TWMS J. App. Eng. Math. V.1, N.2, 2011, pp. 167-172

ON EXISTENCE OF PERIODIC SOLUTIONS OF THE KDV TYPE EQUATION

T. M. IMANALIEV[1], §

Abstract. We prove the existence of the periodic solutions of the KDV type of equation with respect to time variable with periodic external force. Using the integral substitution we reduce the original equation to the integral equation. The existence of the unique solution of the equation is proved by the fixed point theorem. Thus we show that the solitons are periodic in time as well as in space dimension.

Keywords: Periodic solution, KDV type equation, nonlinear integral equation, contractive operator.

AMS Subject Classification: 35B10, 35A22

1. Introduction

In this paper we investigate the properties of Korteweg-de Vries type equation with the external force. We show that the solitary wave under effect of the periodic force becomes periodic. We investigated "inhomogeneous" KDV type equations in [1]- [4]. In that paper the equation is reduced to the system of integral equations. The method of solving was named as the Additional Argument Method. We noticed that there is no much investigations on equations with the right hand side terms. This paper is a part of the research on this topic.

2. Main results

We investigate the properties of the solutions of KDV type equations of the form

$$u_t(t,x) + u(t,x)u_x(t,x) + u_{xxx}(t,x) = f(t,x,u(t,x)), \tag{1}$$

with the known function$f(t,x,u) \in C[R_+, R, R]$, $f(t,x,u) = f(t+T,x,u)$, T=const,

$$\|f(t,x,u)\| = \sup_{\substack{0\le t\le T,\\ -\infty<x,u<\infty}} |f(t,x,u)| \le M = const,$$

We investigate periodic solution of (1) alsatz

$$u(t,x) = \frac{1}{2\pi}\int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)-(w^2+s^2+1)(t-\nu)}}{e^{(w^2+s^2+1)T}-1}Q(\nu,s)dsdwd\nu \tag{2}$$

with the new unknown function $Q(t,x) \in C[R_+, R, R]$,$Q(t+T,x) = Q(t,x)$.

[1] Department of Applied Mathematics Kyrgyz National University, Bishkek, Kyrgyzstan,
e-mail: talaibek@gmail.com

§ Manuscript received 27 July 2011.

Theorem 1. *If $f(t,x,u)$ is periodic then for the certain ball R equation (1) possess unique periodic solution with respect to t.*

We prove the theorem in the form of Lemmas

Lemma 1. *Function (2) is periodic with respect to t.*

Proof. Indeed

$$u(t+T,x)=\frac{1}{2\pi}\int_{t+T}^{t+2T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)-(w^2+s^2+1)(t+T-\nu)}}{e^{(w^2+s^2+1)T}-1}Q(\nu,s)dsdwd\nu. \quad (3)$$

Using the substitution $\nu=T+\rho$ we obtain from (3),

$$u(t+T,x)=\frac{1}{2\pi}\int_{t}^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)-(w^2+s^2+1)(t-\rho)}}{e^{(w^2+s^2+1)T}-1}Q(\nu,\rho)d\rho dwd\nu\equiv u(t,x). \quad (4)$$

□

To define the unknown function $Q(t,x)$ we substitute (2) in (1). Differentiation with respect to t we obtain from (2), the following

$$u_t(t,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)}\left[e^{-(w^2+s^2+1)(t-t-T)}-1\right]}{e^{(w^2+s^2+1)T}-1}Q(t,s)dsdw$$

$$-\frac{1}{2\pi}\int_{t}^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)-(w^2+s^2+1)(t-\nu)}(w^2+s^2+1)}{e^{(w^2+s^2+1)T}-1}Q(\nu,s)dsdwd\nu \quad (5)$$

$$=Q(t,x)-\frac{1}{2\pi}\int_{t}^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)-(w^2+s^2+1)(t-\nu)}(w^2+s^2+1)}{e^{(w^2+s^2+1)T}-1}Q(\nu,s)dsdwd\nu.$$

Tripple differentiation with respect to x of (2), gives us

$$u_{xxx}(t,x)=\frac{1}{2\pi}\int_{t}^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{iw^3e^{-iw(x-s)-(w^2+s^2+1)(t-\nu)}}{e^{(w^2+s^2+1)T}-1}Q(\nu,s)dsdwd\nu. \quad (6)$$

We have

$$u(t,x)u_x(t,x)$$

$$=-\frac{1}{4\pi^2}\int_{t}^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)-(w^2+s^2+1)(t-\nu)}}{e^{(w^2+s^2+1)T}-1}Q(\nu,s)dsdwd\nu \quad (7)$$

$$\cdot\int_{t}^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)-(w^2+s^2+1)(t-\nu)}}{e^{(w^2+s^2+1)T}-1}iwQ(\nu,s)dsdwd\nu.$$

Adding right and left hand sides of (5), (6) and (7) taking into account (1), we obtain

$$u_t(t,x)+uu_x(t,x)+u_{xxx}(t,x)=f(t,x,u)$$

$$=Q(t,x)-\frac{1}{2\pi}\int_{t}^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)-(w^2+s^2+1)(t-\nu)}(w^2+s^2+1)}{e^{(w^2+s^2+1)T}-1}Q(\nu,s)dsdwd\nu$$

$$+\frac{1}{2\pi}\int_{t}^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{iw^{3}e^{-iw(x-s)-(w^{2}+s^{2}+1)(t-\nu)}}{e^{(w^{2}+s^{2}+1)T}-1}Q(\nu,s)dsdwd\nu \tag{8}$$
$$-\frac{1}{4\pi^{2}}\int_{t}^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)-(w^{2}+s^{2}+1)(t-\nu)}}{e^{(w^{2}+s^{2}+1)T}-1}Q(\nu,s)dsdwd\nu$$
$$\cdot\int_{t}^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)-(w^{2}+s^{2}+1)(t-\nu)}}{e^{(w^{2}+s^{2}+1)T}-1}iwQ(\nu,s)dsdwd\nu.$$

To define the unknown periodic function $Q(t,x)$, we have the following nonlinear integral equation

$$Q(t,x)=f\left(t,x,\frac{1}{2\pi}\int_{t}^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)-(w^{2}+s^{2}+1)(t-\nu)}}{e^{(w^{2}+s^{2}+1)T}-1}Q(\nu,s)dsdwd\nu\right)$$
$$+\frac{1}{2\pi}\int_{t}^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)-(w^{2}+s^{2}+1)(t-\nu)}(w^{2}+s^{2}+1)}{e^{(w^{2}+s^{2}+1)T}-1}Q(\nu,s)dsdwd\nu \tag{9}$$
$$+\frac{1}{4\pi^{2}}\int_{t}^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)-(w^{2}+s^{2}+1)(t-\nu)}}{e^{(w^{2}+s^{2}+1)T}-1}Q(\nu,s)dsdwd\nu$$
$$\cdot\int_{t}^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)-(w^{2}+s^{2}+1)(t-\nu)}}{e^{(w^{2}+s^{2}+1)T}-1}iwQ(\nu,s)dsdwd\nu.$$

Lemma 2. *The function $Q(t,x)$ is periodic $Q(t+T,x)=Q(t,x)$.*

Proof. Direct substitution gives us

$$Q(t+T,x)=f\left(t+T,x,\frac{1}{2\pi}\int_{t+T}^{t+2T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)-(w^{2}+s^{2}+1)(t+T-\nu)}}{e^{(w^{2}+s^{2}+1)T}-1}Q(\nu,s)dsdwd\nu\right)$$
$$+\frac{1}{2\pi}\int_{t+T}^{t+2T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)-(w^{2}+s^{2}+1)(t+T-\nu)}(1+w^{2}+s^{2}-iw^{3})}{e^{(w^{2}+s^{2}+1)T}-1}Q(\nu,s)dsdwd\nu$$
$$+\frac{1}{4\pi^{2}}\int_{t+T}^{t+2T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)-(w^{2}+s^{2}+1)(t+T-\nu)}}{e^{(w^{2}+s^{2}+1)T}-1}Q(\nu,s)dsdwd\nu \tag{10}$$
$$\cdot\int_{t+T}^{t+2T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)-(w^{2}+s^{2}+1)(t+T-\nu)}}{e^{(w^{2}+s^{2}+1)T}-1}iwQ(\nu,s)dsdwd\nu.$$

Applying the substitution $\nu=T+s$ we obtain from (10) the following

$$Q(t+T,x)=f\left(t,x,\frac{1}{2\pi}\int_{t}^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)-(w^{2}+s^{2}+1)(t-\rho)}}{e^{(w^{2}+s^{2}+1)T}-1}Q(\rho,s)dsdwd\rho\right)$$
$$+\frac{1}{2\pi}\int_{t}^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)-(w^{2}+s^{2}+1)(t-\rho)}(1+w^{2}+s^{2}-iw^{3})}{e^{(w^{2}+s^{2}+1)T}-1}Q(\rho,s)dsdwd\rho$$
$$+\frac{1}{4\pi^{2}}\int_{t}^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)-(w^{2}+s^{2}+1)(t-\rho)}}{e^{(w^{2}+s^{2}+1)T}-1}Q(\rho,s)dsdwd\rho$$

$$\cdot \int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)-(w^2+s^2+1)(t-\rho)}}{e^{(w^2+s^2+1)T}-1}iwQ(\rho,s)dsdwd\rho = Q(t,x).$$

□

Lemma 3. *The equation (9) possess periodic solution if*

$$f(t,x,u)\in Lip\{G_0\{0\le t\le T,-\infty<x,u<\infty\}\},$$

$$\|f(t,x,u_2)-f(t,x,u_1)\|\le N\|u_2-u_1\|\left[(N+3)e^{-T}+\frac{M}{\sqrt{2}}e^{-2T}\right]\le\frac{1}{2}.$$

Proof. Consider the space of continuous functions $C\{Q(t,x)\}$ with the norm

$$\|Q(t,x)-G(t,x)\| = \sup_{\substack{0\le t\le T\\ -\infty<x<\infty}}|Q(t,x)-G(t,x)|.$$

The right hand side of equation (9) we consider as operator $H[Q]:G_0\to G_0$. We have

$$\|H[Q]\|\le\|f(t,x,0)\|$$

$$+\left\|f\left(t,x,\frac{1}{2\pi}\int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)-(w^2+s^2+1)(t-\nu)}}{e^{(w^2+s^2+1)T}-1}Q(\nu,s)dsdwd\nu\right)-f(t,x,0)\right\|$$

$$+\frac{1}{2\pi}\int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\left|e^{-iw(x-s)}\right|e^{-(w^2+s^2+1)(t-\nu)}(|w|^3+w^2+s^2+1)}{e^{(w^2+s^2+1)T}-1}\|Q(\nu,s)\|dsdwd\nu$$

$$+\frac{1}{4\pi^2}\int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\left|e^{-iw(x-s)}\right|e^{-(w^2+s^2+1)(t-\nu)}}{e^{(w^2+s^2+1)T}-1}\|Q(\nu,s)\|dsdwd\nu$$

$$\cdot\int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\left|e^{-iw(x-s)}\right|e^{-(w^2+s^2+1)(t-\nu)}}{e^{(w^2+s^2+1)T}-1}|w|\|Q(\nu,s)\|dsdwd\nu$$

$$\le M+\frac{N}{2\pi}\int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\left|e^{-iw(x-s)}\right|e^{-(w^2+s^2+1)(t-\nu)}}{e^{(w^2+s^2+1)T}-1}dsdwd\nu\|Q(\nu,s)\|$$

$$+\frac{1}{2\pi}\int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-(w^2+s^2+1)T}(|w|^3+w^2+s^2+1)}{1-e^{-(w^2+s^2+1)T}}dsdwd\nu\|Q(\nu,s)\|$$

$$+\frac{1}{4\pi^2}\int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-(w^2+s^2+1)T}}{1-e^{-(w^2+s^2+1)T}}dsdwd\nu$$

$$\cdot\int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-(w^2+s^2+1)T}}{1-e^{-(w^2+s^2+1)T}}|w|dsdwd\nu\|Q(\nu,s)\|^2$$

$$\le M+\frac{N}{2\pi}\int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(w^2+s^2+1)T}dwdsd\nu R$$

$$+\frac{e^{-T}}{2\pi}\int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[4e^{-\frac{w^2+s^2}{2}T}\right]dwdsd\nu R$$

$$+\frac{R^2}{4\pi^2}\int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(w^2+s^2+1)T}dwdsd\nu\int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(w^2+s^2+1)T}\left|w\right|dsdwd\nu$$

$$\leq M+\frac{Ne^{-T}\pi}{2\pi}R+\frac{4e^{-T}}{2\pi}\int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\frac{w^2+s^2}{2}T}dwdsd\nu R$$

$$+\frac{R^2e^{-2T}}{4\pi^2}\frac{\sqrt{\pi}\sqrt{\pi}T}{\sqrt{T}\sqrt{T}}\int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{\frac{-w^2-s^2}{2}}dwdsd\nu$$

$$\leq M+\frac{Ne^{-T}}{2}R+\frac{4e^{-T}\sqrt{2}\sqrt{2}T\sqrt{\pi}\sqrt{\pi}}{2\pi\sqrt{T}\sqrt{T}}R+\frac{2R^2e^{-2T}}{4\pi}\frac{\sqrt{\pi}\sqrt{\pi}\sqrt{2}T}{\sqrt{T}\sqrt{T}}$$

$$\leq M+\left[(N+4)e^{-T}+\frac{2Re^{-2T}}{2\sqrt{2}}\right]R$$

$$=M+\left[(N+3)e^{-T}+Re^{-2T}\right]R$$

$$=M+\frac{1}{2}R=R;R=2M.(N+3)e^{-T}+2Me^{-T}\leq\frac{1}{2};$$

Thus, the operator $H\left[Q\right]$ mapps the points of the ball with the radius R=2M onto the same ball. We now show that the operator $H\left[Q\right]$ is the contractive operator. Indeed

$$\left\|H\left[Q\right]-H\left[G\right]\right\|\leq\left\|f\left(t,x,\frac{1}{2\pi}\int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)-(w^2+s^2+1)(t-\nu)}}{e^{(w^2+s^2+1)T}-1}Q(\nu,s)dwdsd\nu\right)\right.$$

$$\left.-f\left(t,x,\frac{1}{2\pi}\int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-iw(x-s)-(w^2+s^2+1)(t-\nu)}}{e^{(w^2+s^2+1)T}-1}G(\nu,s)dsdwd\nu\right)\right\|$$

$$+\frac{1}{2\pi}\int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\left|e^{-iw(x-s)}\right|e^{-(w^2+s^2+1)(t-\nu)}(\left|w\right|^3+w^2+s^2+1)}{e^{(w^2+s^2+1)T}-1}dsdwd\nu\left\|Q-G\right\|$$

$$+\frac{1}{4\pi^2}\int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\left|e^{-iw(x-s)}\right|e^{-(w^2+s^2+1)(t-\nu)}}{e^{(w^2+s^2+1)T}-1}dsdwd\nu$$

$$\cdot\int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\left|e^{-iw(x-s)}\right|e^{-(w^2+s^2+1)(t-\nu)}}{e^{(w^2+s^2+1)T}-1}\left|w\right|\left\|Q(\nu,s)\right\|dsdwd\nu\left\|Q-G\right\|$$

$$+\frac{1}{4\pi^2}\int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\left|e^{-iw(x-s)}\right|e^{-(w^2+s^2+1)(t-\nu)}}{e^{(w^2+s^2+1)T}-1}\left\|G(\nu,s)\right\|dsdwd\nu$$

$$\cdot\int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\left|e^{-iw(x-s)}\right|e^{-(w^2+s^2+1)(t-\nu)}}{e^{(w^2+s^2+1)T}-1}\left|w\right|dwdsd\nu\left\|Q-G\right\|$$

$$\leq\frac{N}{2\pi}\int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\left|e^{-iw(x-s)}\right|e^{-(w^2+s^2+1)(t-\nu)}}{e^{(w^2+s^2+1)T}-1}dsdwd\nu\left\|Q-G\right\|$$

$$+\frac{1}{2\pi}\int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-(w^2+s^2+1)T}(\left|w\right|^3+w^2+s^2+1)}{1-e^{-(w^2+s^2+1)T}}dsdwd\nu\left\|Q-G\right\|$$

$$+\frac{2R}{4\pi^2}\int_t^{t+T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-(w^2+s^2+1)T}}{1-e^{-(w^2+s^2+1)T}}dsdwd\nu$$

$$\cdot \int_t^{t+T} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-(\frac{w^2}{2}+s^2)T}}{1-e^{-(\frac{w^2}{2}+s^2+1)T}} dwdsd\nu \, \|Q-G\|$$

$$\leq \left[\frac{N}{2\pi} \int_t^{t+T} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-(w^2+s^2+1)T}}{1-e^{-(w^2+s^2+1)T}} dsdwd\nu + \frac{4e^{-T}}{2\pi} \int_t^{t+T} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{w^2+s^2}{2}T} dsdwd\nu \right.$$

$$\left. + \frac{e^{-2T}R}{2\pi^2} \int_t^{t+T} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(w^2+s^2)T} dsdwd\nu \int_t^{t+T} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(\frac{w^2}{2}+s^2)T} dwdsd\nu \right] \|Q-G\|$$

$$\leq \left[Ne^{-T} + 4e^{-T} + \frac{Re^{-2T}}{\sqrt{2}} \right] \|Q-G\| \leq \frac{1}{2} \|Q-G\|$$

Thus we proved that operator $H[Q]$ is contractive. Therefore nonlinear integral equation (9) has the unique periodic solution $Q(t,x)$, and

$$\|Q(t,x)\| \leq R = 2M;$$

□

Hence we proved that the equation (1) also has the unique periodic solution with

$$\|u(t,x)\| \leq R = 2M;$$

References

[1] Imanaliev, M., Pankov, P. S. and Imanaliev,T. M., (1995), On The Theory of Partial Differential Equations of KDV type, 342(1), 17-19.

[2] Imanaliev, M., Pankov, P. S. and Imanaliev,T. M., (1995), On The Theory of Nonlinear Partial Wave Equations, Doklady Academii Nauk, 343(5), 596-598.

[3] Imanaliev,T. M., (1998), On the Theory of Multiperiodic Solutions of Nonlinear Partial Equations of KDV Type, Research on Integro-Differential Equations. - Bishkek: Ilim, 27, 26-30.

[4] Imanaliev,T. M., (1998), On Periodic Solutions of KDV Type Equation, Research on Integro-Differential Equations. - Bishkek: Ilim, 27, 31-37.

Talaibek M. Imanaliev was born in Frunze city, Kyrgyzstan, 1965. He graduated from the Mathematics Department of Kyrgyz National University (Honours). He got the PhD degree from National Academy of Sciences of Kyrgyz Republic in 1993 and DsC degree from National Attestation Commission of Kyrgyzstan in 2001. He is a Professor of Kyrgyz National University. He is the author of more than 80 papers in the field of general theory of PDE and soliton waves. He is one of authors of the new Additional Argument Method to reduce transport type of equations to integral equation. He supervised five PhD and number of MSc theses.