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ON VARIOUS CHARACTERIZATIONS OF SUPERCOMPLETE MAPPINGS

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Abstract. The supercomlete uniformly continuous mappings has been introduced and its some properties has been studied. The category characterization of the supercomplete uniformly continuous mappings has been found.

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1. Introduction

In 1983 at the seminar on topology (Karlov University, Prague) Z.Frolik had set the task: To give a category characterization of the main classes of a uniform spaces. In 2007 academician A.A.Borubayev on one of the seminars under its management (Kyrgyz National University named after J.Balasagyn, Bishkek) has solved the above setted problem where has given the category characterization of compact, complete uniform spaces and topological groups complete in Raikov sense ([2]).

In a context of the put problems as the space can be considered as a mapping special case, identifying this space with its mapping in a point, there is an idea of distribution on mappings of concepts and the statements which are available for spaces that allows to generalize many results. So, the category characterization of supercomplete spaces by means of supercomplete uniformly continuous mapping introduced by us is obtained and the supercomplete uniformly continuous mappings are studied.

2. Main results

Let $(X, \mathscr{U})$ be a uniform space, $(\exp X, \exp \mathscr{U})$ be a space of the closed subsets with Hausdorf uniformity ([1]).

Definition 2.1. *Uniform space $(X, \mathscr{U})$ is called supercomplete, if its hyperspace $(\exp X, \exp \mathscr{U})$ is complete ([3]).*

Definition 2.2. *Filter $\mathscr{F}$ in $(X, \mathscr{U})$ is called stable, if for any covering $\alpha \in \mathscr{U}$ there exists such $F' \in \mathscr{F}$ that $F' \subset \alpha(F)$ for any $F \in \mathscr{F}$ ([3]).*

Proposition 2.1. *Every Cauchy filter is stable.*

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Proof. Let $\mathscr{F}$ be an arbitrary Cauchy filter in $(X, \mathscr{U})$. For an arbitrary covering $\alpha \in \mathscr{U}$ there exist $\mathrm{A} \in \alpha$ and fixed $\mathrm{F}_\alpha \in \mathscr{F}$ such, that $\mathrm{F}_\alpha \subset \mathrm{A}$ and $\mathrm{F}_\alpha \in \alpha$. For any $\mathrm{F} \in \mathscr{F}$ take place $\mathrm{F} \cap \mathrm{F}_\alpha \neq \varnothing$, it means $\mathrm{F} \cap \alpha(\mathrm{F}_\alpha) \neq \varnothing$. As above mentioned element F_α is an element of the both filter $\mathscr{F}$, and the covering α, then $\mathrm{F}_\alpha \subset \alpha(\mathrm{A})$. Hence, $\mathrm{A} \subset \alpha(\mathrm{F})$ and $\mathrm{F}_\alpha \subset \alpha(\mathrm{F})$ for any $\mathrm{F} \in \mathscr{F}$, it means the stability of the Cauchy filter $\mathscr{F}$. The proof is completed. □

Let $f : (X, \mathscr{U}) \to (Y, \mathscr{V})$ be a uniformly continuous mapping of the uniform space $(X, \mathscr{U})$ into the uniform space $(Y, \mathscr{V})$.

Proposition 2.2. *Uniformly continuous image $f\mathscr{F}$ of any stable filter $\mathscr{F}$ in $(X, \mathscr{U})$ is a basis for some stable filter in $(Y, \mathscr{V})$.*

Proof. Let $\mathscr{F}$ be a stable Cauchy filter in $(X, \mathscr{U})$ and $f\mathscr{F} = \{f(\mathrm{F}) : \mathrm{F} \in \mathscr{F}\}$. Let $\mathbf{\Phi}$ be a filter, induced by the uniformly continuous image $f\mathscr{F}$ in $(Y, \mathscr{V})$, i.e. $\mathbf{\Phi} = \{\mathbf{\Phi} \subset \mathbf{Y} :$ there exists $\mathrm{F} \in \mathscr{F}$ such that $\mathbf{f}(\mathrm{F}) \subset \mathbf{\Phi}\}$ and for any covering $\beta \in \mathscr{V}$ there exists a set $\Phi' \in \mathbf{\Phi}$ such, that $\Phi' \subset \beta(\Phi)$ for any element $\Phi' \in \mathbf{\Phi}$. As the mapping f is uniformly continuous as for any uniformly covering $\alpha \in \mathscr{U}$ there is such uniformly covering $\beta \in \mathscr{V}$ that $f\alpha \succ \beta$. As filter $\mathscr{F}$ is stable there is such element F' $\in \mathscr{F}$, that F' $\subset \alpha(\mathrm{F})$ for any set $\mathrm{F} \in \mathscr{F}$, hence $f(\mathrm{F'}) \subset f(\alpha(\mathrm{F'})) = f(\alpha)(f(\mathrm{F})) \subset \beta(f(\mathrm{F})) \subset \beta(\Phi)$ is provided. Suppose, then $\Phi' \in \mathbf{\Phi}$, it means, by the Definition 2, $\mathbf{\Phi}$ is a stable filter in Y. Thus, the uniformly continuous image $f\mathscr{F}$ of the Cauchy filter $\mathscr{F}$ is a basis for some stable Cauchy filter $\mathbf{\Phi}$ in the uniform space $(Y, \mathscr{V})$. Proposition is proved. □

Definition 2.3. *Uniformly continuous mapping $f : (X, \mathscr{U}) \to (Y, \mathscr{V})$ is called supercomplete, if for any stable filter $\mathscr{F}$ in $(X, \mathscr{U})$, for which it $\bigcap \left\{\overline{f(F)} : F \in \mathscr{F}\right\} \neq \varnothing$ follows $\bigcap \left\{\overline{F} : F \in \mathscr{F}\right\} \neq \varnothing$.*

Proposition 2.3. *Every supercomplete mapping is complete.*

Proof. Let $\mathscr{F}$ be an arbitrary Cauchy filter in $(X, \mathscr{U})$ such, that $\bigcap \left\{\overline{f(\mathrm{F})} : \mathrm{F} \in \mathscr{F}\right\} \neq \varnothing$. By Proposition 1, the Cauchy filter $\mathscr{F}$ is a stable. By Proposition 2, its $f\mathscr{F}$ is a basis of the stable filter. From condition $\bigcap \left\{\overline{f(\mathrm{F})} : \mathrm{F} \in \mathscr{F}\right\} = \{y\}$ for any $y \in Y$, by supercomplete mapping f we have $\bigcap \left\{\overline{\mathrm{F}} : \mathrm{F} \in \mathscr{F}\right\} \neq \varnothing$, and as $\mathscr{F}$ is a Cauchy filter, then any its coherent point is its limit, i.e. the filter $\mathscr{F}$ is converging. So, uniformly continuous mapping f is complete. Proposition is proved. □

Proposition 2.4. *Every close subspace of a supercomplete space is supercomplete.*

Proof. Let $(X, \mathscr{U})$ be a supercomplete space and $A \subset X$, then $(A, \mathscr{U}_A)$ - its close subspace. Every stable Cauchy filter $\mathscr{F}'$ in A is a basis for some stable Cauchy filter $\mathscr{F}$ in X and $\bigcap \left\{\overline{\mathrm{F}} : \mathrm{F} \in \mathscr{F}\right\} \bigcap A \neq \varnothing$. Hence, $\mathscr{F}' \subset \mathscr{F}$, it means the stable Cauchy filter $\mathscr{F}'$ is hyperconverging to some point $x \in X$, i.e. subspace $(A, \mathscr{U}_A)$ is supercomplete. □

Proposition 2.5. *Let $f : (X, \mathscr{U}) \to (Y, \mathscr{V})$ be a uniformly continuous supercomplete mapping of a uniform space $(X, \mathscr{U})$ onto a uniform space $(Y, \mathscr{V})$. Then $\left(f^{-1}(y), \mathscr{U}|_{f^{-1}(y)}\right)$ is a supercomplete for any point $y \in Y$.*

Proof. Let $y \in Y$ be an arbitrary point and $\mathscr{U}|_{f^{-1}(y)}$ be a uniformity, induced $f^{-1}(y)$ of the uniformity $\mathscr{U}$. Let $\mathscr{F}$ be an arbitrary stable Cauchy filter in $\left(f^{-1}(y), \mathscr{U}|_{f^{-1}(y)}\right)$, i.e. for any covering $\alpha \in \mathscr{U}$ and any set $\mathrm{F} \in \mathscr{F}$ there exists $\mathrm{F}_\alpha \in \mathscr{F}$ such, that $\mathrm{F}_\alpha \subset \alpha(\mathrm{F})$. By Proposition 2, stable Cauchy filter $\mathscr{F}$ is a basis for some stable Cauchy filter in $(X, \mathscr{U})$.

Let $\mathscr{F}' = \{A \subset X : \text{there exists} \, \mathrm{F}_A \in \mathscr{F} : \mathrm{F}_A \subset A\}$ be a stable Cauchy filter in $(X, \mathscr{U})$, induced by stable Cauchy filter $\mathscr{F}$. Then $\mathrm{F}_\alpha \subset \alpha(A)$ for any $A \in \mathscr{F}'$. By the construction, $\mathscr{F} \subset \mathscr{F}'$, we have $\bigcap \{f(A) : A \in \mathscr{F}'\} \subset \bigcap \{f(\mathrm{F}) : \mathrm{F} \in \mathscr{F}\} = \{y\}$ for any point $y \in Y$. From Proposition 2 it follows $f(\mathscr{F}')$ is stable Cauchy filter and it is hyperconverging in $(Y, \mathscr{V})$. As the mapping f is supercomplete, then the filter $\mathscr{F}'$ is hyperconverging in $(X, \mathscr{U})$, i.e. the next formula take place $\varnothing \neq \bigcap \left\{\bar{A} : A \in \mathscr{F}'\right\} \subset \bigcap \left\{\bar{\mathrm{F}} : \mathrm{F} \in \mathscr{F}\right\}$. Hence, $\bigcap \left\{\bar{\mathrm{F}} : \mathrm{F} \in \mathscr{F}\right\} \neq \varnothing$, the stable Cauchy filter $\mathscr{F}$ is hyperconverging in $\left(f^{-1}(y), \mathscr{U}|_{f^{-1}(y)}\right)$, it means $\left(f^{-1}(y), \mathscr{U}|_{f^{-1}(y)}\right)$ is supercomplete for any point $y \in Y$. Q.E.D. $\square$

We formulate the next technique lemma from [1].

Lemma 2.1. (1) *Every Cauchy net in a uniform space* $(\exp X, \exp \mathscr{U})$ *induce a stable Cauchy filter* $\mathscr{F}$ *in a uniform space* $(X, \mathscr{U})$.

(2) *Every stable Cauchy filter* $\mathscr{F}$ *in a uniform space* $(X, \mathscr{U})$ *induce a Cauchy net in a uniform space* $(\exp X, \exp \mathscr{U})$.

Proof. (1) Let $\{\mathrm{H}_i : i \in I\}$ be a Cauchy net in a uniform space $(\exp X, \exp \mathscr{U})$, i.e. for any covering $\alpha \in \mathscr{U}$ there exists such index $i_0 \in I$, that for any $i, j \geq i_0$ $H_i \subseteq \alpha(\mathrm{H}_j)$ $\mathrm{H}_j \subseteq \alpha(\mathrm{H}_i)$ provided. The family $\mathfrak{B} = \{\bigcup \{H_j : j \geq i\} : i \in I\}$ is a basis for some filter, by construction, the first, $\mathfrak{B}$ is nonempty and, the second, identifying element H_j of the net with element F_i of the filter, i.e. for any there exists such index $i_0 \in I$, that $i_0 \geq j$, $i_0 \geq k$ take place $\mathrm{F}_{i_0} \subseteq \mathrm{F}_j \cap \mathrm{F}_k$. Moreover, for any covering $\alpha \in \mathscr{U}$ take place $\mathrm{F}_{i_0} \subseteq \alpha(\mathrm{F}_i)$ for any $i \in I$.It means the family $\{\mathrm{F}_i : i \in I\}$ is a stable Cauchy filter $\mathscr{F}$ in a uniform space $(X, \mathscr{U})$, induced by a Cauchy net $\{H_i : i \in I\}$ in a uniform space $(\exp X, \exp \mathscr{U})$.

(2) Let $\mathscr{F}$ be a stable Cauchy filter in a uniform space $(\exp X, \exp \mathscr{U})$. We consider a family of closed sets $\left\{\bar{\mathrm{A}}_{\mathrm{F}} : \mathrm{F} \in \mathscr{F}\right\}$, which is a Cauchy net in a uniform space $(\exp X, \exp \mathscr{U})$. Identifying element F of the stable Cauchy filter $\mathscr{F}$ with element A_{F} of the Cauchy net $\left\{\bar{\mathrm{A}}_{\mathrm{F}} : \mathrm{F} \in \mathscr{F}\right\}$, we obtain directed set $\mathscr{F} = \{\mathrm{A}_{\mathrm{F}} = \mathrm{F} \colon \mathrm{F} \in \mathscr{F}\}$ in $(X, \mathscr{U})$, i.e. the stable Cauchy filter $\mathscr{F}$ in a uniform space $(X, \mathscr{U})$, induce a Cauchy net in a uniform space $(\exp X, \exp \mathscr{U})$.

Lemma is proved completely.

$\square$

We define complete mapping in nets terms.

Definition 2.4. *Uniformly continuous mapping* $f : (X, \mathscr{U}) \to (Y, \mathscr{V})$ *is called complete, if for any Cauchy net* $\{x_i \colon x_i \in X, i \in I\}$ *in* $(X, \mathscr{U})$*, the net* $\{f(x_i) : x_i \in X,\, i \in I\}$ *is converging in* $(Y, \mathscr{V})$*, then* $\{x_i \colon x_i \in X, i \in I\}$ *is converging in* $(X, \mathscr{U})$.

Every uniformly continuous mapping $f : (X, \mathscr{U}) \to (Y, \mathscr{V})$ of a uniform space $(X, \mathscr{U})$ into uniform space $(Y, \mathscr{V})$ induce uniformly continuous mapping $\exp f : (\exp X, \exp \mathscr{U}) \to (\exp Y, \exp \mathscr{V})$, defined by the rule $\exp f(\mathrm{N}) = \overline{f(\mathrm{N})}$ for any $\mathrm{N} \in \exp X$, when $(\exp X, \exp \mathscr{U})$ be a space of the closed subsets with Hausdorf uniformity ([3], p. 31).

Theorem 2.1. *Uniformly continuous mapping* $f : (X, \mathscr{U}) \to (Y, \mathscr{V})$ *is a supercomplete, if and only if the mapping* $\exp f : (\exp X, \exp \mathscr{U}) \to (\exp Y, \exp \mathscr{V})$ *is complete.*

Proof. Necessity: Let $f : (X, \mathscr{U}) \to (Y, \mathscr{V})$ be a supercomplete uniformly continuous mapping. Let $\{\mathrm{F}_i : i \in \mathrm{I}\}$ be a Cauchy net in $(\exp X, \exp \mathscr{U})$ such, that the net $\{\exp f(\mathrm{F}_i) : i \in \mathrm{I}\}$ is converging in $(Y, \mathscr{V})$. We show, the Cauchy net $\{\mathrm{F}_i : i \in \mathrm{I}\}$ is converging in $(\exp X, \exp \mathscr{U})$.

For any index $j \in I$ we suppose set $A_j = \bigcup \{\mathrm{F}_i : i \geq j\}$. The family $\{A_j : j \in I\}$ is a basis of the stable Cauchy filter in $(X, \mathscr{U})$. As the net $\{\exp f(\mathrm{F}_i) : i \in I\}$ is converging in $(\exp Y, \exp \mathscr{V})$, the net $\left\{\overline{f(\mathrm{F}_i)} : i \in I\right\}$ is converging to F in $(\exp Y, \exp \mathscr{V})$. By the construction, the family $\{f(\mathrm{A}_j) : j \in I\}$ is a basis of the stable Cauchy filter, which is hyperconverging in $(\exp Y, \exp \mathscr{V})$, i.e. F$=\bigcap\left\{\overline{f(\mathrm{A}_j)} : j \in I\right\}$. As the mapping f is supercomplete, we have $\bigcap\left\{\bar{A}_j : j \in I\right\} = N \neq \varnothing$ in $(\exp X, \exp \mathscr{U})$, $\{A_j : j \in I\}$ is hyperconverging in $(X, \mathscr{U})$, hence, the Cauchy net $\{\mathrm{F}_i : i \in I\}$ is converging to N in $(\exp X, \exp \mathscr{U})$. It means the mapping $\exp f$ is complete.

Sufficiency: Let uniformly continuous mapping $\exp f : (\exp X, \exp \mathscr{U}) \to (\exp Y, \exp \mathscr{V})$ be a complete. By Lemma, every Cauchy net in $(\exp X, \exp \mathscr{U})$ induce in $(X, \mathscr{U})$ the stable Cauchy filter $\mathscr{L}$. Every Cauchy filter $\mathscr{L}$ in $(\exp X, \exp \mathscr{U})$, for its $\exp(f)(\mathscr{L})$ is converging in $(\exp Y, \exp \mathscr{V})$, is converging in $(\exp X, \exp \mathscr{U})$, i.e. from formula

$$\bigcap\left\{\overline{\exp(f)(\mathrm{L})} : \mathrm{L} \in \mathscr{L}, \quad \mathscr{L} \in \exp X\right\} \neq \varnothing$$

follows $\bigcap\left\{\overline{(\mathrm{L})} : \mathrm{L} \in \mathscr{L}, \mathscr{L} \in \exp X\right\} \neq \varnothing$ (Theorem 0.6.5. [1]). It means the mapping $f : (X, \mathscr{U}) \to (Y, \mathscr{V})$ is supercomplete. Theorem 1 is proved completely. □

Definition 2.5. *Let **K** be an arbitrary category, $\mathscr{A}$ be a some morphisms class of a category **K**. Object X of a category **K** is called $\mathscr{A}$-closed, if very morphism $f : X \to Y$ for an arbitrary object Y, belongs to a class $\mathscr{A}$ ([2]).*

By analogue with [2] we characterize a supercomplete uniformly continuous mappings in category terms.

Theorem 2.2. *Let **Unif** be a category of a uniform spaces and uniformly continuous mappings, $\mathscr{A}$ be a class of a uniformly continuous supercomplete mappings. In order that a uniform space $(X, \mathscr{U})$ is supercomplete, it is necessary and sufficiently, that the object $(X, \mathscr{U})$ of a category **Unif** is $\mathscr{A}$-closed.*

Proof. Necessity: Let $(X, \mathscr{U})$ be a supercomplete uniform space, and $f : (X, \mathscr{U}) \to (Y, \mathscr{V})$ be a uniformly continuous mapping of a uniform space $(X, \mathscr{U})$ into an arbitrary uniform space $(Y, \mathscr{V})$. We show, the mapping f is supercomplete. Let $\mathscr{F}$ be an arbitrary stable Cauchy filter in $(X, \mathscr{U})$. As the uniform space $(X, \mathscr{U})$ is complete, the stable Cauchy filter $\mathscr{F}$ is hyperconverging in $(X, \mathscr{U})$ independently from that its image $f\mathscr{F}$ is hyperconverging in $(Y, \mathscr{V})$ or not. So, f is a supercomplete mapping, and an object $(X, \mathscr{U})$ is $\mathscr{A}$ - closed.

Sufficiency: Let be an object $(X, \mathscr{U})$ of a category ***Unif*** is $\mathscr{A}$ - closed, i.e. every uniformly continuous mapping $f : (X, \mathscr{U}) \to (Y, \mathscr{V})$ of a uniform space $(X, \mathscr{U})$ into an arbitrary uniform space $(Y, \mathscr{V})$ is supercomplete. We show that, a uniform space $(X, \mathscr{U})$ is supercomplete. Let $\mathscr{F}$ be an arbitrary stable Cauchy filter in $(X, \mathscr{U})$. As $(Y, \mathscr{V})$ be an arbitrary uniform space, then we suppose it is one-point. Then $f : (X, \mathscr{U}) \to (Y, \mathscr{V})$ is a constant mapping and image $f\mathscr{F}$ of the Cauchy filter is hyperconverging in $(Y, \mathscr{V})$. Then, by the mapping f is supercomplete, the uniform space $(X, \mathscr{U})$ is supercomplete. Theorem 2.2 is proved completely. □

Theorem 2.2 immediately implies the uniform analogue of Isbell Theorem ([3]) on paracompact spaces with maximal uniformity $\mathscr{U}_X$.

Corollary 2.1. *Let **Unif** be a category of a uniform spaces and uniformly continuous mappings, and $\mathscr{A}$ be a class of a uniformly continuous supercomplete mappings. In order that a uniform space $(X, \mathscr{U})$ is uniformly paracompact and $\lambda\mathscr{U} = \mathscr{U}_X$, it is necessary and sufficiently, that the object $(X, \mathscr{U})$ of the category **Unif** is $\mathscr{A}$ - closed ([3]).*

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