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LIE-POISSON INTEGRATORS FOR A RIGID SATELLITE ON A CIRCULAR ORBIT

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Abstract. In the last two decades, many structure preserving numerical methods like Poisson integrators have been investigated in numerical studies. Since the structure matrices are different in many Poisson systems, no integrator is known yet to preserve the Poisson structure of any Poisson system. In the present paper, we propose Lie–Poisson integrators for Lie–Poisson systems whose structure matrix is different from the ones studied before. In particular, explicit Lie-Poisson integrators for the equations of rotational motion of a rigid body (the satellite) on a circular orbit around a fixed gravitational center have been constructed based on the splitting. The splitted parts have been composed by a first, a second and a third order compositions. It has been shown that the proposed schemes preserve the quadratic invariants of the equation. Numerical results reveal the preservation of the energy and agree with the theoretical treatment that the invariants lie on the sphere in long–term with different orders of accuracy.

Keywords:Lie-Poisson integrators, symplectic integrators, rigid body equations, splittings.

AMS Subject Classification: 70G45, 65P10, 70E17, 70H07

1. Introduction

Geometric integrators, which are well designed numerical schemes preserving some invariant quantities of a differential equation, have been developed by many authors for particular types of differential equation namely, Hamiltonian ordinary differential equations (ODEs) and Hamiltonian partial differential equations (PDEs). For canonical Hamiltonian system ODEs, different kinds of symplectic integrators have been developed (see [1, Chapter VI], [2, 3, 4]). For Hamiltonian ODEs in a non-canonical form with a linear structure matrix, i.e. Lie-Poisson ODEs, Lie–Poisson integrators were constructed [5, 6]. In [7, 8] explicit Lie-Poisson integrators have been constructed for some class of Lie-Poisson systems. For Poisson systems, i.e. Hamiltonian systems in a non-canonical form with a non-linear structure matrix, Poisson integrators were developed. In [9], it has been shown that the midpoint rule preserves the Poisson structure up to second order accuracy. In [10], preservation of the Poisson structure by the second order Lobatto IIIA-B method has been proven for the Volterra lattice equations. In [11] it has been shown that symplectic Runge-Kutta methods preserve the Poisson structure when the Poisson tensor is constant. It has also been shown that the Poisson structure could be preserved for non-constant Poisson tensor using a nonlinear change of coordinates. In [12] Poisson

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integrators for completely integrable hamiltonian systems are discussed. In [13] an explicit Poisson integrator for symmetric rigid bodies in space is derived. For an overview of Poisson integrators, see [14] and references therein. For Hamiltonian PDEs, multisymplectic integrators have been constructed (see [15, 16, 17] and references therein). A survey for multisymplectic integrators is given in [18, Chapter 12].

Dynamical systems in celestial mechanics and dynamical astronomy attract much attention. Most of these dynamical systems are Hamiltonian ODEs. One of the important questions concerning these systems is the problem of integrability. A system is integrable if it has an enough number of first integrals. Unfortunately, most of the dynamical systems in celestial mechanics and dynamical astronomy are non-integrable [19, 20, 21]. Therefore, numerical studies are essential to understand the long time behavior of these dynamical systems.

We will consider the following system of ODEs [22]

$$\begin{aligned} \frac{d}{dt}\mathbf{m} &= \mathbf{m}\times\omega + 3\Omega\gamma\times\mathbf{I}\gamma \\ \frac{d}{dt}\gamma &= \gamma\times(\omega-\Omega\mathbf{n}) \\ \frac{d}{dt}\mathbf{n} &= \mathbf{n}\times\omega \end{aligned} \tag{1}$$

which describes the rotational motion of a rigid body (the satellite) on a circular orbit around a fixed gravitational center. Here $\Omega^2 = gq/r^3$, $\quad \mathbf{m} = \mathbf{I}\omega$, $\quad \mathbf{I} = \mathrm{diag}(I_1, I_2, I_3)$ with g is the gravitational constant, q is the mass of the central body, r is the radius of the orbit, I_1, I_2, I_3 are the principal moments of inertia of the body, $\mathbf{m} = (m_1, m_2, m_3)^T$ is the vector of the angular momentum, $\omega = (\omega_1, \omega_2, \omega_3)^T$ is the vector of the total angular velocity of the satellite, $\gamma = (\gamma_1, \gamma_2, \gamma_3)^T$ and $\mathbf{n} = (n_1, n_2, n_3)^T$ are unit vectors along the radius vector and the normal to the orbital plane, respectively. Components of all vectors are taken here with respect to the principal axis reference frame.

Equation (1) possesses four integrals, namely the energy

$$H(\mathbf{z}) = \frac{1}{2} < \mathbf{m}, \mathbf{I}^{-1}\mathbf{m} > + \frac{3}{2}\Omega < \gamma, \mathbf{I}\gamma > -\Omega < \mathbf{m}, \mathbf{n} >, \tag{2}$$

and three geometrical integrals

$$G(\mathbf{z}) = < \gamma, \gamma >, \qquad N(\mathbf{z}) = < \mathbf{n}, \mathbf{n} >, \qquad K(\mathbf{z}) = < \gamma, \mathbf{n} >, \tag{3}$$

where $\mathbf{z} = (\mathbf{m}^T, \gamma^T, \mathbf{n}^T)$ and $< \cdot, \cdot >$ stands for the standard inner product in $\mathbf{R}^3$. For the real system we consider [22]

$$G(\mathbf{z}) = N(\mathbf{z}) = 1, \text{ and } K(\mathbf{z}) = 0. \tag{4}$$

In this paper, explicit Lie-Poisson integrators are constructed for the numerical solution of the system (1). In section 2, the Lie-Poisson equations and the Lie-Poisson integrators are reviewed. The Lie-Poisson structure of the system (1) is also presented. In section 3, symplectic integrators like the implicit midpoint and the multi–stage implicit symplectic scheme are discussed. In section 4, it is shown that the numerical methods presented in section 3 are conservative Lie-Poisson integrators for the system (1). Section 5 is devoted to some numerical experiments.

2. Lie-Poisson Equations and Lie-Poisson Integrators

The system considered in this paper is the Hamiltonian system of odes in $\mathbf{R}^n$

$$\frac{d\mathbf{x}}{dt} = \Lambda(\mathbf{x})\nabla H(\mathbf{x}), \tag{5}$$

with the Hamiltonian function $H : \mathbf{R}^n \to \mathbf{R}$. Here $\Lambda(\mathbf{x})$ is a skew-symmetric matrix, i.e., $\Lambda^T(\mathbf{x}) = -\Lambda(\mathbf{x})$. $\nabla H(\mathbf{x})$ is defined with respect to the standard inner product in $\mathbf{R}^n$.

If $\Lambda(\mathbf{x})$ is linear in $\mathbf{x}$, then the system (5) is called a Hamiltonian system on a Lie-Poisson structure or a Lie-Poisson equation. For the smooth functions $F, G : \mathbf{R}^n \to \mathbf{R}$, we can define the Poisson bracket

$$\{F, G\}(\mathbf{x}) = \nabla F(\mathbf{x})\Lambda(\mathbf{x})\nabla G^T(\mathbf{x}). \tag{6}$$

In view of (6), the Lie-Poisson equation (5) can be written as

$$\frac{d\mathbf{x}}{dt} = \{\mathbf{x}, H\}.$$

Let $\Psi_h(\mathbf{x}(t)) = \mathbf{x}(t+h)$ denote the phase flow of the equation (5). The Poisson bracket (6) is preserved by the phase flow $\Psi_h(\mathbf{x}(t))$, i.e.,

$$\{F, G\} \circ \Psi_h = \{F \circ \Psi_h, G \circ \Psi_h\}$$

or equivalently

$$\left(\frac{\partial \Psi_h}{\partial \mathbf{x}}\right) \Lambda(\mathbf{x}) \left(\frac{\partial \Psi}{\partial \mathbf{x}}\right)^T = \Lambda\left(\Psi(\mathbf{x})\right). \tag{7}$$

When Λ is a symplectic matrix,

$$\Lambda = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \tag{8}$$

where I is an $n \times n$ identity matrix, we obtain the symplecticness condition

$$\left(\frac{\partial \Psi_h}{\partial \mathbf{x}}\right) \Lambda \left(\frac{\partial \Psi}{\partial \mathbf{x}}\right)^T = \Lambda.$$

But, in general, this property is not satisfied when Λ is not the symplectic matrix [11]. Therefore, these properties have to be reflected in the numerical integration. In a view of Lie-Poisson setting, a difference scheme for the system (5) is said to be a Lie-Poisson integrator if, and only if it preserves the Lie-Poisson structure (7), i.e.

$$\left(\frac{\partial \mathbf{x}^1}{\partial \mathbf{x}^0}\right) \Lambda(\mathbf{x}^0) \left(\frac{\partial \mathbf{x}^1}{\partial \mathbf{x}^0}\right)^T = \Lambda(\mathbf{x}^1). \tag{9}$$

where the notations $\mathbf{x}^0 := \mathbf{x}^n$ and $\mathbf{x}^1 := \mathbf{x}^{n+1}$ have been used for simplicity.

The equation (1), rotational motion of a rigid body (satellite) on a circular orbit around a fixed gravitational center, is a Lie-Poisson system [23]. It can be written in the form of (5) with the Hamiltonian function (2), and the 9×9 structure matrix

$$\Lambda(\mathbf{x}) = \begin{pmatrix} J(\mathbf{m}) & J(\gamma) & J(\mathbf{n}) \\ J(\gamma) & 0 & 0 \\ J(\mathbf{n}) & 0 & 0 \end{pmatrix} \tag{10}$$

where $\mathbf{x} = (\mathbf{m}^T, \gamma^T, \mathbf{n}^T)$ and $J(u)$ is the 3×3 matrix

$$\mathrm{J}(\mathbf{u}) = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}.$$

Since the structure matrices are different in many Poisson systems, no integrator is known yet to preserve the Poisson structure of any Poisson system. In this paper, Lie–Poisson integrators are proposed for the structure matrix (10) for which a Lie–Poisson integrator has never been studied in the literature before.

To prove that a numerical scheme is a Lie-Poisson integrator for the structure matrix (10) of the system (1), we have to show that (9) is satisfied, i.e.,

$$\begin{pmatrix} \frac{\partial \mathbf{m}^1}{\partial \mathbf{m}^0} & \frac{\partial \mathbf{m}^1}{\partial \gamma^0} & \frac{\partial \mathbf{m}^1}{\partial \mathbf{n}^0} \\ \frac{\partial \gamma^1}{\partial \mathbf{m}^0} & \frac{\partial \gamma^1}{\partial \gamma^0} & \frac{\partial \gamma^1}{\partial \mathbf{n}^0} \\ \frac{\partial \mathbf{n}^1}{\partial \mathbf{m}^0} & \frac{\partial \mathbf{n}^1}{\partial \gamma^0} & \frac{\partial \mathbf{n}^1}{\partial \mathbf{n}^0} \end{pmatrix} \begin{pmatrix} J(\mathbf{m}^0) & J(\gamma^0) & J(\mathbf{n}^0) \\ J(\gamma^0) & 0 & 0 \\ J(\mathbf{n}^0) & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \mathbf{m}^1}{\partial \mathbf{m}^0} & \frac{\partial \mathbf{m}^1}{\partial \gamma^0} & \frac{\partial \mathbf{m}^1}{\partial \mathbf{n}^0} \\ \frac{\partial \gamma^1}{\partial \mathbf{m}^0} & \frac{\partial \gamma^1}{\partial \gamma^0} & \frac{\partial \gamma^1}{\partial \mathbf{n}^0} \\ \frac{\partial \mathbf{n}^1}{\partial \mathbf{m}^0} & \frac{\partial \mathbf{n}^1}{\partial \gamma^0} & \frac{\partial \mathbf{n}^1}{\partial \mathbf{n}^0} \end{pmatrix}^T$$

$$= \begin{pmatrix} J(\mathbf{m}^1) & J(\gamma^1) & J(\mathbf{n}^1) \\ J(\gamma^1) & 0 & 0 \\ J(\mathbf{n}^1) & 0 & 0 \end{pmatrix}. \tag{11}$$

A numerical method for the system (1) preserves the geometric integrals (3) if the conditions

$$G(\mathbf{z}^1) = G(\mathbf{z}^0), \quad N(\mathbf{z}^1) = N(\mathbf{z}^0), \quad K(\mathbf{z}^1) = K(\mathbf{z}^0). \tag{12}$$

are hold.

3. Numerical Methods

In this section, some numerical schemes are presented to integrate the system (1) numerically, using a Lie-Poisson splitting.

3.1. **The Implicit Midpoint Rule.** We discretize the equation (5) using the implicit midpoint rule

$$x^1 = x^0 + h\Lambda\left(\frac{x^1 + x^0}{2}\right)\nabla H\left(\frac{x^1 + x^0}{2}\right) \tag{13}$$

If $\Lambda(x) \equiv \Lambda$ is a constant matrix, then the midpoint rule (13) preserves the Poisson structure (7) [9]. For the canonical case, i.e. $\Lambda(x) \equiv \Lambda$ in (8), the scheme (13) is also a symplectic integrator [24]. When $\Lambda(x)$ is not a constant matrix, in general, the midpoint rule (13) does not preserve the Poisson structure (6). In a Lie-Poisson setting, i.e. when $\Lambda(x)$ is linear in x, (13) is not a Lie-Poisson integrator but it preserves the Lie-Poisson structure up to second order [9]. However, for a separable Hamiltonian system, explicit Lie-Poisson integrators can be constructed as for the canonical Hamiltonian systems [7, 25].

3.2. **Multi-Stage Implicit Symplectic Schemes.** For a Hamiltonian system of the form

$$\frac{dx}{dt} = J^{-1}\nabla H(z)$$

some implicit symplectic schemes can be obtained in [26]. Since the Hamiltonian H is quadratic in this work, we can write the vector field $J^{-1}\nabla H(z) = Wz$, and implement the schemes as follows:

$$\left\{ \begin{array}{lcl} U_1 & = & x^0 + \frac{1}{2}hWU_1 \\ x^1 & = & 2U_1 - x^0 \end{array} \right. \tag{14}$$

$$\left\{ \begin{array}{lcl} U_1 & = & x^0 + \frac{1}{4}hWU_1 \\ U_2 & = & 2U_1 - x^0 + \frac{1}{4}hWU_2 \\ x^1 & = & 2(U_2 - U_1) + x^0 \end{array} \right. \tag{15}$$

The schemes (14) and (15) are first order and second order, respectively. Third and fourth order schemes can be obtained in [27].

4. Lie-Poisson Integrators for Satellite Equation

In this section, it is shown that the equation of the motion (1) can be numerically integrated by means of explicit Lie-Poisson integrators. Any system of differential equations can be integrated by a so-called 'splitting method' if only one can split the vector field in such a way that each of the splitted term generates an explicitly solvable dynamics. An overview of the splitting methods can be found in [28].

We consider the Satellite equation (1) in Lie-Poisson form

$$\frac{d\mathbf{x}}{dt} = \Lambda(\mathbf{x})\nabla H(\mathbf{x}), \quad \mathbf{x} = (\mathbf{m}^T, \gamma^T, \mathbf{n}^T) \tag{16}$$

with the structure matrix (10) and the Hamiltonian function (2). We split the Hamiltonian (2) in to nine pieces

$$H = H_1 + H_2 + \cdots + H_9 \tag{17}$$

where $H_i = \frac{1}{2}\frac{m_i^2}{I_i}$ for $i = 1, 2, 3$, $H_i = -\Omega n_i m_i$ for $i = 4, 5, 6$ and $H_i = \frac{3\Omega}{2} I_i \gamma_i^2$ for $i = 7, 8, 9$. Substituting $H = H_i$, $i = 1, 2\cdots, 9$ into (16), we get nine different subsystems each of which is completely integrable; in other words an exact solution can be found.

For example, substituting $H = H_1$ in (16), the solution evolves according to

$$\begin{array}{lllllllll} \frac{dm_1}{dt} &=& 0, & \frac{d\gamma_1}{dt} &=& 0, & \frac{dn_1}{dt} &=& 0, \\ \frac{dm_2}{dt} &=& am_1m_3, & \frac{d\gamma_2}{dt} &=& am_1\gamma_3, & \frac{dn_2}{dt} &=& am_1n_3, \\ \frac{dm_3}{dt} &=& -am_1m_2, & \frac{d\gamma_3}{dt} &=& -am_1\gamma_2, & \frac{dn_3}{dt} &=& -am_1n_2, \end{array} \tag{18}$$

where $a = 1/I_1$. We notice that m_1, γ_1 and n_1 are constant. Therefore, an exact solution can be obtained.

Theorem 4.1. *The midpoint rule for the system (18) is Poisson.*

Proof. We discretize (18) using the implicit midpoint rule and obtain

$$\begin{array}{lllllllll} m_1^1 &=& m_1^0, & \gamma_1^1 &=& \gamma_1^0, & n_1^1 &=& n_1^0, \\ m_2^1 &=& m_2^0 + ham_1^0 m_3^{\frac{1}{2}}, & \gamma_2^1 &=& \gamma_2^0 + ham_1^0\gamma_3^{\frac{1}{2}}, & n_2^1 &=& n_2^0 + ham_1^0 n_3^{\frac{1}{2}}, \\ m_3^1 &=& m_3^0 - ham_1^0 m_2^{\frac{1}{2}}, & \gamma_3^1 &=& \gamma_3^0 - ham_1^0\gamma_2^{\frac{1}{2}}, & n_3^1 &=& n_3^0 - ham_1^0 n_2^{\frac{1}{2}}. \end{array} \tag{19}$$

where $y^{\frac{1}{2}} := \frac{1}{2}(y^1 + y^0)$. Solving for m_i^1, γ_i^1, n_i^1, $i = 1, 2, 3$ we get

$$\begin{aligned}
m_1^1 &= m_1^0, \\
m_2^1 &= \frac{4 - a^2h^2m_1^{02}}{4 + a^2h^2m_1^{02}} m_2^0 + \frac{4ahm_1^0}{4 + a^2h^2m_1^{02}} m_3^0, \\
m_3^1 &= \frac{4 - a^2h^2m_1^{02}}{4 + a^2h^2m_1^{02}} m_3^0 - \frac{4ahm_1^0}{4 + a^2h^2m_1^{02}} m_2^0, \\
\gamma_1^1 &= \gamma_1^0, \\
\gamma_2^1 &= \frac{4 - a^2h^2m_1^{02}}{4 + a^2h^2m_1^{02}} \gamma_2^0 + \frac{4ahm_1^0}{4 + a^2h^2m_1^{02}} \gamma_3^0, \\
\gamma_3^1 &= \frac{4 - a^2h^2m_1^{02}}{4 + a^2h^2m_1^{02}} \gamma_3^0 - \frac{4ahm_1^0}{4 + a^2h^2m_1^{02}} \gamma_2^0 \\
n_1^1 &= n_1^0, \\
n_2^1 &= \frac{4 - a^2h^2m_1^{02}}{4 + a^2h^2m_1^{02}} n_2^0 + \frac{4ahm_1^0}{4 + a^2h^2m_1^{02}} n_3^0, \\
n_3^1 &= \frac{4 - a^2h^2m_1^{02}}{4 + a^2h^2m_1^{02}} n_3^0 - \frac{4ahm_1^0}{4 + a^2h^2m_1^{02}} n_2^0.
\end{aligned} \tag{20}$$

We notice the Jacobi matrices $\frac{\partial \mathbf{m}^1}{\partial \gamma^0} = \frac{\partial \mathbf{m}^1}{\partial \mathbf{n}^0} = \frac{\partial \gamma^1}{\partial \mathbf{n}^0} = \frac{\partial \mathbf{n}^1}{\partial \gamma^0} = 0$. The matrices $\frac{\partial \mathbf{m}^1}{\partial \mathbf{m}^0}, \frac{\partial \gamma^1}{\partial \mathbf{m}^0}, \frac{\partial \mathbf{n}^1}{\partial \mathbf{m}^0}, \frac{\partial \gamma^1}{\partial \gamma^0}$ and $\frac{\partial \mathbf{n}^1}{\partial \mathbf{n}^0}$ can be easily obtained from (20). For example,

$$\frac{\partial \mathbf{n}^1}{\partial \mathbf{n}^0} = \begin{pmatrix} \frac{\partial n_1^1}{\partial n_1^0} & \frac{\partial n_1^1}{\partial n_2^0} & \frac{\partial n_1^1}{\partial n_3^0} \\ \frac{\partial n_2^1}{\partial n_1^0} & \frac{\partial n_2^1}{\partial n_2^0} & \frac{\partial n_2^1}{\partial n_3^0} \\ \frac{\partial n_3^1}{\partial n_1^0} & \frac{\partial n_3^1}{\partial n_2^0} & \frac{\partial n_3^1}{\partial n_3^0} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{4-h^2a^2m_1^{02}}{4+h^2a^2m_1^{02}} & \frac{4ham_1^0}{4+h^2a^2m_1^{02}} \\ 0 & -\frac{4ham_1^0}{4+h^2a^2m_1^{02}} & \frac{4-h^2a^2m_1^{02}}{4+h^2a^2m_1^{02}} \end{pmatrix}.$$

It is seen that (11) is satisfied by substituting these Jacobian matrices into (11). This proves that the implicit midpoint rule is a Lie-Poisson integrator for the subsystem (18).

Note that the system (20) is an explicit scheme which is an advantage in contrast to the implicit scheme which requires several Newton steps in order to find an approximate solution. □

Similarly, one can show that the implicit midpoint rule produces an explicit Lie-Poisson integrator for the subsystems obtained by replacing $H = H_i$, $i = 2, 3, \cdots, 9$ in (16).

Let us now consider the symplectic integration of the Lie–Poisson system (16). Lie–Poisson integrators can also be obtained from the symplectic integration of Hamiltonian systems [29, 30].

Theorem 4.2. *The multi-stage implicit symplectic scheme (14) for the system (18) is Poisson.*

Proof. Writing the system (18) as

$$\frac{d\mathbf{z}}{dx} = W\mathbf{z}$$

with $\mathbf{z} = (\mathbf{m}^T, \gamma^T, \mathbf{n}^T)$ and

$$W = \begin{pmatrix} B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & am_1 \\ 0 & -am_1 & 0 \end{pmatrix}$$

where $a = 1/I_1$, we apply the multi-stage symplectic scheme (14) and obtain

$$\begin{array}{rcl} m_1^1 &=& m_1^0 \\ m_2^1 &=& (\frac{8}{4+h^2a^2m_1^{0\,2}} - 1)m_2^0 + 4ha\frac{m_1^0}{4+h^2a^2m_1^{0\,2}}m_3^0 \\ m_3^1 &=& (\frac{8}{4+h^2a^2m_1^{0\,2}} - 1)m_3^0 - 4ha\frac{m_1^0}{4+h^2a^2m_1^{0\,2}}m_2^0 \\ \gamma_1^1 &=& \gamma_1^0 \\ \gamma_2^1 &=& (\frac{8}{4+h^2a^2m_1^{0\,2}} - 1)\gamma_2^0 + 4ha\frac{m_1^0}{4+h^2a^2m_1^{0\,2}}\gamma_3^0 \\ \gamma_3^1 &=& (\frac{8}{4+h^2a^2m_1^{0\,2}} - 1)\gamma_3^0 - 4ha\frac{m_1^0}{4+h^2a^2m_1^{0\,2}}\gamma_2^0 \\ n_1^1 &=& n_1^0 \\ n_2^1 &=& (\frac{8}{4+h^2a^2m_1^{0\,2}} - 1)n_2^0 + 4ha\frac{m_1^0}{4+h^2a^2m_1^{0\,2}}n_3^0 \\ n_3^1 &=& (\frac{8}{4+h^2a^2m_1^{0\,2}} - 1)n_3^0 - 4ha\frac{m_1^0}{4+h^2a^2m_1^{0\,2}}n_2^0 \end{array} \tag{21}$$

We note the Jacobian matrices $\frac{\partial \mathbf{m}^1}{\partial \gamma^0} = \frac{\partial \mathbf{m}^1}{\partial \mathbf{n}^0} = \frac{\partial \gamma^1}{\partial \mathbf{n}^0} = \frac{\partial \mathbf{n}^1}{\partial \gamma^0} = 0$. The matrices $\frac{\partial \mathbf{m}^1}{\partial \mathbf{m}^0}, \frac{\partial \gamma^1}{\partial \mathbf{m}^0}, \frac{\partial \mathbf{n}^1}{\partial \mathbf{m}^0}, \frac{\partial \gamma^1}{\partial \gamma^0}$ and $\frac{\partial \mathbf{n}^1}{\partial \mathbf{n}^0}$ can be obtained easily. For example,

$$\frac{\partial \gamma^1}{\partial \gamma^0} = \begin{pmatrix} \frac{\gamma_1^1}{\gamma_1^0} & \frac{\gamma_1^1}{\gamma_2^0} & \frac{\gamma_1^1}{\gamma_3^0} \\ \frac{\gamma_2^1}{\gamma_1^0} & \frac{\gamma_2^1}{\gamma_2^0} & \frac{\gamma_2^1}{\gamma_3^0} \\ \frac{\gamma_3^1}{\gamma_1^0} & \frac{\gamma_3^1}{\gamma_2^0} & \frac{\gamma_3^1}{\gamma_3^0} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{4-h^2a^2m_1^{0\,2}}{4+h^2a^2m_1^{0\,2}} & \frac{4ham_1^0}{4+h^2a^2m_1^{0\,2}} \\ 0 & -\frac{4ham_1^0}{4+h^2a^2m_1^{0\,2}} & \frac{4-h^2a^2m_1^{0\,2}}{4+h^2a^2m_1^{0\,2}} \end{pmatrix}.$$

Substituting these matrices into (11), we see that (11) is satisfied. This proves that the multi-stage implicit symplectic scheme (14) is an explicit Lie-Poisson integrator for the system (18). □

Similarly, one can show that the multi-stage implicit symplectic scheme (14) is an explicit Lie-Poisson integrator for the subsystems obtained by replacing $H = H_i$, $i = 2, 3, \cdots, 9$ in (16).

Theorem 4.3. *The multi-stage implicit symplectic scheme (15) for the system (18) is Poisson.*

Proof. Similar to the proof of Theorem 2. □

Besides being Lie–Poisson integrators, the schemes considered in this section preserve the quadratic invariants (3) for the subsystem (18)[31]. For this, we have to show that (12) is satisfied under midpoint discretization. From (20), we can write

$$\gamma_2^1 = A\gamma_2^0 + B\gamma_3^0, \text{ and } \gamma_3^1 = A\gamma_3^0 - B\gamma_2^0,$$

where $A = \frac{4-a^2h^2m_1^{0\,2}}{4+a^2h^2m_1^{0\,2}}$ and $B = \frac{4ahm_1^0}{4+a^2h^2m_1^{0\,2}}$. Then

$$\begin{array}{rcl} (\gamma_1^1)^2 + (\gamma_2^1)^2 + (\gamma_3^1)^2 &=& (\gamma_1^0)^2 + (A\gamma_2^0 + B\gamma_3^0)^2 + (A\gamma_3^0 - B\gamma_2^0)^2 \\ &=& (\gamma_1^0)^2 + (A^2 + B^2)(\gamma_2^0)^2 + (A^2 + B^2)(\gamma_3^0)^2 \\ &=& (\gamma_1^0)^2 + (\gamma_2^0)^2 + (\gamma_3^0)^2 \end{array} \tag{22}$$

where we have used the fact that $(A^2 + B^2) = 1$. This shows that the condition $G(\mathbf{z}^1) = G(\mathbf{z}^0)$ in (12) is satisfied under midpoint discretization for the subsystem (18). Similarly, one can show that midpoint discretization for the subsystem (18) satisfies the other conditions $N(\mathbf{z}^1) = N(\mathbf{z}^0)$ and $K(\mathbf{z}^1) = K(\mathbf{z}^0)$ in (12) where $N(\mathbf{z})$ and $K(\mathbf{z})$ in (3). It can be shown that the implicit midpoint discretization satisfies the conditions in (3) for the subsystem obtained by replacing $H = H_i$, $i = 2, 3, \cdots, 9$ in (16).

Similarly, one can show that the conditions in (3) are satisfied by each subsystem obtained by replacing $H = H_i,\ i = 1, 2, \cdots, 9$ in (16) and discretizing by the schemes (14) and (15).

Consider the subsystems obtained by replacing $H = H_i,\ i = 1, 2, \cdots, 9$ in (16) where $H_i,\ i = 1, 2, \cdots, 9$ are defined in (17). Let $\mathcal{H}_1[h], \mathcal{H}_2[h], \cdots, \mathcal{H}_9[h]$ denote the discrete flow obtained by applying the implicit midpoint rule for these subsystems. For example, we have proven in Theorem 1 that the discrete flow $\mathcal{H}_1[h]$ is a Lie-Poisson integrator for the subsystem (18). In a similar way, it can be shown that the discrete flows $\mathcal{H}_2[h], \mathcal{H}_3[h], \cdots, \mathcal{H}_9[h]$ are also Lie-Poisson integrators. Thus, an explicit Lie-Poisson integrators can be constructed by composing the discrete flows $\mathcal{H}_1[h], \mathcal{H}_2[h], \cdots, \mathcal{H}_9[h]$ on the bases of the implicit midpoint rule.

A simple composition is given by the Lie-Trotter formula [1]

$$x^{n+1} = \mathcal{H}_1[h] \circ \mathcal{H}_2[h] \circ \cdots \circ \mathcal{H}_9[h] x^n. \tag{23}$$

which is first-order. A more accurate composition is the Strang splitting [1]

$$x^{n+1} = \mathcal{H}_1[\frac{h}{2}] \circ \cdots \circ \mathcal{H}_8[\frac{h}{2}] \circ \mathcal{H}_9[h] \circ \mathcal{H}_8[\frac{h}{2}] \circ \cdots \circ \mathcal{H}_1[\frac{h}{2}] x^n. \tag{24}$$

which is second-order and symplectic. Higher-order compositions can be obtained using higher–order decompositions, such as the fourth–order Suzuki–Trotter decomposition [32]

$$x^{n+1} = \left(\prod_{i=1}^{5} \mathcal{H}_1[p_i \frac{h}{2}] \circ \cdots \circ \mathcal{H}_8[p_i \frac{h}{2}] \circ \mathcal{H}_9[p_i h] \circ \mathcal{H}_8[p_i \frac{h}{2}] \circ \cdots \circ \mathcal{H}_1[p_i \frac{h}{2}] \right) x^n \tag{25}$$

where $p_1 = p_2 = p_4 = p_5 = p = \frac{1}{4-4^{\frac{1}{3}}},\ p_3 = 1 - 4p$.

Here, we point out that the integrators (23), (24) and (25) are first, second and fourth–order Lie-Poisson integrators for system (1) respectively based on the implicit midpoint discretization. Let $\mathcal{H}_1, \cdots, \mathcal{H}_9$ denote the discrete flow obtained by the one-stage (14) and two-stage (15) schemes. Then one can construct a first, a second or a fourth–order Lie-Poisson integrators based on the one-stage (14) and two-stage (15) schemes for the system (1) using the Theorem 2 and Theorem 3 respectively.

5. Numerical Results

In this section, numerical results for the solution of the rigid body equation (1) are presented. Three Lie–Poisson integrators, based on the Hamiltonian splitting namely the implicit midpoint rule, the one-stage and the two-stage implicit symplectic schemes, have been used for numerical integration. The splitted parts are then composed to obtain the first–order (23),the second–order (24) and the fourth–order (25) integrators for each case.

All computations are done on the time interval $[0, 1000]$ with the uniform grids $t_m = t_0 + m\Delta t$,$m = 0, 1, \cdots, 1000/\Delta t$ for different time step length Δt. Figures 1–3 show the numerical results for the implicit midpoint rule. The results for the one-stage and two-stage implicit symplectic schemes are similar to the results of the implicit midpoint rule, therefore they are not shown here.

The accuracy of the methods is tested by looking at the four invariants, namely the energy (2) and the three quadratic first integrals (3). The errors are computed using the L_∞ norm defined by

$$\begin{aligned} Err(H) &= \|H(z^0) - H(z^m)\|_\infty, \\ Err(G) &= \|G(z^0) - G(z^m)\|_\infty, \\ Err(N) &= \|N(z^0) - N(z^m)\|_\infty, \\ Err(K) &= \|K(z^0) - K(z^m)\|_\infty, \end{aligned} \tag{26}$$

		1^{st} order		2^{nd} order		4^{th} order	
	Δt	$Err(H)$	p	$Err(H)$	p	$Err(H)$	p
MP	$\frac{1}{10}$	$7.418E-1$	–	$9.199E-2$	–	$2.024E-3$	–
	$\frac{1}{20}$	$3.414E-1$	1.119	$2.159E-2$	2.091	$1.138E-4$	4.153
	$\frac{1}{40}$	$1.582E-1$	1.109	$5.370E-3$	2.008	$6.980E-6$	4.027
	$\frac{1}{80}$	$7.640E-2$	1.050	$1.337E-3$	2.006	$4.337E-7$	4.009
	$\frac{1}{160}$	$3.756E-2$	1.024	$3.340E-4$	2.000	$2.710E-8$	4.000
1-STG	$\frac{1}{10}$	$7.418E-1$	–	$9.199E-2$	–	$2.024E-3$	–
	$\frac{1}{20}$	$3.414E-1$	1.119	$2.159E-2$	2.091	$1.138E-4$	4.152
	$\frac{1}{40}$	$1.582E-1$	1.109	$5.370E-3$	2.008	$6.980E-6$	4.027
	$\frac{1}{80}$	$7.640E-2$	1.050	$1.337E-3$	2.006	$4.337E-7$	4.009
	$\frac{1}{160}$	$3.756E-2$	1.024	$3.340E-4$	2.000	$2.710E-8$	4.000
2-STG	$\frac{1}{10}$	$7.517E-1$	–	$8.712E-2$	–	$2.111E-3$	–
	$\frac{1}{20}$	$3.350E-1$	1.166	$2.027E-2$	2.104	$1.186E-4$	4.153
	$\frac{1}{40}$	$1.589E-1$	1.076	$4.962E-3$	2.030	$7.271E-6$	4.028
	$\frac{1}{80}$	$7.653E-2$	1.054	$1.234E-3$	2.008	$4.516E-7$	4.009
	$\frac{1}{160}$	$3.761E-2$	1.025	$3.084E-4$	2.000	$2.825E-8$	4.000

TABLE 1. Comparison of the convergence rate from L_∞-errors for the first-order, second-order and fourth-order compositions. MP:Midpoint rule, 1-STG: one–stage scheme, 2-STG: two–stage scheme.

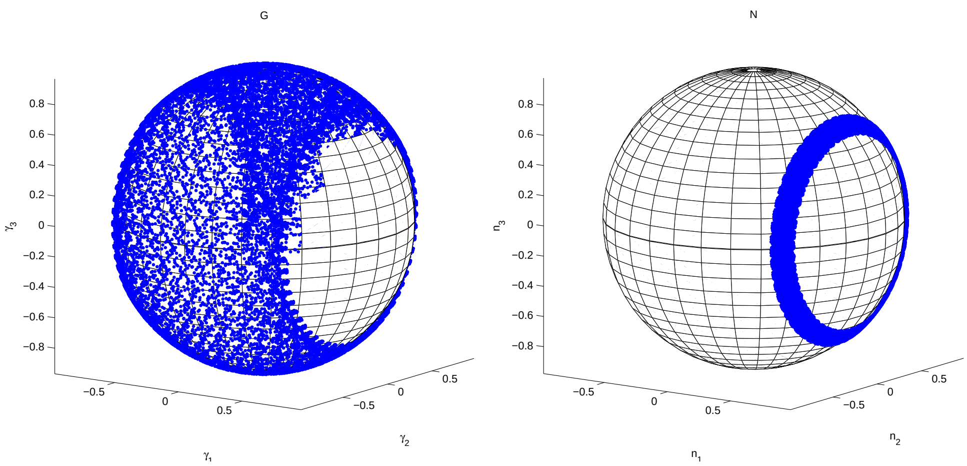

FIGURE 1. Preservation of the geometric invariants G and N in (3) on the unit sphere. The implicit midpoint rule with the first–order composition.

where $H(z^0)$, $G(z^0)$, $N(z^0)$ and $K(z^0)$ are the values of the invariants at $t=0$ and $H(z^m)$, $G(z^m)$, $N(z^m)$ and $K(z^m)$ are the values of the invariants at $t = z^0 + m\Delta t$. We take $\Omega = 1$, the inertia parameters $I_1 = 1.1$, $I_2 = 2.1$, $I_3 = 2.5$ and the initial values [22]

$$\begin{array}{llllllllll} m_1^0 &=& -10, & m_2^0 &=& 0.1, & m_3^0 &=& 0.2, \\ \gamma_1^0 &=& 0.1, & \gamma_2^0 &=& -0.3, & \gamma_3^0 &\simeq& 0.94898, \\ n_1^0 &\simeq& 0.6993786, & n_2^0 &=& n_1^0, & n_3^0 &\simeq& 0.14744. \end{array} \tag{27}$$

Figure 1 provides the results obtained by the first–order composition (23) based on the implicit midpoint rule with the time step length $\Delta t = 1/20$. Similar results are obtained by the second–order (24) and the fourth–order (25) compositions. From the figure, we

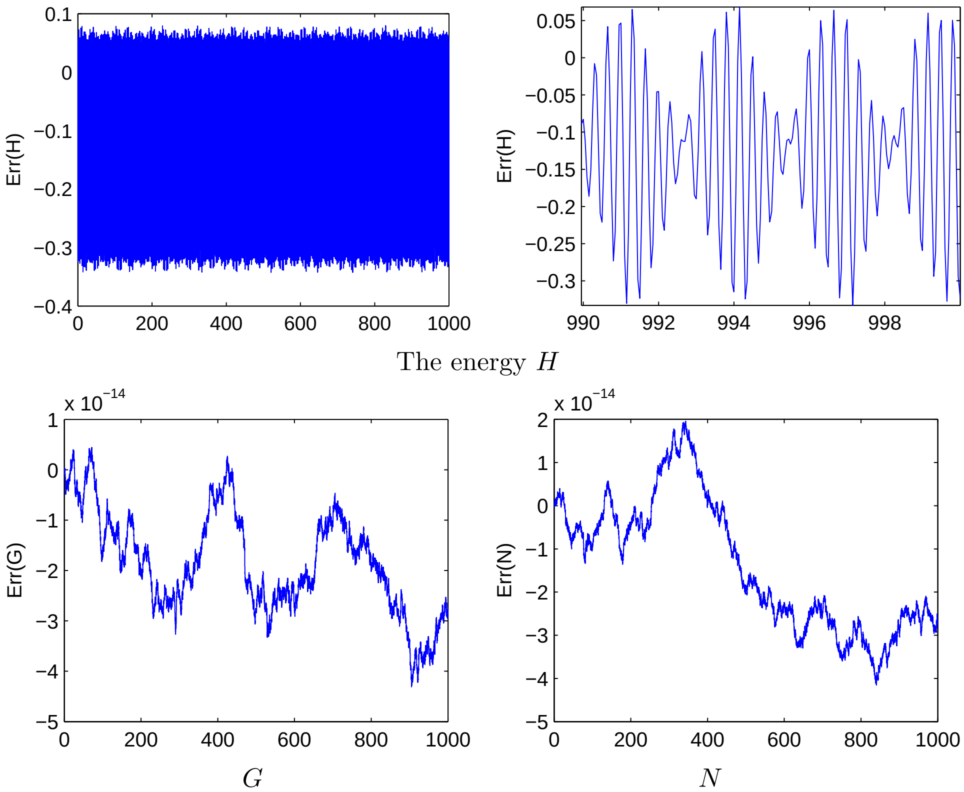

FIGURE 2. Numerical errors for the preservation of the energy H and the geometric invariants G, N and K in (3). The implicit midpoint rule with the first–order composition.

see that the invariants $G(\mathbf{z})$ and $N(\mathbf{z})$ in (3) remain on the unit sphere. The errors in (26) can be found in Figure 2. Notice that the error in the energy (2) is oscillating and does not grow in time which is a typical behavior for a symplectic integrator. However, it is not preserved exactly though it is a quadratic invariant. It can be eliminated by using a higher–order composition instead of a simple first–order composition. The error in the energy for the second–order and the fourth–order compositions for various time step lengths Δt can be found in Table 1. Note that increasing the order of the composition decreases the error in conserved quantities.

The rate of the convergence of the integrators, discussed in this paper, can be calculated from the formula [33]

$$p = \log\left(\frac{\|H(z^0) - H(z^m)_{\Delta t_1}\|}{\|H(z^0) - H(z^m)_{\Delta t_2}\|}\right) / \log\left(\frac{\Delta t_1}{\Delta t_2}\right) \tag{28}$$

where the value p is called the rate of convergence. The errors in the Hamiltonian H and the rate of convergence p of the methods are shown in Table 1 for decreasing time step lengths Δt. We present the L_∞ error for the terminating time $t = 32$. Notice that halving the time step Δt results in a decrease in the maximum error in H by a factor of 2^{-1} in the first–order, 2^{-2} in the second–order and 2^{-4} in the fourth–order methods. It can be seen from the Table 1 that the Hamiltonian H is preserved more accurately by increasing the order of the composition method. Furthermore, the rate of convergence p is almost equal

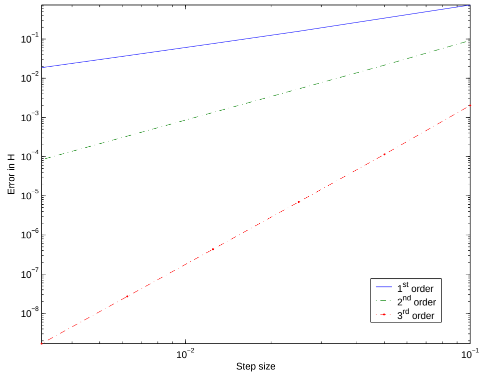

FIGURE 3. Numerical errors for the preservation of the energy H in (3). The implicit midpoint rule with the first, second and fourth–order composition for different step sizes.

to 1, 2 and 4 for the first, second and fourth–order compositions, respectively. Figure 3 justifies the expected rate of convergence.

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