose that the nonempty neutrosophic $\in$-subsets $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are subalgebras of X for all $\alpha \in (\frac{1 - k_T}{2}, 1]$, $\beta \in (\frac{1 - k_I}{2}, 1]$ and $\gamma \in [0, \frac{1 - k_F}{2})$. If there are $a, b \in X$ such that $A_T(a * b) \vee \frac{1 - k_T}{2} < A_T(a) \wedge A_T(b) := \alpha$, then $\alpha \in (\frac{1 - k_T}{2}, 1]$ and $a, b \in T_\in(A; \alpha)$. It follows that $a * b \in T_\in(A; \alpha)$, that is, $A_T(a * b) \geq \alpha$ since $T_\in(A; \alpha)$ is a subalgebra of X. This is a contradiction, and so $A_T(x * y) \vee \frac{1 - k_T}{2} \geq A_T(x) \wedge A_T(y)$ for all $x, y \in X$. By the similar way, we know that $A_I(x * y) \vee \frac{1 - k_I}{2} \geq A_I(x) \wedge A_I(y)$ for all $x, y \in X$. Now, assume that $A_F(a * b) \wedge \frac{1 - k_F}{2} > A_F(a) \vee A_F(b)$ for some $a, b \in X$. Then $a, b \in F_\in(A; \gamma)$ and $\gamma \in [0, \frac{1 - k_F}{2})$ where $\gamma = A_F(a) \vee A_F(b)$. But $a * b \notin F_\in(A; \gamma)$, a contradiction. Thus $A_F(x * y) \wedge \frac{1 - k_F}{2} \leq A_F(x) \vee A_F(y)$ for all $x, y \in X$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in $X \in \mathcal{B}(X)$ which satisfies the condition (11). Let $a, b, x, y \in X$, $\alpha \in (\frac{1 - k_T}{2}, 1]$ and $\beta \in (\frac{1 - k_I}{2}, 1]$ be such that $x, y \in T_\in(A; \alpha)$ and $a, b \in I_\in(A; \beta)$. Then

$$A_T(x * y) \vee \tfrac{1 - k_T}{2} \geq A_T(x) \wedge A_T(y) \geq \alpha > \tfrac{1 - k_T}{2},$$
$$A_I(a * b) \vee \tfrac{1 - k_I}{2} \geq A_I(a) \wedge A_I(b) \geq \beta > \tfrac{1 - k_I}{2}.$$

It follows that $A_T(x * y) \geq \alpha$ and $A_I(a * b) \geq \beta$, that is, $x * y \in T_\in(A; \alpha)$ and $a * b \in I_\in(A; \beta)$. Now, let $x, y \in F_\in(A; \gamma)$ for $x, y \in X$ and $\gamma \in [0, \frac{1 - k_F}{2})$. Then

$$A_F(x * y) \wedge \tfrac{1 - k_F}{2} \leq A_F(x) \vee A_F(y) \leq \gamma < \tfrac{1 - k_F}{2},$$

and so $A_F(x*y) \leq \gamma$. Hence $x*y \in F_\in(A;\gamma)$. Therefore $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are subalgebras of X for all $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2})$. □

Theorem 3.6. *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, if the nonempty neutrosophic q_k-subsets $T_{q_{k_T}}(A;\alpha)$, $I_{q_{k_I}}(A;\beta)$ and $F_{q_{k_F}}(A;\gamma)$ are subalgebras of X for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$, then the following assertion is valid.*

$$(\forall x, y \in X) \left(\begin{array}{l} x \in T_{q_{k_T}}(A;\alpha_x), y \in T_{q_{k_T}}(A;\alpha_y) \ \Rightarrow x*y \in T_\in(A;\alpha_x \vee \alpha_y) \\ x \in I_{q_{k_I}}(A;\beta_x), y \in I_{q_{k_I}}(A;\beta_y) \ \Rightarrow x*y \in I_\in(A;\beta_x \vee \beta_y) \\ x \in F_{q_{k_F}}(A;\gamma_x), y \in F_{q_{k_F}}(A;\gamma_y) \ \Rightarrow x*y \in F_\in(A;\gamma_x \wedge \gamma_y) \end{array} \right). \tag{12}$$

for all $x, y \in X$, $\alpha_x, \alpha_y \in (0, \frac{1-k_T}{2}]$, $\beta_x, \beta_y \in (0, \frac{1-k_I}{2}]$ and $\gamma_x, \gamma_y \in [\frac{1-k_F}{2}, 1)$.

Proof. Let $x, y \in X$ and $\alpha_x, \alpha_y \in (0, \frac{1-k_T}{2}]$ be such that $x \in T_{q_{k_T}}(A;\alpha_x)$ and $y \in T_{q_{k_T}}(A;\alpha_y)$. Then $x, y \in T_{q_{k_T}}(A;\alpha_x \vee \alpha_y)$. Since $\alpha_x \vee \alpha_y \in (0, \frac{1-k_T}{2}]$, it follows from the hypothesis that $x*y \in T_{q_{k_T}}(A;\alpha_x \vee \alpha_y)$. Hence

$$A_T(x*y) > 1 - (\alpha_x \vee \alpha_y) - k_T \geq \alpha_x \vee \alpha_y,$$

and so $x*y \in T_\in(A;\alpha_x \vee \alpha_y)$. Similarly, we can verify that if $x \in I_{q_{k_I}}(A;\beta_x)$ and $y \in I_{q_{k_I}}(A;\beta_y)$, then $x*y \in I_\in(A;\beta_x \vee \beta_y)$. Now, let $x, y \in X$ and $\gamma_x, \gamma_y \in [\frac{1-k_F}{2}, 1)$ be such that $x \in F_{q_{k_F}}(A;\gamma_x)$ and $y \in F_{q_{k_F}}(A;\gamma_y)$. Then $x, y \in F_{q_{k_F}}(A;\gamma_x \wedge \gamma_y)$ since $\gamma_x \wedge \gamma_y \in [\frac{1-k_F}{2}, 1)$, which implies from hypothesis that $x*y \in F_{q_{k_F}}(A;\gamma_x \wedge \gamma_y)$. Thus

$$A_F(x*y) < 1 - (\gamma_x \wedge \gamma_y) - k_F \leq \tfrac{1-k_F}{2} \leq \gamma_x \wedge \gamma_y$$

and hence $x*y \in F_\in(A;\gamma_x \wedge \gamma_y)$. □

Corollary 3.4 ([1])**.** *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, if the nonempty neutrosophic q-subsets $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$, then the following assertion is valid.*

$$(\forall x, y \in X) \left(\begin{array}{l} x \in T_q(A;\alpha_x), y \in T_q(A;\alpha_y) \ \Rightarrow x*y \in T_\in(A;\alpha_x \vee \alpha_y) \\ x \in I_q(A;\beta_x), y \in I_q(A;\beta_y) \ \Rightarrow x*y \in I_\in(A;\beta_x \vee \beta_y) \\ x \in F_q(A;\gamma_x), y \in F_q(A;\gamma_y) \ \Rightarrow x*y \in F_\in(A;\gamma_x \wedge \gamma_y) \end{array} \right).$$

for all $x, y \in X$, $\alpha_x, \alpha_y \in (0, 0.5]$ and $\gamma_x, \gamma_y \in [0.5, 1)$.

Theorem 3.7. *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, if the nonempty neutrosophic $\in \vee\, q_k$-subsets $T_{\in \vee\, q_{k_T}}(A;\alpha)$, $I_{\in \vee\, q_{k_I}}(A;\beta)$ and $F_{\in \vee\, q_{k_F}}(A;\gamma)$ are subalgebras of X for all $\alpha \in (0, \frac{1-k_T}{2}]$, $\beta \in (0, \frac{1-k_I}{2}]$ and $\gamma \in [\frac{1-k_F}{2}, 1)$, then the following assertion is valid.*

$$(\forall x, y \in X) \left(\begin{array}{l} x \in T_{q_{k_T}}(A;\alpha_x), y \in T_{q_{k_T}}(A;\alpha_y) \ \Rightarrow x*y \in T_{\in \vee\, q_{k_T}}(A;\alpha_x \vee \alpha_y) \\ x \in I_{q_{k_I}}(A;\beta_x), y \in I_{q_{k_I}}(A;\beta_y) \ \Rightarrow x*y \in I_{\in \vee\, q_{k_I}}(A;\beta_x \vee \beta_y) \\ x \in F_{q_{k_F}}(A;\gamma_x), y \in F_{q_{k_F}}(A;\gamma_y) \ \Rightarrow x*y \in F_{\in \vee\, q_{k_F}}(A;\gamma_x \wedge \gamma_y) \end{array} \right). \tag{13}$$

for all $x, y \in X$, $\alpha_x, \alpha_y \in (0, \frac{1-k_T}{2}]$, $\beta_x, \beta_y \in (0, \frac{1-k_I}{2}]$ and $\gamma_x, \gamma_y \in [\frac{1-k_F}{2}, 1)$.

Proof. Assume that $x \in I_{q_{k_I}}(A;\beta_x)$ and $y \in I_{q_{k_I}}(A;\beta_y)$ for $x, y \in X$ and $\beta_x, \beta_y \in (0, \frac{1-k_I}{2}]$. Then $x, y \in I_{\in \vee\, q_{k_I}}(A;\beta_x \vee \beta_y)$ where $\beta_x \vee \beta_y \in (0, \frac{1-k_I}{2}]$. It follows from the assumption that $x*y \in I_{\in \vee\, q_{k_I}}(A;\beta_x \vee \beta_y)$. By the similar way, we know that if $x \in T_{q_{k_T}}(A;\alpha_x)$ and $y \in T_{q_{k_T}}(A;\alpha_y)$, then $x*y \in T_{\in \vee\, q_{k_T}}(A;\alpha_x \vee \alpha_y)$. Let $x, y \in X$ and $\gamma_x, \gamma_y \in [\frac{1-k_F}{2}, 1)$

be such that $x \in F_{q_{k_F}}(A;\gamma_x)$ and $y \in F_{q_{k_F}}(A;\gamma_y)$. Then $x, y \in F_{\in\vee\, q_{k_F}}(A;\gamma_x \wedge \gamma_y)$ with $\gamma_x \wedge \gamma_y \in [\frac{1-k_F}{2}, 1)$. Since $F_{\in\vee\, q_{k_F}}(A;\gamma_x \wedge \gamma_y)$ is a subalgebra of X by hypothesis, we have $x * y \in F_{\in\vee\, q_{k_F}}(A;\gamma_x \wedge \gamma_y)$. □

Corollary 3.5 ([1]). *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, if the nonempty neutrosophic $\in \vee\, q$-subsets $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$, then the following assertion is valid.*

$$(\forall x, y \in X) \left(\begin{array}{l} x \in T_q(A;\alpha_x), y \in T_q(A;\alpha_y) \ \Rightarrow x * y \in T_{\in\vee\, q}(A;\alpha_x \vee \alpha_y) \\ x \in I_q(A;\beta_x), y \in I_q(A;\beta_y) \ \Rightarrow x * y \in I_{\in\vee\, q}(A;\beta_x \vee \beta_y) \\ x \in F_q(A;\gamma_x), y \in F_q(A;\gamma_y) \ \Rightarrow x * y \in F_{\in\vee\, q}(A;\gamma_x \wedge \gamma_y) \end{array} \right).$$

for all $x, y \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 0.5]$ and $\gamma_x, \gamma_y \in [0.5, 1)$.

Theorem 3.8. *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, if the nonempty neutrosophic $\in \vee\, q_k$-subsets $T_{\in\vee\, q_{k_T}}(A;\alpha)$, $I_{\in\vee\, q_{k_I}}(A;\beta)$ and $F_{\in\vee\, q_{k_F}}(A;\gamma)$ are subalgebras of X for all $\alpha \in (\frac{1-k_T}{2}, 1]$, $\beta \in (\frac{1-k_I}{2}, 1]$ and $\gamma \in [0, \frac{1-k_F}{2})$, then $A = (A_T, A_I, A_F)$ satisfies* (13) *for all $\alpha_x, \alpha_y \in (\frac{1-k_T}{2}, 1]$, $\beta_x, \beta_y \in (\frac{1-k_I}{2}, 1]$ and $\gamma_x, \gamma_y \in [0, \frac{1-k_F}{2})$.*

Proof. It is similar to the proof of Theorem 3.7. □

Corollary 3.6 ([1]). *Given a neutrosophic set $A = (A_T, A_I, A_F)$ in $X \in \mathcal{B}(X)$, if the nonempty neutrosophic $\in \vee\, q$-subsets $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$, then the following assertion is valid.*

$$(\forall x, y \in X) \left(\begin{array}{l} x \in T_q(A;\alpha_x), y \in T_q(A;\alpha_y) \ \Rightarrow x * y \in T_{\in\vee\, q}(A;\alpha_x \vee \alpha_y) \\ x \in I_q(A;\beta_x), y \in I_q(A;\beta_y) \ \Rightarrow x * y \in I_{\in\vee\, q}(A;\beta_x \vee \beta_y) \\ x \in F_q(A;\gamma_x), y \in F_q(A;\gamma_y) \ \Rightarrow x * y \in F_{\in\vee\, q}(A;\gamma_x \wedge \gamma_y) \end{array} \right).$$

for all $x, y \in X$, $\alpha_x, \alpha_y, \beta_x, \beta_y \in (0.5, 1]$ and $\gamma_x, \gamma_y \in [0, 0.5)$.

4. Conclusion

In this paper, we investigate further properties of $(\in, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra, $(q_{(k_T,k_I,k_F)}, \in \vee q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra, $(\in, q_{(k_T,k_I,k_F)})$-neutrosophic subalgebra and $(q_{(k_T,k_I,k_F)}, \in)$-neutrosophic subalgebra in BCK/BCI-algebras. We hope that this work will provide a deep impact on the upcoming research in this field and other related areas to open up new horizons of interest and innovations. Indeed, this work may serve as a foundation for further study of neutrosophic subalgebras in BCK/BCI-algebras. To extend these results, one can further study the neutrosophic set theory of different algebras such as MTL-algerbas, BL-algebras, MV-algebras, EQ-algebras, R0-algebras and Q-algebras etc. One may also apply this concept to study some applications in many fields like decision making, knowledge base systems, medical diagnosis, data analysis and graph theory etc.

5. Acknowledgements

The authors are very thankful to the reviewers for careful detailed reading and helpful comments/suggestions that improved the overall presentation of this paper.

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