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NEW APPROACH TO THE SOLUTIONS OF THE PIB EQUATION

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Abstract. In this paper, based on the Exp-function method and mathematical derivation, we obtain several explicit and exact traveling wave solutions for the PIB equation.

Keywords: The Exp-function method, PIB equation, Traveling wave solution

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1. Introduction

It is well known that many important phenomena and dynamic processes in physics, mechanics, chemistry, biology and etc can be represented by nonlinear partial differential equations. For decades, mathematicians and physicists have devoted considerable effort to the study of solutions of nonlinear partial differential equations. The study of exact solutions of nonlinear evolution equations plays an important role in soliton theory and explicit formulas of such exact solutions play an essential role in the nonlinear science. Also, the explicit formulas may provide physical information and help us to understand the mechanism of related physical models.

In recent years, many kinds of powerful methods have been proposed to find solutions of nonlinear partial differential equations, e.g., the inverse scattering method [1], the variational iteration method [2], the homotopy perturbation method [3, 4, 5], Bäcklund transformation method [6, 7], the tanh-method [8], the sinh-method [9], the homogeneous balance method [10], the F-expansion method [11], algebraic geometric method [12]. One may find a complete review in [13].

J.H.He in [14] suggested a novel method, so-called Exp-function method, to search for solitary solutions, compact-like solutions and periodic solutions of various nonlinear wave equations. The basic idea of the Exp-function method was provided in [15] and one may find several applications of the Exp-function method over various areas in [14, 16, 17, 18, 19, 20].

we consider non-traveling wave solutions of the two dimensional Painleve integrable Burgers equation (PIB) [21, 22, 23]

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$$\begin{aligned} &u_t = uu_y + \lambda v u_x + \mu u_{yy} + \lambda\mu u_{xx} \\ &u_x = v_y, \end{aligned} \tag{1}$$

where λ and μ are nonzero constants. Eq. (1) was derived from the generalized Painleve integrability classification by Hong et al. [22]. Some explicitly exact solutions of Eq. (1) have been obtained via variable separation approach [22, 23] and multiple Riccati equations rational expansion method [21].

The outline of this paper is as follows. In the following section we review the Exp-function method and then we apply the method to find explicit formulas of solution of the PIB equation in Section 3, We present a brief conclusion in Section 4.

2. The Exp-function method

We consider the following nonlinear partial differential equation

$$N(\chi, \chi_x, \chi_y, \chi_z, \chi_t, \chi_{xx}, \chi_{yy}, \chi_{zz}, \chi_{xy}, \chi_{xt}, \chi_{yt}, \cdots) = 0. \tag{2}$$

By using transformation

$$\eta = a\,x + b\,y + c\,z + d\,t + \gamma,$$

where a, b, c, d and γ are constants, we can convert (2) to the following nonlinear ordinary differential equation

$$M(\chi, \chi', \chi'', \chi''', \cdots) = 0, \tag{3}$$

where the prime denotes the differentiation with respect to η.

Adopting the Exp-function method given in [14], and assuming that the traveling wave solution can be expressed in the following form

$$\chi(\eta) = \frac{\displaystyle\sum_{n=-N_a}^{N_b} a_n \exp(n\,\eta)}{\displaystyle\sum_{m=-M_a}^{M_b} b_m \exp(m\,\eta)}, \tag{4}$$

where M_a, M_b, N_a and N_b are positive integers which could be freely chosen, and a_n and b_m are unknown coefficients to be determine. The equation (4) can be rewritten in the expanded form such as

$$\chi(\eta) = \frac{a_{N_b}\,\exp(N_b\,\eta) + \cdots + a_{-N_a}\exp(-N_a\,\eta)}{b_{M_b}\,\exp(M_b\,\eta) + \cdots + b_{-M_a}\exp(-M_a\,\eta)}. \tag{5}$$

In order to determine the values of N_a and M_a, we balance the linear terms of the highest order in equation (3) with the highest order nonlinearity. Similarly, to determine the values of N_b and M_b, we balance the linear terms of the lowest order in equation (3) with the lowest order nonlinear terms. For more details see [14, 19].

3. Explicit formula of solutions of the PIB equation

In order to obtain the traveling wave solutions of the equation (1), by using the transformation $u(\eta) = u(x,t,y)$, $\eta = kx + \omega t + \beta y$, the equation (1) can be converted to following system:

$$\begin{aligned} &\omega u' = \beta u u' + \lambda k v u' + \mu\beta^2 u'' + \lambda\mu k^2 u'' \\ &k u' = \beta v', \end{aligned} \tag{6}$$

from second relation in(6)we have:

$$v = (\frac{k}{\beta}u + c_1), \tag{7}$$

where c_1 is a constant. Substituting (7) into the first relation (6), we get the following ordinary differential equation:

$$(\beta + \tfrac{\lambda k^2}{\beta})uu' + (\lambda k c_1 - \omega)u' + (\mu\beta^2 + \lambda\mu k^2)u'' = 0. \tag{8}$$

Using equation (5) in (8) and according to the homogeneous balance principle yields that

$$M_a = N_a \quad \text{and} \quad M_b = N_b.$$

In the following subsections, we consider some arbitrary values of the numbers N_a and N_b to derive explicit analytic solutions of (8). One may choose the numbers N_a and N_b arbitrary , but the resultant solutions do not strongly depend upon such choice (see [14, 19]).

3.1. **Case 1:** $N_a = 1$ **and** $N_b = 1$**.** For simple case of these choice, the trial function (5) becomes

$$u(\eta) = \frac{a_1\exp(\eta) + a_0 + a_{-1}\exp(-\eta)}{b_1\exp(\eta) + b_0 + b_{-1}\exp(-\eta)}. \tag{9}$$

For convenience, set $b_1 = 1$. Substituting equation (9) into equation (8) and using some mathematical calculations we can derive the following relations:

case I:

$$a_0 = a_0, \qquad a_1 = -\frac{-2\omega\beta b_0 + 2\beta\lambda k c_1 b_0 + \beta^2 a_0 + \lambda k^2 a_0}{b_0(\beta^2 + \lambda k^2)}, \qquad a_{-1} = 0,$$
$$b_0 = b_0, \qquad b_{-1} = 0, \qquad \beta = \beta, \qquad \lambda = \lambda, \qquad k = k,$$
$$\mu = -\frac{\lambda k^2 a_0 + \beta\lambda k c_1 b_0 + \beta^2 a_0 - \beta\omega b_0}{\beta b_0(\beta^2 + \lambda k^2)}, \qquad \omega = \omega, \qquad c_1 = c_1.$$

Then we have the following solitonary solution $u(\eta)$

$$u(\eta) = \frac{-\frac{-2\omega\beta b_0 + 2\beta\lambda k c_1 b_0 + \beta^2 a_0 + \lambda k^2 a_0}{b_0(\beta^2 + \lambda k^2)}\exp(\eta) + a_0}{\exp(\eta) + b_0},$$

where a_0,b_0, λ,β,k,ω and c_1 are arbitrary parameters.

case II:

$$a_0 = 0, \qquad a_1 = a_1, \qquad a_{-1} = a_1 a_{-1} - 4a_{-1}\mu\beta,$$
$$b_0 = 0, \qquad b_{-1} = b_{-1}, \qquad \beta = \beta, \qquad \lambda = \lambda, \qquad k = k,$$
$$\mu = \mu, \qquad \omega = \frac{a_1\beta^2 + a_1\lambda k^2 + \beta\lambda k c_1 - 2\mu\beta^3 - 2\mu\beta\lambda k^2}{\beta}, \qquad c_1 = c_1.$$

Then we have the following solitonary solution $u(\eta)$

$$u(\eta) = a_1 - \frac{4\mu a_{-1}\beta\exp(-\eta)}{\exp(\eta) + a_{-1}\exp(-\eta)},$$

where a_1, a_{-1}, μ and β are arbitrary parameters.

3.2. **Case 2:** $N_a = 2$ **and** $N_b = 2$**.** In this case, we set $N_a = M_a = 2$ and $N_b = M_b = 2$, then the trial function (5) becomes

$$u(\eta) = \frac{a_2\exp(2\,\eta) + a_1\exp(\eta) + a_0 + a_{-1}\exp(-\,\eta) + a_{-2}\exp(-2\,\eta)}{b_2\exp(2\,\eta) + b_1\exp(\eta) + b_0 + b_{-1}\exp(-\eta) + b_{-2}\exp(-2\,\eta)}. \tag{10}$$

There are some arbitrary parameters in the above equation. We also set $a_1 = b_1 = 0$ for convenience, then the trial function (10) is simplified as:

$$u(\eta) = \frac{a_2\exp(2\,\eta) + a_0 + a_{-1}\exp(-\,\eta) + a_{-2}\exp(-2\,\eta)}{b_2\ \exp(2\,\eta) + b_0 + b_{-1}\exp(-2\,\eta) + b_{-2}\exp(-2\,\eta)}. \tag{11}$$

Substituting equation (11) into equation (8), we can derive the following relations:
case I:

$$a_2 = a_2, \qquad a_{-1} = 0, \qquad a_0 = 0, \qquad a_{-2} = a_2 b_{-2} - 8\mu\beta b_{-2}, \qquad b_{-1} = 0,$$
$$b_0 = 0 \qquad b_{-2} = b_{-2}, \qquad \beta = \beta, \qquad \lambda = \lambda, \qquad \mu = \mu, \qquad k = k$$
$$\omega = -\frac{-\beta k\lambda c_1 + 4\mu\beta\lambda k^2 - \lambda k^2 a_2 + 4\mu\beta^3 - \beta^2 a_2}{\beta}, \qquad c_1 = c_1.$$

Then, we have the following solitonary solution $u(\eta)$

$$u(\eta) = a_2 - \frac{8\mu\beta b_{-2}\exp(-2\,\eta)}{\exp(2\,\eta) + b_{-2}\exp(-2\,\eta)},$$

where a_2, b_{-2}, μ and β are arbitrary parameters.
case II:

$$a_2 = \frac{6\mu\beta b_{-1} + a_{-1}}{b_{-1}}, \qquad a_{-1} = a_{-1}, \qquad a_0 = 0, \qquad a_{-2} = 0, \qquad b_{-1} = b_{-1}$$
$$b_0 = 0 \qquad b_{-2} = 0, \qquad \beta = \beta, \qquad \lambda = \lambda, \qquad \mu = \mu, \qquad k = k$$
$$\omega = \frac{3\mu\beta^3 b_{-1} + \beta^2 a_{-1} + 3\mu\beta\lambda k^2 b_{-1} + \lambda k^2 a_{-1} + \beta\lambda k c_1 b_{-1}}{\beta b_{-1}}, \qquad c_1 = c_1.$$

Then, we have the following solitonary solution $u(\eta)$

$$u(\eta) = a_{-1} + \frac{6\mu\beta\exp(2\,\eta)}{\exp(2\,\eta) + b_{-1}\exp(-\,\eta)},$$

where a_{-1} , b_{-1} , μ and β are arbitrary parameters.

4. Conclusions

In the two previous cases we obtained the exact solution for equation (1)which are satisfied and having the meaningful physical interpretation. This approach give us some more implicit solutions, but all these solutions may not satisfy in the given equation .

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