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LUCAS POLYNOMIAL SOLUTION FOR NEUTRAL DIFFERENTIAL EQUATIONS WITH PROPORTIONAL DELAYS

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Abstract. This paper proposes a combined operational matrix approach based on Lucas and Taylor polynomials for the solution of neutral type differential equations with proportional delays. The advantage of the proposed method is the ease of its application. The method facilitates the solution of the given problem by reducing it to a matrix equation. Illustrative examples are validated by means of absolute errors. Residual error estimation is presented to improve the solutions. Presented in graphs and tables the results are compared with the existing methods in literature.

Keywords: Neutral differential equations, Lucas and Taylor polynomials, collocation and matrix methods.

AMS Subject Classification: 34K40, 65L60, 40C05

1. Introduction

The neutral type differential equations with proportional delays is one of the essential classes of delay differential equations. Usually used in modeling of physical phenomena, they play an important role in other fields of science such as physics, mechanics, electrodynamics, biology, astrophysics, quantum mechanics, and biomathematics. Consequently, such equations have received significant attention in the last decades. The main difficulty in studying delay differential equations is that they have a special transcendental nature; therefore many of them cannot be solved by well-known exact methods.

Various numerical approaches have been used to approximate the solutions of neutral differential equations (NDEs). The variational iteration method for solving NDEs with proportional delays is studied by Chen and Wang [8]. Modified variational iteration method is studied by Ghaneai et.al. [11]. They introduced an auxiliary parameter into the well-known variational iteration algorithm. A new homotopy perturbation method and Padé approximation is introduced by Abolhasani et.al. [1]. Homotopy perturbation

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method is used by Biazar and Ghanbari [6]. Sakar [19] improved the Homotopy analysis method with optimal determination of auxiliary parameter with the use of residual error function. A semi-analytical approach based on Taylor method is used by Rebenda et. al. [18]. They combined this method with the differential transformation method (DTM). Bhrawy et. al. [4, 5] used Legendre-Gauss collocation method and the shifted Jacobi-Gauss-Lobatto pseudo spectral (SJGLP) method. Bezier surface form is applied by Ghomanjani and Farahi [12]. The reproducing kernel Hilbert space method is used by Lv and Gao [17]. Cheng et.al. [9] studied an algorithm based on reproducing kernel theory. In their study, approximate solution is obtained by truncating the n-term of the exact solution. Ibis and Bayram [14] presented a collocation method based on Hermite polynomials. Polynomial least squares method is used by Căruntu and Bota [7] to find analytical approximate solutions for a general class of nonlinear delay differential equations. A two-stage order-one Runge-Kutta method is applied by Bellen and Zennaro [3]. One-leg θ-method is used by Wang and Li [20]. Yüzbaşı and Sezer [21] employed the shifted Legendre method for the approximate solution of pantograph-delay type differential equations, which are one class of the neutral differential equations.

In this study, we propose the application of a novel numerical method, developed by Sezer et. al. [13, 2, 10], to find the approximate solution of NDEs with proportional delays in the truncated Lucas series form. The application of the method reduces the solution of the given problem to a matrix equation solution, corresponding to a system of algebraic equations with unknown Lucas coefficients.

2. Lucas Polynomial Formulation of the Problem

A general class of neutral differential equations with proportional delays is given as

$$(y(t) + cy(\alpha_m t))^{(m)} + dy(t) + \sum_{k=0}^{m-1} P_k(t)\, y^{(k)}(\alpha_k t)) = g(t), \quad 0 \leq t \leq b \tag{1}$$

with the initial conditions,

$$\sum_{k=0}^{m-1} a_{ik}\, y^{(k)}(0) = \lambda_i, \quad i = 0, 1, \cdots, m-1 \tag{2}$$

where $P_k(t)$ and $g(t)$ are the given analytical functions defined on the interval $0 \leq t \leq b$, $0 \leq \alpha_m < 1$. c, d, a_{ik}, b and λ_i are suitable constants; $y(t)$ is unknown function to be determined.

The solution of Eq. (1) with the initial conditions in Eq. (2) is approximated by the Lucas series

$$y(t) \cong y_N(t) = \sum_{n=0}^{N} a_n\, L_n(t), \quad 0 \leq t \leq b \tag{3}$$

where a_n, $n = 0, 1, ..., N$ are unknown coefficients to be determined and $L_n(t)$, $n = 0, 1, ..., N$; $N > m$, are the Lucas polynomials defined recursively as [16]

$$\begin{aligned} &L_0(t) = 2 \\ &L_1(t) = t \\ &L_{n+1}(t) = t\, L_n(t) + L_{n-1}(t), \qquad n \geq 1. \end{aligned}$$

The unknown function $y(t)$, the proportional delay term $y(\alpha_m t)$ and their derivatives in Eq. (1) can be written in the matrix form as follows

$$y(t) \cong y_N(t) = \mathbf{L}(t)\,\mathbf{A} = \underbrace{\mathbf{T}(t)\,\mathbf{M}}_{\mathbf{L}(t)}\,\mathbf{A} \tag{4}$$

where

$$\mathbf{A} = \begin{bmatrix} a_o & a_1 & \cdots & a_N \end{bmatrix}^T, \quad \mathbf{T}(t) = \begin{bmatrix} 1 & t & \cdots & t^N \end{bmatrix}, \quad \text{and}$$

$$\mathbf{M}^T = \begin{bmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{1}\binom{1}{0} & 0 & \cdots & 0 \\ \frac{2}{1}\binom{1}{1} & 0 & \frac{2}{2}\binom{2}{0} & \cdots & 0 \\ 0 & \frac{3}{2}\binom{2}{1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \frac{n}{\left(\frac{n+1}{2}\right)}\binom{\frac{n+1}{2}}{\frac{n-1}{2}} & 0 & \cdots & \left.\frac{n}{n}\binom{n}{0}\right\} n \text{ odd} \\ \frac{n}{\left(\frac{n}{2}\right)}\binom{\frac{n}{2}}{\frac{n}{2}} & 0 & \frac{n}{\left(\frac{n+2}{2}\right)}\binom{\frac{n+2}{2}}{\frac{n-2}{2}} & \cdots & \left.\frac{n}{n}\binom{n}{0}\right\} n \text{ even} \end{bmatrix}.$$

The derivatives of $y(t)$ in Eq. (4) can be approximated by the fundamental matrix relations of Taylor and Lucas matrix methods as in [2],

$$y^{(k)}(t) \cong y_N^{(k)}(t) = \mathbf{L}^{(k)}(t)\,\mathbf{A} = \mathbf{T}^{(k)}(t)\,\mathbf{M}\,\mathbf{A} = \mathbf{T}(t)\,\mathbf{B}^k\,\mathbf{M}\,\mathbf{A} \tag{5}$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B}^0 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Replacing t by $\alpha_m t$ in Eq. (5) yields

$$y^{(k)}(\alpha_m t) \cong y_N^{(k)}(\alpha_m t) = \mathbf{T}^{(k)}(\alpha_m t)\,\mathbf{M}\,\mathbf{A} = \mathbf{T}(\alpha_m t)\,\mathbf{B}^k\,\mathbf{M}\,\mathbf{A} = \mathbf{T}(t)\,\mathbf{S}(\alpha_m)\,\mathbf{B}^k\,\mathbf{M}\,\mathbf{A} \tag{6}$$

where

$$\mathbf{S}(\alpha_m) = \begin{bmatrix} (\alpha_m)^0 & 0 & 0 & \cdots & 0 \\ 0 & (\alpha_m)^1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (\alpha_m)^N \end{bmatrix}.$$

The matrix equation for the problem defined in Eq. (1) can be obtained by substituting the matrix relations in Eqs. (5) and (6) into Eq. (1) as follows

$$\left[\mathbf{T}(t)\mathbf{B}^m\mathbf{M}+c\mathbf{T}(t)\mathbf{S}(\alpha_m)\mathbf{B}^m\mathbf{M}+d\mathbf{T}(t)\mathbf{M}+\sum_{k=0}^{m-1}\mathbf{P}_k(t)\mathbf{T}(t)\mathbf{S}(\alpha_k)\mathbf{B}^k\mathbf{M}\right]\mathbf{A}=\mathbf{G}. \quad (7)$$

Inserting the collocation points ($t_j=\frac{b}{N}j, \quad j=0,1,\cdots,N$) in Eq. (7) gives

$$\underbrace{\left[\mathbf{T}\mathbf{B}^m+\mathbf{C}\mathbf{T}\mathbf{S}(\alpha_m)\mathbf{B}^m+\mathbf{D}\mathbf{T}+\sum_{k=0}^{m-1}\mathbf{P}_k\mathbf{T}\mathbf{S}(\alpha_k)\mathbf{B}^k\right]\mathbf{M}}_{\mathbf{W}}\mathbf{A}=\mathbf{G} \quad (8)$$

where

$$\mathbf{T}=\begin{bmatrix} T(t_0)\\ T(t_1)\\ \vdots\\ T(t_N)\end{bmatrix}=\begin{bmatrix} 1 & t_0 & \cdots & t_0^N\\ 1 & t_1 & \cdots & t_1^N\\ \vdots & \vdots & \ddots & \vdots\\ 1 & t_N & \cdots & t_N^N\end{bmatrix}$$

$$\mathbf{D}=diag\,[d]$$
$$\mathbf{C}=diag\,[c]$$
$$\mathbf{P}_k=diag\begin{bmatrix} P_k(t_0) & P_k(t_1) & \cdots & P_k(t_N)\end{bmatrix}$$
$$\mathbf{G}=\begin{bmatrix} g(t_0) & g(t_1) & \cdots & g(t_N)\end{bmatrix}^T$$

Similarly, we obtain the corresponding matrix form of the initial conditions given in Eq. (2), by using the relation in Eq. (5) as

$$\mathbf{U}\mathbf{A}=\boldsymbol{\lambda} \quad (9)$$

where

$$\mathbf{U}=\begin{bmatrix} U_0\\ U_1\\ \vdots\\ U_{m-1}\end{bmatrix}, \quad \boldsymbol{\lambda}=\begin{bmatrix} \lambda_0\\ \lambda_1\\ \vdots\\ \lambda_{m-1}\end{bmatrix}$$

and

$$U_j=\sum_{k=0}^{m-1}a_{jk}\mathbf{T}(0)\mathbf{B}^k\mathbf{M}=[u_{jr}],\ j=0,1,\ldots,m-1,\ r=0,1,\ldots,N.$$

In order to find the solution of Eq. (1)-(2), we replace the m row of the augmented matrix ($[\mathbf{U};\boldsymbol{\lambda}]$) of Eq. (9) by any m rows of the augmented matrix ($[\mathbf{W};\mathbf{G}]$) of Eq. (8), then we solve this new augmented matrix, which has the form

$$\left[\widetilde{\mathbf{W}}\ ;\ \widetilde{\mathbf{G}}\right]=\left[\begin{array}{ccccccc} w_{00} & w_{01} & \cdots & w_{0N} & ; & g\left(t_0\right) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & g\left(t_1\right) \\ \vdots & \vdots & \vdots & \vdots & \vdots; & \vdots \\ w_{N-m,0} & w_{N-m,1} & \cdots & w_{N-m,N} & ; & g\left(t_{N-m}\right) \\ u_{00} & u_{01} & \cdots & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \lambda_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots; & \vdots \\ u_{m-1,0} & u_{m-1,1} & \cdots & u_{m-1,N} & ; & \lambda_{m-1} \end{array}\right].$$

If rank $\widetilde{\mathbf{W}} = rank\left[\ \widetilde{\mathbf{W}}\ \ ;\ \ \widetilde{\mathbf{G}}\ \right] = N+1$, then the coefficient matrix $\mathbf{A}$ is uniquely determined and the solution of Eq. (1)-(2) can be obtained from Eq. (4).

3. Residual error estimation

We know from [15, 21] that the residual error estimation is an efficient numerical scheme to improve the approximate solutions. Thus, we here aim to improve the Lucas polynomial solutions, constituting the residual error estimation based on Lucas polynomial and residual function. First, we obtain the residual function $R_N(t)$ via the Lucas polynomial solution (3) as

$$R_N(t) = (y_N(t) + cy_N(\alpha_m t))^{(m)} + dy_N(t) + \sum_{k=0}^{m-1} P_k(t)\, y_N^{(k)}(\alpha_k t) - g(t),$$

and $e_N(t)$ is error function, which is defined to be

$$e_N(t) = y(t) - y_N(t) = -R_N(t). \tag{10}$$

On the other hand, the initial conditions is taken as the homogenous form

$$\sum_{k=0}^{m-1} a_{ik}\, y^{(k)}(0) = 0. \tag{11}$$

By following the procedure described in 2, we solve the error problem, which is combination of Eqs. (10) and (11). Then, we have

$$e_{N,M}(t) = \sum_{n=0}^{M} \hat{a}_n\, L_n(t),\ \left(M > N\right),$$

where $e_{N,M}(t)$ is an estimated error function.
Thus, we improve the Lucas polynomial solution (3) as the following:

$$y_{N,M}(t) = y_N(t) + e_{N,M}(t),$$

where $y_{N,M}(t)$ is a corrected Lucas polynomial solution. In addition, the corrected error function is described as $E_{N,M}(t) = y(t) - y_{N,M}(t)$.

4. Numerical Examples

In this section, results obtained from the application of the method to three prototype examples are given. In the first two problems, exact solutions of the problems are obtained. Other numerical results of the second problem are also presented. The last problem is solved numerically, and the results are compared with the exact solution as well as other numerical solutions.

4.1. **Example 1:**

As a first example, we consider the following second-order neutral differential equation with proportional delays

$$\begin{cases} y''(t) = \frac{3}{4}y(t) + y\left(\frac{t}{2}\right) + y'\left(\frac{t}{2}\right) + \frac{1}{2}y''\left(\frac{t}{2}\right) - t^2 - t + 1, \quad 0 \le t \le 1 \\ y(0) = y'(0) = 0 \end{cases}$$

where $\begin{cases} P_0(t) = -1, \quad P_1(t) = -1, \\ \alpha_0 = \frac{1}{2}, \quad \alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{1}{2} \\ g(t) = -t^2 - t + 1, \quad c = -\frac{1}{2}, \quad d = -\frac{3}{4}. \end{cases}$

The collocation points for $N = 2$ are computed as $\{t_0 = 0,\ t_1 = \frac{1}{2},\ t_2 = 1\}$. Following the procedure in the previous section, the fundamental matrix equation of the given example becomes

$$\underbrace{\left[\mathbf{T}\,\mathbf{B}^2 + \mathbf{C}\mathbf{T}\,\mathbf{S}(\alpha_2)\mathbf{B}^2 + \mathbf{D}\,\mathbf{T} + \mathbf{P}_0\mathbf{T}\,\mathbf{S}(\alpha_0)\mathbf{B}^0 + \mathbf{P}_1\mathbf{T}\,\mathbf{S}(\alpha_1)\mathbf{B}\right]\mathbf{M}}_{\mathbf{W}}\mathbf{A} = \mathbf{G}, \qquad (12)$$

where the matrices are

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} \\ 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} -\frac{3}{4} & 0 & 0 \\ 0 & -\frac{3}{4} & 0 \\ 0 & 0 & -\frac{3}{4} \end{bmatrix}, \quad \mathbf{P}_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{P}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\mathbf{B}^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{S}(\alpha_0) = \mathbf{S}(\alpha_1) = \mathbf{S}(\alpha_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}, \ \mathbf{M} = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So the augmented matrix of Eq. (12) is

$$\left[\ \mathbf{W} \ ; \ \mathbf{G}\ \right] = \begin{bmatrix} -\frac{7}{2} & -1 & -\frac{5}{2} & ; & 1 \\ 0 & -\frac{5}{4} & -1 & ; & \frac{1}{4} \\ 0 & 0 & -1 & ; & -1 \end{bmatrix}.$$

Now, let's find the augmented matrix of the initial conditions:

$$y(0)=0 \quad \Rightarrow \quad y(0)=\mathbf{T}(0)\mathbf{MA}=0$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{A}=0$$

$$\Rightarrow \quad \begin{bmatrix} 2 & 0 & 2 & ; & 0 \end{bmatrix},$$

$$y'(0)=0 \quad \Rightarrow \quad y'(0)=\mathbf{T}(0)\mathbf{BMA}=0$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{A}=0$$

$$\Rightarrow \quad \begin{bmatrix} 0 & 1 & 0 & ; & 0 \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{U} & ; & \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 & ; & 0 \\ 0 & 1 & 0 & ; & 0 \end{bmatrix}$$

.

Interchanging the last two rows of $[\mathbf{W};\mathbf{G}]$ by these two rows yields

$$\begin{bmatrix} \widetilde{\mathbf{W}} & ; & \widetilde{\mathbf{G}} \end{bmatrix} = \begin{bmatrix} -\frac{7}{2} & -1 & -\frac{5}{2} & ; & 1 \\ 2 & 0 & 2 & ; & 0 \\ 0 & 1 & 0 & ; & 0 \end{bmatrix}$$

Solving this system for $\mathbf{A}$ gives $\mathbf{A} = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T$. From Eq. (4), $y(t)$ is obtained as

$$y(t)=\mathbf{T}(t)\mathbf{MA}=\begin{bmatrix} 1 & t & t^2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, the solution of the problem becomes

$$y(t)=t^2$$

which is the exact solution, illustrating the accuracy, efficiency and applicability of the present method.

4.2. **Example 2:**

Consider the following third-order neutral differential equation with proportional delays [8]

$$\begin{cases} y'''(t)=y(t)+y'\left(\frac{t}{2}\right)+y''\left(\frac{t}{3}\right)+\frac{1}{2}y'''\left(\frac{t}{4}\right)-t^4-\frac{t^3}{2}-\frac{4}{3}t^2+21t, & 0<t<1 \\ y(0)=y'(0)=y''(0)=0 \end{cases}$$

$$\text{where} \begin{cases} P_0(t)=0, \quad P_1(t)=-1, \quad P_2(t)=-1, \\ \\ \alpha_0=0, \quad \alpha_1=\frac{1}{2}, \quad \alpha_2=\frac{1}{3}, \quad \alpha_3=\frac{1}{4} \\ \\ g(t)=-t^4-\frac{t^3}{2}-\frac{4t^2}{3}+21t, \quad c=-\frac{1}{2}, \quad d=-1. \end{cases}$$

The exact solution of this problem is $y(t) = t^4$. Lucas matrix method has been applied, following the same steps described in Section 2. The resulting augmented matrix is as follows

$$[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}] = \begin{bmatrix} -2 & -1 & -4 & 0 & -10 & ; & 0 \\ -2 & -1.25 & -4.3125 & -1.3125 & -6.09505 & ; & 5.15495 \\ 2 & 0 & 2 & 0 & 2 & ; & 0 \\ 0 & 1 & 0 & 3 & 0 & ; & 0 \\ 0 & 0 & 1 & 0 & 4 & ; & 0 \end{bmatrix}.$$

Once we solve this system for $\mathbf{A}$, we get $\mathbf{A} = \begin{bmatrix} 3 & 0 & -4 & 0 & 1 \end{bmatrix}^T$. Thus, the solution can be obtained from Eq. (4) as

$$y(t) = \mathbf{L}(t)\mathbf{A} = \begin{bmatrix} 2 & t & t^2+2 & t^3+3t & t^4+4t^2+2 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -4 \\ 0 \\ 1 \end{bmatrix} = t^4$$

which is the exact solution. This problem has also been solved by two-stage order-one Runge-Kutta method [3] and variational iteration method [8] using $n = 4, 5$, and 6. The two-stage order-one Runge-Kutta method has five decimal place accuracy only at $t = 0.1$. The accuracy decreases to two decimal places at $t = 1$. The variational iteration method has four decimal places accuracy at $t = 1$ when $n = 4$, and this increases to six decimal places when $n = 6$. When the results are compared, one can assume that the present method is both effective and accurate.

4.3. **Example 3:** The last example is the first-order neutral differential equation with proportional delay [20]

$$\begin{cases} y'(t) = -y(t) + 0.1y\,(0.8t) + 0.5y'(0.8t) + (0.32t - 0.5)e^{-0.8t} + e^{-t}, \ 0 \le t \le T \\ y(0) = 0 \end{cases}$$

which has the exact solution $y(t) = te^{-t}$. This problem is solved by applying the present method and residual error estimation with $N = 4$; $M = 5$, 6 and $T = 1$, 7. Table 1 shows the comparison of the present method with the two-stage order-one Runge-Kutta method [3], One-leg θ-method [20] and variational iteration method with $n = 5$ and $n = 6$ in terms of absolute errors. Based on results presented in Table 1, one can see that the present method has better accuracy even with $N = 4$.

Table 1. Comparison of accuracy of the present method with other numerical methods in terms of absolute errors

t_i	Two-stage order-one Runge-Kutta meth. $\lvert e(t_i)\rvert$ [3]	One-leg θ-meth. $\lvert e(t_i)\rvert$ [20]	Variational iteration method $\lvert e_5(t_i)\rvert$ [8]	Variational iteration meth. $\lvert e_6(t_i)\rvert$ [8]	Present meth. $\lvert e_4(t_i)\rvert$	Present meth. $\lvert E_{4,5}(t_i)\rvert$	Present meth. $\lvert E_{4,6}(t_i)\rvert$
0.1	8.68e–04	4.65e–03	2.62e–03	1.30e–03	4.78e–05	4.01e–06	2.64e–07
0.2	1.49e–03	1.45e–02	4.36e–03	2.14e–03	1.04e–04	6.93e–06	3.67e–07
0.3	1.90e–03	2.57e–02	5.40e–03	2.63e–03	1.11e–04	5.88e–06	2.74e–07
0.4	2.16e–03	3.60e–02	5.89e–03	2.84e–03	7.86e–05	3.88e–06	2.12e–07
0.5	2.28e–03	4.43e–02	5.96e–03	2.83e–03	4.31e–05	3.30e–06	1.95e–07
0.6	2.31e–03	5.03e–02	5.71e–03	2.67e–03	3.78e–05	3.42e–06	1.37e–07
0.7	2.27e–03	5.37e–02	5.23e–03	2.39e–03	6.22e–05	1.89e–06	9.25e–08
0.8	2.17e–03	5.47e–02	4.59e–03	2.04e–03	5.67e–05	6.14e–07	1.90e–07
0.9	2.03e–03	5.35e–02	3.84e–03	1.64e–03	1.20e–04	7.38e–06	1.07e–07
1.0	1.86e–03	5.03e–02	3.04e–03	1.22e–03	7.12e–04	5.85e–05	3.74e–06

Figure 1 presents the numerical and the exact solution of the problem on the interval $[0, 7]$. It could be seen that the numerical results agree perfectly with the exact solution.

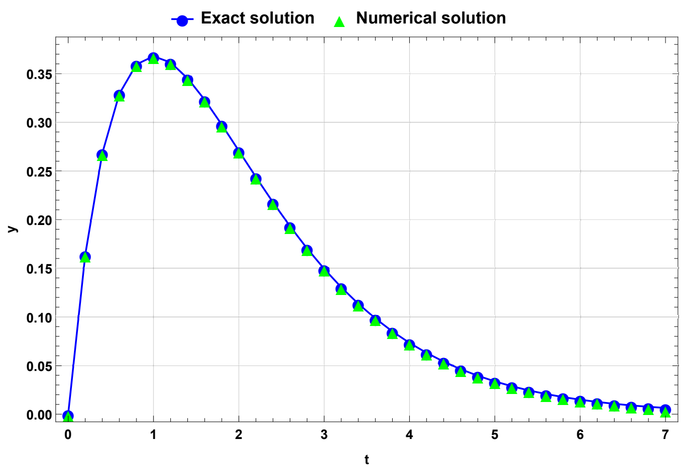

Figure 1. Comparison of numerical and exact solutions on $[0, 7]$.

5. Conclusion

In this paper, we have demonstrated the applicability of the Lucas polynomial approach in solving neutral differential equations with proportional delays. The said method has been applied to three test problems, and the method yields high-accuracy approximate, or even exact, solutions. One can see that the results of the Lucas matrix method have better accuracy than those generated by some other numerical methods. Hence, it is an effective and convenient approach to solve the indicated problems, producing approximate and exact solutions.

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