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APPROXIMATION OF PERIODIC FUNCTIONS BY SUB-MATRIX MEANS OF THEIR FOURIER SERIES

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Abstract. Some results on approximation of periodic functions are extended in two directions: Improving the degree of approximation of periodic functions by sub-matrix means of its Fourier series and such degree is applicable for a wider class of summbility matrices in the sense of their entries in which class of sequences belongs to.

Keywords: Fourier series, degree of approximation, periodic functions, non-decreasing sequences, Cesàro submethod.

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1. Introduction

Over seven decades considerable attention has been paid in the literature to the problem of studying the error estimates of the 2π-periodic functions belonging Lipschitz classes through the summability means of Fourier series, referred as Fourier approximation. As applications of such means we would like to mention here that engineers and scientists use properties of Fourier approximation for designing digital filters (see [1] and references therein). Newly, some similar results about Fourier approximation for a class of more extensive functions are obtained in [2]. To restate those results and our new results, we need to introduce some notations and definitions.

Let f be a 2π-periodic function, $f \in L^p := L^p[0, 2\pi]$, $p \geq 1$. The L^p-norm of $f \in L^p$ is defined by

$$\|f\|_p := \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p dx\right)^{\frac{1}{p}} \quad \text{and} \quad \|f\|_\infty := \sup_{x\in[0,2\pi]} |f(x)|.$$

The partial sum of the Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx),$$

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usually is written as

$$s_n(f;x) := \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx), \quad n \in \mathbb{N},$$

and

$$t_n(f;x) := \sum_{k=0}^{n} a_{n,k} s_k(f;x)$$

denotes the transformation of the sequence $\{s_k(f;x)\}$ by a lower triangular matrix $(a_{n,k})$ with non-negative entries such that $\sum_{k=0}^{n} a_{n,k} = 1$.

Besides that, the degree of approximation $E_n(f)$ of a function $f \in L^p$ by a trigonometric polynomial $T_n(x)$ of degree n is given by

$$E_n(f) = \min_{T_n} \|f - T_n\|_p.$$

Throughout this paper we denote

$$\phi_x(t) := f(x+t) + f(x-t) - 2f(x),$$

and $\tau := [1/t]$ the integral part of $1/t$.

The following result already has been proved.

Theorem 1.1 ([2])**.** *Let $(a_{n,k})$ be a lower triangular matrix with non-negative entries and non-decreasing entries with respect to k. Then, the degree of approximation of a 2π-periodic function $f(x)$ by matrix means of its Fourier series is given by*

$$|t_n(f;x) - f(x)| = \mathcal{O}\left[(n+1)^{\frac{1}{p}}\left(\omega\left(\frac{\pi}{n+1}\right) + (n+1)^{-\sigma}\right)\right], \quad p > 1,$$

provided a positive increasing function $\omega(t)$ satisfies the following conditions:

(i) *There exists a real number σ $(1/p < \sigma < 1)$ such that $\omega(t)/t^{\sigma}$ is an increasing function,*

(ii) $\frac{\phi_x(t)}{t^{-\frac{1}{p}}\omega(t)}$ *is a bounded function of t, also bounds uniformly in x.*

Another result, proved in the same paper, uses the notion of the hump matrix, see for example [1] or [3]: A lower triangular matrix T is called a hump matrix if, for each n, there exists an integer $k_0 = k_0(n)$ such that $a_{n,k} \le a_{n,k+1}$ for $0 \le k < k_0$ and $a_{n,k} \ge a_{n,k+1}$ for $k_0 \le k < n$.

Theorem 1.2 ([2])**.** *Let $(a_{n,k})$ be a hump matrix with non-negative entries and satisfies condition $(n+1)\max_k\{a_{n,k}\} = \mathcal{O}(1)$. Then the degree of approximation of a 2π-periodic function $f(x)$ by matrix means of its Fourier series is given by*

$$|t_n(f;x) - f(x)| = \mathcal{O}\left[(n+1)^{\frac{1}{p}}\left(\omega\left(\frac{\pi}{n+1}\right) + (n+1)^{-\sigma}\right)\right], \quad p > 1,$$

provided a positive increasing function $\omega(t)$ satisfies conditions (i) and (ii).

In the sequel we need to recall a class of sequences introduced in [4].

A sequence $\mathbf{c} := \{c_n\}$ of nonnegative numbers tending to zero is called of Rest Bounded Variation, or briefly $\mathbf{c} \in RBVS$, if it has the property

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \le K(\mathbf{c})c_m \tag{1}$$

for all natural numbers m, where $K(\mathbf{c})$ is a constant depending only on $\mathbf{c}$.

Let $\mathbb{F}$ be an infinite subset of $\mathbb{N}$ and $\mathbb{F}$ as range of strictly increasing sequence of positive integers, say $\mathbb{F}=\{\lambda(n)\}_{n=1}^{\infty}$. The Cesàro sub-method C_λ is defined as

$$(C_\lambda x)_n=\frac{1}{\lambda(n)}\sum_{k=1}^{\lambda(n)}x_k,\quad (n=1,2,\dots),$$

where $\{x_k\}$ is a sequence of real or complex numbers. Therefore, C_λ-method yields a subsequence of the Cesàro method C_1 and hence it is regular for any λ. C_λ-matrix is obtained by deleting a set of rows from Cesàro matrix. The basic properties of C_λ-method can be found in [5] and [6].

More general method than C_λ-method has been considered, see [9], and the following transformation has also been defined:

$$t_n^\lambda(f;x):=\sum_{k=0}^{\lambda(n)}a_{\lambda(n),k}s_k(f;x),$$

where $(a_{n,k})$ is an infinite lower triangular regular matrix with non-negative entries and row sums 1.

It is the purpose of this paper to prove the analogues of Theorems 1.1 and 1.2 using the polynomials $t_n^\lambda(f;x)$ and among others employing the condition $\{a_{n,k}\}\in RBVS$ with respect to k. Our technique used for the proof of our results has some in common with that in [2], but it has also some differences. As we will see our results give better degree of approximation since those are not expressed in terms of n, but in terms of $\lambda(n)$.

To do this we need next notations

$$A_{\lambda(n),k}:=\sum_{r=0}^{k}a_{\lambda(n),r},\quad K_n^\lambda(t):=\frac{1}{2\pi}\sum_{k=0}^{\lambda(n)}a_{\lambda(n),k}\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}},$$

and some helpful statements given in next section.

Throughout this paper, for two positive quantities u and v, we write $u=\mathcal{O}(v)$ instead of $u\le Kv$, where K is an absolute positive constant.

2. Auxiliary Lemmas

In this section we are going to prove some estimates for $|K_n^\lambda(t)|$, which we need afterwards for the proofs of the main results.

Lemma 2.1. *Let $(a_{n,k})$ be a lower triangular regular matrix with non-negative entries. Then for $0<t\le\frac{1}{\lambda(n)+1}$*

$$|K_n^\lambda(t)|=\mathcal{O}(\lambda(n)+1).$$

Proof. Applying the elemenatry inequality $\sin\alpha\le\alpha$, Jordan's inequality $\sin\beta\ge\frac{2}{\pi}\beta$ for $\beta\in[0,\frac{\pi}{2}]$, and our assumptions, we have

$$\begin{aligned}|K_n^\lambda(t)|&\le\frac{1}{2\pi}\sum_{k=0}^{\lambda(n)}\left|a_{\lambda(n),k}\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right|\\&\le\frac{1}{2}\sum_{k=0}^{\lambda(n)}a_{\lambda(n),k}\frac{\left(k+\frac{1}{2}\right)t}{t}\le\frac{1}{4}(2\lambda(n)+1)\sum_{k=0}^{\lambda(n)}a_{\lambda(n),k}=\mathcal{O}(\lambda(n)+1).\end{aligned}$$

□

Lemma 2.2. *Let $(a_{n,k})$ be a lower triangular regular matrix with non-negative entries and $\{a_{n,k}\} \in RBVS$ with respect to k. Then for $\frac{1}{\lambda(n)+1} < t \leq \pi$*

$$|K_n^\lambda(t)| = \mathcal{O}\left(\frac{A_{\lambda(n),\tau}}{t}\right).$$

Proof. Using Jordan's inequality $\sin\beta \geq \frac{2}{\pi}\beta$ for $\beta \in [0, \frac{\pi}{2}]$, and our assumptions, we have

$$\begin{aligned}
|K_n^\lambda(t)| &= \left|\frac{1}{2\pi}\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k}\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right| \leq \frac{1}{2t}\left|\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k}\operatorname{Im} e^{i\left(k+\frac{1}{2}\right)t}\right| \\
&\leq \frac{1}{2t}\left|e^{\frac{it}{2}}\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k}e^{ikt}\right| = \frac{1}{2t}\left|\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k}e^{ikt}\right|.
\end{aligned}$$

Similar to [1], see Lemma 2, we obtain

$$\begin{aligned}
\left|\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k}e^{ikt}\right| &\leq \left|\sum_{k=0}^{\tau-1} a_{\lambda(n),k}e^{ikt}\right| + \left|\sum_{k=\tau}^{\lambda(n)} a_{\lambda(n),k}e^{ikt}\right| \\
&\leq \sum_{k=0}^{\tau-1} a_{\lambda(n),k} + 2a_{\lambda(n),\tau} \times \max_{\tau\leq k\leq\lambda(n)}\left|\sum_{s=0}^{k} e^{ist}\right| \\
&= A_{\lambda(n),\tau-1} + 2a_{\lambda(n),\tau} \times \max_{\tau\leq k\leq\lambda(n)}\left|\frac{1-e^{i(s+1)t}}{1-e^{it}}\right| \\
&\leq A_{\lambda(n),\tau-1} + \frac{4a_{\lambda(n),\tau}}{\sqrt{(1-\cos t)^2+(\sin t)^2}} \\
&= A_{\lambda(n),\tau-1} + \frac{2a_{\lambda(n),\tau}}{\sin\frac{t}{2}} \\
&\leq A_{\lambda(n),\tau-1} + \frac{2\pi a_{\lambda(n),\tau}}{t} \\
&\leq A_{\lambda(n),\tau-1} + 2\pi(\tau+1)a_{\lambda(n),\tau}.
\end{aligned}$$

Since $\{a_{n,j}\} \in RBVS$ with respect to j, then for $0 \leq s \leq \tau$ we get

$$\begin{aligned}
a_{\lambda(n),\tau} &\leq \sum_{k=\tau}^{\infty}\left|a_{\lambda(n),k} - a_{\lambda(n),k+1}\right| \\
&\leq \sum_{k=s}^{\infty}\left|a_{\lambda(n),k} - a_{\lambda(n),k+1}\right| \leq K(\mathbf{c})a_{\lambda(n),s},
\end{aligned}$$

so $a_{\lambda(n),\tau} \leq K(\mathbf{c})a_{\lambda(n),s}$, and therefore

$$\begin{aligned}|K_n^\lambda(t)| &\leq \frac{1}{2t}\left|\sum_{k=0}^{\lambda(n)} a_{\lambda(n),k}e^{ikt}\right| \leq \frac{1}{2t}\left(A_{\lambda(n),\tau-1}+2\pi(\tau+1)a_{\lambda(n),\tau}\right)\\ &\leq \frac{1}{2t}\left(A_{\lambda(n),\tau-1}+2\pi a_{\lambda(n),\tau}\sum_{s=0}^{\tau}1\right)\\ &\leq \frac{1}{2t}\left(A_{\lambda(n),\tau-1}+2\pi K(\mathbf{c})\sum_{s=0}^{\tau}a_{\lambda(n),s}\right)=\mathcal{O}\left(\frac{A_{\lambda(n),\tau}}{t}\right).\end{aligned}$$

□

Lemma 2.3. *Let $(a_{n,k})$ be a hump matrix with $(\lambda(n)+1)\max_k\{a_{\lambda(n),k}\}=\mathcal{O}(1)$. Then for $\frac{1}{\lambda(n)+1}<t\leq\pi$ holds*

$$|K_n^\lambda(t)|=\mathcal{O}\left(\frac{1}{t^2(\lambda(n)+1)}\right).$$

Proof. We start form the following estimate realized in Lemma 2.2:

$$|K_n^\lambda(t)|\leq\frac{1}{2t}\left|\sum_{k=0}^{\lambda(n)}a_{\lambda(n),k}e^{ikt}\right|.$$

Indeed, using the assumptions $a_{\lambda(n),k_0}=\max\{a_{\lambda(n),0}a_{\lambda(n),1},\dots,a_{\lambda(n),n}\}$, $\max_k a_{\lambda(n),k}=\mathcal{O}\left(\frac{1}{\lambda(n)+1}\right)$, and the well-known Jordan's inequality $\sin\gamma\geq\frac{2}{\pi}\gamma$ for $\beta\in[0,\frac{\pi}{2}]$, we obtain

$$\begin{aligned}|K_n^\lambda(t)| &\leq \frac{1}{2t}\left|\sum_{k=0}^{\lambda(n)}a_{\lambda(n),k}e^{ikt}\right|\\ &\leq \frac{1}{2t}a_{\lambda(n),k_0}\left|\sum_{k=0}^{\lambda(n)}e^{ikt}\right| \leq \frac{1}{2t}a_{\lambda(n),k_0}\left|\frac{1-e^{i(\lambda(n)+1)t}}{1-e^{it}}\right|\\ &\leq \frac{1}{2t}a_{\lambda(n),k_0}\frac{1}{|\sin\frac{t}{2}|}\leq\frac{a_{\lambda(n),k_0}}{2t^2}=\frac{\max_k\{a_{\lambda(n),k}\}}{2t^2}=\mathcal{O}\left(\frac{1}{t^2(\lambda(n)+1)}\right).\end{aligned}$$

□

Remark 2.1. *If we take $\lambda(n)=n$ in Lemmas 2.1 and 2.3, then we obtain Lemmas 1 and 3 proved in* [1].

3. Main Results

At the beginning, we prove the following main result.

Theorem 3.1. *Let $(a_{n,k})$ be a lower triangular matrix with non-negative entries and $\{a_{n,k}\}\in RBVS$ with respect to k. Then the degree of approximation of a 2π-periodic function $f(x)$ by polynomials $t_n^\lambda(f;x)$ is given by*

$$|t_n^\lambda(f;x)-f(x)|=\mathcal{O}\left[(\lambda(n)+1)^{\frac{1}{p}}\left(\omega\left(\frac{\pi}{\lambda(n)+1}\right)+(\lambda(n)+1)^{-\sigma}\right)\right],\quad p>1,$$

provided a positive increasing function $\omega(t)$ satisfies the following conditions:

(i) *There exists a real number σ $(1/p<\sigma<1)$ such that $\omega(t)/t^\sigma$ is an increasing function, and*

(ii) $\frac{\phi_x(t)}{t^{-\frac{1}{p}}\omega(t)}$ *is a bounded function of* t*, also bounds uniformly in* x*.*

Proof. Using the well-known equality

$$s_n(f;x) - f(x) = \int_0^\pi \phi_x(t) \frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}} dt,$$

we write

$$t_n^\lambda(f;x) - f(x) = \frac{1}{2\pi}\int_0^\pi \phi_x(t) \sum_{k=0}^{\lambda(n)} a_{\lambda(n),k} \frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}} dt$$

$$\begin{aligned}
&= \int_0^\pi \phi_x(t) K_n^\lambda(t) dt \\
&= \int_0^{\frac{\pi}{\lambda(n)+1}} \phi_x(t) K_n^\lambda(t) dt + \int_{\frac{\pi}{\lambda(n)+1}}^\pi \phi_x(t) K_n^\lambda(t) dt \qquad (2)\\
&:= I_1(t) + I_2(t), \quad \text{say.}
\end{aligned}$$

In the sequel, we estimate separately $|I_1(t)|$ and $|I_2(t)|$.
Using Lemma 2.1 and assumptions of our theorem we have

$$\begin{aligned}
|I_1(t)| &= \mathcal{O}(\lambda(n)+1)\int_0^{\frac{\pi}{\lambda(n)+1}} |\phi_x(t)| dt \\
&= \mathcal{O}(\lambda(n)+1)\int_0^{\frac{\pi}{\lambda(n)+1}} \left|\frac{\phi_x(t)}{t^{-\frac{1}{p}}\omega(t)} \cdot \frac{\omega(t)}{t^\sigma}\right| t^{\sigma-\frac{1}{p}} dt \\
&= \mathcal{O}(\lambda(n)+1)\int_0^{\frac{\pi}{\lambda(n)+1}} \frac{\omega(t)}{t^\sigma} \cdot t^{\sigma-\frac{1}{p}} dt \qquad (3)\\
&= \mathcal{O}(\lambda(n)+1)(\lambda(n)+1)^\sigma \omega\left(\frac{\pi}{\lambda(n)+1}\right)\int_0^{\frac{\pi}{\lambda(n)+1}} t^{\sigma-\frac{1}{p}} dt \\
&= \mathcal{O}(\lambda(n)+1)(\lambda(n)+1)^\sigma \omega\left(\frac{\pi}{\lambda(n)+1}\right)(\lambda(n)+1)^{\frac{1}{p}-\sigma-1} \\
&= \mathcal{O}\left[(\lambda(n)+1)^{\frac{1}{p}}\omega\left(\frac{\pi}{\lambda(n)+1}\right)\right].
\end{aligned}$$

Now we estimate $|I_2(t)|$. Indeed, since the function $f_n(t) = \frac{\pi}{(\lambda(n)+1)t}$ is continuous over interval $\left[\frac{\pi}{\lambda(n)+1}, \pi\right]$, then there exists a positive constant, say M, so that

$$A_{\lambda(n),\tau} \leq A_{\lambda(n),\lambda(n)} = 1 = \mathcal{O}\left(\frac{\pi}{(\lambda(n)+1)t}\right),$$

and whence, using Lemma 2.2 and assumptions of the theorem, we find that

$$\begin{aligned}
|I_2(t)| &= \mathcal{O}(1)\int_{\frac{\pi}{\lambda(n)+1}}^{\pi}\left|\frac{\phi_x(t)A_{\lambda(n),\tau}}{t}\right|dt\\
&= \mathcal{O}(1)\int_{\frac{\pi}{\lambda(n)+1}}^{\pi}|\phi_x(t)|\cdot\frac{\pi}{(\lambda(n)+1)t^2}dt\\
&= \mathcal{O}(1)\frac{\pi}{\lambda(n)+1}\int_{\frac{\pi}{\lambda(n)+1}}^{\pi}\left|\frac{\phi_x(t)}{t^{-\frac{1}{p}}\omega(t)}\right|\cdot\frac{t^{-\frac{1}{p}}\omega(t)}{t^2}dt\\
&= \mathcal{O}(1)\frac{\pi}{\lambda(n)+1}\int_{\frac{\pi}{\lambda(n)+1}}^{\pi}\frac{\omega(t)}{t^{\sigma}}\cdot t^{\sigma-\frac{1}{p}-2}dt\\
&= \mathcal{O}(1)\frac{\pi}{\lambda(n)+1}\cdot\frac{\omega(\pi)}{\pi^{\sigma}}\int_{\frac{\pi}{\lambda(n)+1}}^{\pi}t^{\sigma-\frac{1}{p}-2}dt\\
&= \mathcal{O}(1)\frac{\pi}{\lambda(n)+1}(\lambda(n)+1)^{1+\frac{1}{p}-\sigma}\\
&= \mathcal{O}\left[(\lambda(n)+1)^{\frac{1}{p}-\sigma}\right].
\end{aligned}\tag{4}$$

Finally, inserting (3) and (4) into (2), we obtain

$$|t_n^{\lambda}(f;x)-f(x)| = \mathcal{O}\left[(\lambda(n)+1)^{\frac{1}{p}}\left(\omega\left(\frac{\pi}{\lambda(n)+1}\right)+(\lambda(n)+1)^{-\sigma}\right)\right].$$

The proof is completed. □

Remark 3.1. *If NIS denotes the class of non-negative and non-increasing sequences, then the following inclusion $NIS \subset RBVS$ holds true, see* [9]. *Subsequently, the results of Theorem 3.1 covers as well the case of NIS class.*

Taking $\lambda(n) = n$ in Theorem 3.1, we have:

Corollary 3.1. *Let $(a_{n,k})$ be a lower triangular matrix with non-negative entries and $\{a_{n,k}\} \in RBVS$ with respect to k. Then the degree of approximation of a 2π-periodic function $f(x)$ by polynomials $t_n(f;x)$ is given by*

$$|t_n(f;x)-f(x)| = \mathcal{O}\left[(n+1)^{\frac{1}{p}}\left(\omega\left(\frac{\pi}{n+1}\right)+(n+1)^{-\sigma}\right)\right],\quad p>1,$$

provided a positive increasing function $\omega(t)$ satisfies conditions (i) and (ii).

For $a_{\lambda(n),k} = \frac{p_k}{P_{\lambda(n)}}$, $k = 0,1,\dots,\lambda(n)$, $P_{\lambda(n)} = p_0+p_1+\dots+p_{\lambda(n)} \neq 0$, $n \geq 0$, we obtain the means $R_n^{\lambda}(f;x) = \frac{1}{P_{\lambda(n)}}\sum_{m=0}^{\lambda(n)} p_m s_m(f;x)$ introduced in [8].

Hence, we have:

Corollary 3.2. *Let $\{p_k\} \in RBVS$. Then the degree of approximation of a 2π-periodic function $f(x)$ by polynomials $R_n^{\lambda}(f;x)$ is given by*

$$|R_n^{\lambda}(f;x)-f(x)| = \mathcal{O}\left[(\lambda(n)+1)^{\frac{1}{p}}\left(\omega\left(\frac{\pi}{\lambda(n)+1}\right)+(\lambda(n)+1)^{-\sigma}\right)\right],\quad p>1,$$

provided a positive increasing function $\omega(t)$ satisfies conditions (i) and (ii).

Remark 3.2. *For $\lambda(n) = n$ in Corollary 3.2, we obtain ordinary Riesz means $R_n = \frac{1}{P_n}\sum_{m=0}^{n} p_m s_m(f;x)$, as well as its deviation from the 2π-periodic function $f(x)$, expressed in terms of n.*

Remark 3.3. *For $p_k = 1$ for all $k \geq 0$ in Corollary 3.2, we obtain sub-Cesáro means $\sigma_n^\lambda(f;x) = \frac{1}{\lambda(n)+1}\sum_{m=0}^{\lambda(n)} s_m(f;x)$, as well as its deviation from the 2π-periodic function $f(x)$, expressed in terms of $\lambda(n)$.*

Theorem 3.2. *Let $(a_{n,k})$ be a hump matrix with non-negative entries and satisfies condition $(\lambda(n)+1)\max_k\{a_{\lambda(n),k}\} = \mathcal{O}(1)$. Then the degree of approximation of a 2π-periodic function $f(x)$ by polynomial $t_n^\lambda(f;x)$ is given by*

$$|t_n^\lambda(f;x) - f(x)| = \mathcal{O}\left[(\lambda(n)+1)^{\frac{1}{p}}\left(\omega\left(\frac{\pi}{\lambda(n)+1}\right) + (\lambda(n)+1)^{-\sigma}\right)\right], \quad p > 1,$$

provided a positive increasing function $\omega(t)$ satisfies conditions (i) and (ii).

Proof. For the proof we use some parts of the proof of Theorem 3.1. Indeed, in obvious inequality

$$|t_n^\lambda(f;x) - f(x)| \leq |I_1'(t)| + |I_2'(t)|, \quad \text{say.} \tag{5}$$

we proved that

$$|I_1'(t)| = \mathcal{O}\left[(\lambda(n)+1)^{\frac{1}{p}}\omega\left(\frac{\pi}{\lambda(n)+1}\right)\right]. \tag{6}$$

Now, using Lemma 2.3 and assumptions of the theorem, we again find that

$$\begin{aligned}
|I_2'(t)| &= \mathcal{O}(1)\int_{\frac{\pi}{\lambda(n)+1}}^{\pi}\left|\frac{\phi_x(t)}{t^2(\lambda(n)+1)}\right|dt \\
&= \mathcal{O}(1)\frac{\pi}{\lambda(n)+1}\int_{\frac{\pi}{\lambda(n)+1}}^{\pi}\left|\frac{\phi_x(t)}{t^{-\frac{1}{p}}\omega(t)}\right|\cdot\frac{t^{-\frac{1}{p}}\omega(t)}{t^2}dt \\
&= \mathcal{O}(1)\frac{\pi}{\lambda(n)+1}\int_{\frac{\pi}{\lambda(n)+1}}^{\pi}\frac{\omega(t)}{t^\sigma}\cdot t^{\sigma-\frac{1}{p}-2}dt \\
&= \mathcal{O}(1)\frac{\pi}{\lambda(n)+1}\cdot\frac{\omega(\pi)}{\pi^\sigma}\int_{\frac{\pi}{\lambda(n)+1}}^{\pi}t^{\sigma-\frac{1}{p}-2}dt \\
&= \mathcal{O}(1)\frac{\pi}{\lambda(n)+1}(\lambda(n)+1)^{1+\frac{1}{p}-\sigma} \\
&= \mathcal{O}\left[(\lambda(n)+1)^{\frac{1}{p}-\sigma}\right].
\end{aligned} \tag{7}$$

Finally, inserting (6) and (7) into (5) we obtain

$$|t_n^\lambda(f;x) - f(x)| = \mathcal{O}\left[(\lambda(n)+1)^{\frac{1}{p}}\left(\omega\left(\frac{\pi}{\lambda(n)+1}\right) + (\lambda(n)+1)^{-\sigma}\right)\right].$$

□

Remark 3.4. *If we take $\lambda(n) = n$ in Theorem 3.2 we exactly obtain Theorem 1.2 of* [2].

Remark 3.5. *Since $\lambda(n) \geq n$, by assumption on the sequence $\{\lambda(n)\}$, and the function $\omega(t)$ is non-decreasing, then we have*

$$\omega\left(\frac{\pi}{\lambda(n)+1}\right) \leq \omega\left(\frac{\pi}{n+1}\right) \quad \text{and} \quad (\lambda(n)+1)^{-\sigma} \leq (n+1)^{-\sigma},$$

So, our results give sharper estimates than those proved earlier in [1] *and* [2].

Remark 3.6. *Similar corollaries as Corollaries 3.1–3.2 as well as Remarks 3.1–3.3, can be derived form Theorem 3.2.*

References

[1] Srivastava, Shailesh Kumar; Singh, Uaday, (2014), Trigonometric approximation of periodic functions belonging to $Lip(\omega(t), p)$-class. J. Comput. Appl. Math. 270, 223–230.

[2] Zhang, Renjiang, (2015), A note on the trigonometric approximation of $Lip(\omega(t), p)$-class. Appl. Math. Comput. 269, 129–132.

[3] Mishra, Vishnu Narayan; Mishra, Lakshmi Narayan, (2012), Trigonometric approximation of signals (functions) in $\mathrm{L_p}$-norm. Int. J. Contemp. Math. Sci. 7, no. 17-20, 909–918.

[4] Leindler, L, (2004), On the degree of approximation of continuous functions. Acta Math. Hungar. 104, no. 1-2, 105–113.

[5] Armitage, David H.; Maddox, Ivor J, (1989), A new type of Cesàro mean. Analysis 9, no. 1-2, 195–206.

[6] Osikiewicz, Jeffrey A, (2000), Equivalence results for Cesàro submethods. Analysis (Munich) 20, no. 1, 35–43.

[7] Mittal, M. L.; Singh, Mradul Veer, (2016), Applications of Cesàro submethod to trigonometric approximation of signals (functions) belonging to class $\mathrm{Lip}(\alpha, p)$ in L_p-norm. J. Math., Art. ID 9048671, 7 pp.

[8] Mittal, M. L.; Singh, Mradul Veer, (2014), Approximation of signals (functions) by trigonometric polynomials in L_p-norm. Int. J. Math. Math. Sci., Art. ID 267383, 6 pp.

[9] Mohapatra, N. Ram and Szal, Bogdan, On trigonometric approximation of functions in the L^p-norm, https://arxiv.org/pdf/1205.5869.pdf.

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