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AN APPLICATION OF MODIFIED REDUCTIVE PERTURBATION METHOD TO SYMMETRIC REGULARIZED-LONG-WAVE EQUATIONS

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ABSTRACT. In this work, we extended the application of "the modified reductive perturbation method" to symmetrical regularized long waves with quadratic nonlinearity and obtained various form of KdV equations as the governing equations. Seeking a localized travelling wave solutions to these evolution equations we determined the scale parameters g_1 and g_2 so as to remove the possible secularities that might occur. To indicate the power and elegance of the present method, we compared our result with the exact travelling wave solution of the symmetric regularized long-wave equation with quadratic nonlinearity. These results show that for weakly nonlinear case the solutions for both approaches coincide with each other. The present method is seen to be fairly simple as compared to the renormalization method of Kodama and Taniuti [4] and the multiple scale expansion method of Kraenkel et al [6].

Keywords: Modified reductive perturbation method, Ion-acoustic waves, Korteweg-deVries hierarchy.

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1. Introduction

In collisionless cold plasma, in fluid-filled elastic tubes and in shallow-water waves, due to nonlinearity of the governing equations, for weakly dispersive case one obtains the Korteweg-de Vries (KdV) equation for the lowest order term in the perturbation expansion, the solution of which may be described by solitons (Davidson [1]). To study the higher order terms in the perturbation expansion, the reductive perturbation method has been introduced by use of the stretched time and space variables (Taniuti [2]). However, in such an approach some secular terms appear which can be eliminated by introducing some slow scale variables (Sugimoto and Kakutani [3]) or by a renormalization procedure of the velocity of the KdV soliton (Kodama and Taniuti [4]). Nevertheless, this approach remains somewhat artificial, since there is no reasonable connection between the smallness parameters of the stretched variables and the one used in the perturbation expansion of the field variables. The choice of the former parameter is based on the linear wave analysis of the concerned problem and the wave number or the frequency is taken as the

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perturbation parameter (Washimi and Taniuti [5]). On the other hand, at the lowest order, the amplitude and the width of the wave are expressed in terms of the unknown perturbed velocity, which is also used as the smallness parameter. This causes some ambiguity over the correction terms. Another attempt to remove such secularities is made by Kraenkel et al [6] for long water waves by use of the multiple time scale expansion but could not obtain explicitly the correction terms to the wave speed.

In order to remove these uncertainities, Malfliet and Wieers [7] presented a dressed solitary wave approach, which is based on the assumption that the field variables admit localized travelling wave solution. Then, for the longwave limit, they expanded the field variables and the wave speed into a power series of the wave number, which is assumed to be the only smallness parameter, and obtained the explicit solution for various order terms in the expansion. However, this approach can only be used when one studies progressive wave solution to the original nonlinear equations and it does not give any idea about the form of evolution equations governing the various order terms in the perturbation expansion. In our previous paper [8], we have presented a method so called "the modified reductive perturbation method" to examine the contributions of higher order terms in the perturbation expansion and applied it to weakly dispersive ion-acoustic plasma waves and solitary waves in a fluid filled elastic tube [9]. In these works, we have shown that the lowest order term in the perturbation expansion is governed by the nonlinear Korteweg-de Vries equation, whereas the higher order terms in the expansion are governed by the degenerate Korteweg-de Vries equation with non-homogeneous term. By employing the hyperbolic tangent method a progressive wave type of solution was sought and the possible secularities were removed by selecting the scaling parameter in a special way. The basic idea in this method was the inclusion of higher order dispersive effects through the introduction of the scaling parameter g, to balance the higher order nonlinearities with dispersion. The negligence of higher order dispersive effects in the classical reductive perturbation method leads to the imbalance between the nonlinearity and the dispersion, which resulted in some secular terms in the solution of evolution equations. As a matter of fact, the renormalization method presented by Kodama and Taniuti [4] is different but rather involved formulation of the same idea.

In the present work, we extended the application of "the modified reductive perturbation method" to the symmetrical regularized long waves with quadratic nonlinearity and obtained various form of KdV equations as the governing equations. Seeking a localized travelling wave solutions to these evolution equations we determined the scale parameters g_1 and g_2 so as to remove the possible secularities that might occur. To indicate the power and elegance of the present method, we compared our result with the exact travelling wave solution of the symmetric regularized long-wave equation with quadratic nonlinearity. These results show that for weakly nonlinear case the solutions for both approaches coincide with each other. The present method is seen to be fairly simple as compared to the renormalization method of Kodama and Taniuti [4] and the multiple scale expansion method of Kraenkel et al [6].

2. Modified reductive perturbation formalism

In this section we focus our attention to the examination of a symmetrical regular long wave equation [10] given by

$$u_t - v_x + (u^2)_x - u_{xxt} = 0, \quad v_t = u_x, \tag{1}$$

where u is the velocity in the x direction and v is the charge density of the plasma. The equation (1) describes the propagation of ion-plasma acoustic waves in space under weakly nonlinear action. Eliminating v between the equations (1), the following single equation is obtained

$$u_{tt} - u_{xx} + \frac{1}{2}(u^2)_{xt} - u_{xxtt} = 0. \tag{2}$$

The dispersion relation of the linearized form of equation (2) may be given by

$$\omega = k(1+k^2)^{-1/2}, \tag{3}$$

where ω is the angular frequency and k is the wave number. For small wave numbers (long wave length) the dispersion relation reduces to

$$\omega = k(1 - \frac{1}{2}k^2). \tag{4}$$

Motivated with the dispersion relation (4), for the long-wave approximation, it is convenient to introduce the following stretched coordinates

$$\xi = \epsilon^{1/2}(x - ct), \quad \tau = \epsilon^{3/2} g t, \tag{5}$$

where ϵ is a parameter which measures the smallness of nonlinearity, c and g are some constants to be determined from the solution of the field equations. Introducing (5) into the equation (2) gives

$$\begin{aligned}(c^2-1)\frac{\partial^2 u}{\partial \xi^2} - 2cg\epsilon\frac{\partial^2 u}{\partial \xi \partial \tau} + \epsilon^2 g^2 \frac{\partial^2 u}{\partial \tau^2} - \frac{c}{2}\frac{\partial^2 (u^2)}{\partial \xi^2} \\ +\epsilon\frac{g}{2}\frac{\partial^2(u^2)}{\partial \xi \partial \tau} - \epsilon c^2 \frac{\partial^4 u}{\partial \xi^4} + \epsilon^2 2cg\frac{\partial^4 u}{\partial \xi^3 \partial \tau} \\ -\epsilon^3 g^2 \frac{\partial^4 u}{\partial \xi^2 \partial \tau^2} = 0.\end{aligned} \tag{6}$$

Throughout this work we shall assume that the field variable u and the scale parameter g can be expressed as asymptotic series in ϵ as

$$u = \sum_{k=1}^{\infty} \epsilon^k u_k(\xi, \tau), \quad g = 1 + \sum_{k=1}^{\infty} \epsilon^k g_k, \tag{7}$$

where the coefficient functions $u_k(\xi, \tau)$ and the constants g_k are to be determined from the solution of the field equations. Introducing (7) into the equation (6) and setting the coefficients of the like powers of ϵ equal to zero, the following sets of differential equations are obtained:

O(ϵ) equation:

$$(c^2 - 1)\frac{\partial^2 u_1}{\partial \xi^2} = 0. \tag{8}$$

O (ϵ^2) equation:

$$(c^2-1)\frac{\partial^2 u_2}{\partial\xi^2}-2c\frac{\partial^2 u_1}{\partial\xi\partial\tau}-\frac{c}{2}\frac{\partial^2(u_1^2)}{\partial\xi^2}-c^2\frac{\partial^4 u_1}{\partial\xi^4}=0. \tag{9}$$

O(ϵ^3) equation:

$$\begin{aligned}(c^2-1)\frac{\partial^2 u_3}{\partial\xi^2}-2c\frac{\partial^2 u_2}{\partial\xi\partial\tau}-2cg_1\frac{\partial^2 u_1}{\partial\xi\partial\tau}+\frac{\partial^2 u_1}{\partial\tau^2}-c\frac{\partial^2}{\partial\xi^2}(u_1u_2)\\ +\frac{1}{2}\frac{\partial^2(u_1^2)}{\partial\xi\partial\tau}-c^2\frac{\partial^4 u_2}{\partial\xi^4}+2c\frac{\partial^4 u_1}{\partial\xi^3\partial\tau}=0.\end{aligned} \tag{10}$$

O(ϵ^4) equation:

$$\begin{aligned}(c^2-1)\frac{\partial^2 u_4}{\partial\xi^2}-2c\frac{\partial^2 u_3}{\partial\xi\partial\tau}-2cg_1\frac{\partial^2 u_2}{\partial\xi\partial\tau}-2cg_2\frac{\partial^2 u_1}{\partial\xi\partial\tau}\\ +\frac{\partial^2 u_2}{\partial\tau^2}+2g_1\frac{\partial^2 u_1}{\partial\tau^2}-\frac{c}{2}\frac{\partial^2(u_2^2)}{\partial\xi^2}-c\frac{\partial^2(u_1u_3)}{\partial\xi^2}\\ +\frac{\partial^2}{\partial\xi\partial\tau}(u_1u_2)+\frac{g_1}{2}\frac{\partial^2}{\partial\xi\partial\tau}(u_1^2)-c^2\frac{\partial^4 u_3}{\partial\xi^4}\\ +2c\frac{\partial^4 u_2}{\partial\xi^3\partial\tau}+2cg_1\frac{\partial^4 u_1}{\partial\xi^3\partial\tau}-\frac{\partial^4 u_1}{\partial\xi^2\partial\tau^2}=0.\end{aligned} \tag{11}$$

2.1. **Solution of the field equations.** In this sub-section we shall present the solution to the field equations given in (8)-(11). The solution of the equation (8) yields

$$(c^2-1)\frac{\partial^2 u_1}{\partial\xi^2}=0. \tag{12}$$

In order to have a non zero solution for $u_1(\xi,\tau)$ we must have

$$c^2-1=0, \quad \text{or} \quad c=1, \tag{13}$$

where $u_1(\xi,\tau)$ is an unknown function whose governing equation will be obtained later.

Substituting the solution given in (13) into equation (9) it is seen that $u_2(\xi,\tau)$ remains an arbitrary function of its arguments. The remaining part of the equation (9) becomes

$$\frac{\partial}{\partial\xi}\left[\frac{\partial u_1}{\partial\tau}+\frac{1}{2}u_1\frac{\partial u_1}{\partial\xi}+\frac{1}{2}\frac{\partial^3 u_1}{\partial\xi^3}\right]=0. \tag{14}$$

The integration of (14) with respect to ξ yields the following Korteweg-deVries (KdV) equation

$$\frac{\partial u_1}{\partial\tau}+\frac{1}{2}u_1\frac{\partial u_1}{\partial\xi}+\frac{1}{2}\frac{\partial^3 u_1}{\partial\xi^3}=h_1(\tau). \tag{15}$$

Here $h_1(\tau)$ is an arbitrary function of its argument. Without loosing the generality of the problem, for the present case, we may choose it to be zero. Thus, for the first term in the perturbation expansion, the evolution equation reduces to the Korteweg-deVries (KdV) equation

$$\frac{\partial u_1}{\partial\tau}+\frac{1}{2}u_1\frac{\partial u_1}{\partial\xi}+\frac{1}{2}\frac{\partial^3 u_1}{\partial\xi^3}=0. \tag{16}$$

To obtain the solution for $O(\epsilon^3)$ equation, given in (10), we substitute (13) and (16) into (10) to have

$$-2\frac{\partial^2 u_2}{\partial\xi\partial\tau} - \frac{\partial^2}{\partial\xi^2}(u_1 u_2) - \frac{\partial^4 u_2}{\partial\xi^4} - 2g_1\frac{\partial^2 u_1}{\partial\xi\partial\tau}$$

$$+\frac{1}{4}\frac{\partial^2}{\partial\xi\partial\tau}(u_1^2) + \frac{3}{2}\frac{\partial^4 u_1}{\partial\xi^3\partial\tau} = 0. \tag{17}$$

Integrating (17) with respect to ξ and employing the previous reasoning, one gets the following evolution equation

$$\frac{\partial u_2}{\partial\tau} + \frac{1}{2}\frac{\partial}{\partial\xi}(u_1 u_2) + \frac{1}{2}\frac{\partial^3 u_2}{\partial\xi^3} = \frac{\partial}{\partial\xi}[S_1(u_1)], \tag{18}$$

where $S_1(u_1)$ is defined by

$$S_1(u_1) = \frac{g_1}{2}[\frac{1}{2}(u_1)^2 + \frac{\partial^2 u_1}{\partial\xi^2}] - \frac{1}{8}u_1\frac{\partial^2 u_1}{\partial\xi^2}$$

$$+\frac{1}{16}(\frac{\partial u_1}{\partial\xi})^2 + \frac{3}{4}\frac{\partial^2 u_1}{\partial\xi\partial\tau} - \frac{1}{24}u_1^3. \tag{19}$$

The equation (19) is linear in u_2 and contains the inhomogeneous term $S_1(u_1)$. Here, one should note that g_1 remains as an unknown constant and it should be determined from the requirement of removing possible secularities that might occur.

Finally, to obtain the solution for $O(\epsilon^4)$ equation we substitute (13), (16) and (18) into (11) to have

$$-2\frac{\partial^2 u_3}{\partial\xi\partial\tau} - \frac{\partial^2}{\partial\xi^2}(u_1 u_3) - \frac{\partial^4 u_3}{\partial\xi^4} - 2g_1\frac{\partial^2 u_2}{\partial\xi\partial\tau} - 2g_2\frac{\partial^2 u_1}{\partial\xi\partial\tau}$$

$$-\frac{1}{2}\frac{\partial^2}{\partial\xi^2}(u_2^2) + \frac{\partial^2 S_1(u_1)}{\partial\xi\partial\tau} + \frac{3}{2}\frac{\partial^4 u_2}{\partial\xi^3\partial\tau} + g_1\frac{\partial^4 u_1}{\partial\xi^3\partial\tau}$$

$$+\frac{1}{2}\frac{\partial^2}{\partial\xi\partial\tau}(u_1 u_2) - \frac{\partial^4 u_1}{\partial\xi^2\partial\tau^2} = 0. \tag{20}$$

Organizing the terms in equation (20), the following evolution equation is obtained

$$\frac{\partial u_3}{\partial\tau} + \frac{1}{2}\frac{\partial}{\partial\xi}(u_1 u_3) + \frac{1}{2}\frac{\partial^3 u_3}{\partial\xi^3} = S_2(u_1, u_2) \tag{21}$$

where $S_2(u_1, u_2)$ is defined by

$$S_2(u_1, u_2) = -g_1\frac{\partial u_2}{\partial\tau} - g_2\frac{\partial u_1}{\partial\tau} - \frac{1}{4}\frac{\partial}{\partial\xi}(u_2^2) + \frac{1}{2}\frac{\partial S_1(u_1)}{\partial\tau}$$

$$+\frac{3}{4}\frac{\partial^3 u_2}{\partial\xi^2\partial\tau} + \frac{g_1}{2}\frac{\partial^3 u_1}{\partial\xi^2\partial\tau} + \frac{1}{4}\frac{\partial}{\partial\tau}(u_1 u_2)$$

$$-\frac{1}{2}\frac{\partial^3 u_1}{\partial\xi\partial\tau^2}. \tag{22}$$

The equation (21) is linear in u_3 and contains the inhomogeneous term $S_2(u_1, u_2)$. Here one should note that g_1 and g_2 remain as some unknown constants and they should be determined from the requirement of removing possible secularities.

2.2. **Progressive wave solution.** In this sub-section we shall present a progressive wave solution to the evolution equations given in (16), (18) and (21). For this purpose we shall propose a solution of the form

$$u_\alpha = U_\alpha(\zeta), \quad \zeta = \alpha(\xi - \beta\tau), (\alpha = 1, 2, 3), \tag{23}$$

where α, β are two constants to be determined from the solution. Inserting (23) into (16) we have

$$-\beta U_1' + \frac{1}{2} U_1 U_1{}' + \frac{1}{2}\alpha^2 U_1''' = 0. \tag{24}$$

Here a prime denotes the differentiation of the corresponding quantity with respect to ζ. Integrating (24) with respect to ζ and employing the localization condition *i.e.*, U_1 and its various order derivatives vanish as $\zeta \to \pm\infty$, one gets

$$-\beta U_1 + \frac{1}{4} U_1^2 + \frac{1}{2}\alpha^2 U_1'' = 0. \tag{25}$$

Now, we shall seek a solution to the equation (25) of the form

$$U_1 = a \ \mathrm{sech}^2\zeta, \tag{26}$$

where a is the constant wave amplitude. Inserting (26) into (25) and equating the coefficient of various power of sechζ one obtains

$$\alpha = (\frac{a}{12})^{1/2}, \quad \beta = \frac{a}{6}. \tag{27}$$

Here, as usual, it is seen that the wave speed β is proportional to the wave amplitude a.

Inserting (23) and (26) into equation (18) we have

$$-\beta U_2' + \frac{1}{2}(U_1 U_2)' + \frac{1}{2}\alpha^2 U_2''' = S_1'(U_1). \tag{28}$$

Integrating (28) with respect to ζ and employing the localization condition we have

$$U_2'' + 4(3\mathrm{sech}^2\zeta - 1)U_2 = a(4g_1 - a)\mathrm{sech}^2\zeta + a^2\mathrm{sech}^4\zeta. \tag{29}$$

Employing the method of variation of parameters, the general solution of equation (29) may be given by

$$\begin{aligned} U_2 &= d_1\mathrm{sech}^2\zeta\tanh\zeta + (ag_1 - \frac{a^2}{12})\mathrm{sech}^2\zeta + \frac{d_2}{2}\tanh^2\zeta \\ &+ \frac{d_2}{8}\tanh^2\zeta(\cosh^2\zeta + \sinh^2\zeta) + (\frac{15}{8}d_2 - ag_1 + \frac{a^2}{4})\zeta\mathrm{sech}^2\zeta\tanh\zeta. \end{aligned} \tag{30}$$

Here d_1 and d_2 are unknown integration constants. It is seen that the last two terms in equation (30) cause to secularities(see, Sugimoto and Kakutani [3]). In order to avoid the secularities the coefficients of these terms must vanish, i.e.,

$$d_2 = 0, \quad -ag_1 + \frac{a^2}{4} = 0, \quad \text{or}, \quad g_1 = \frac{a}{4}. \tag{31}$$

Here g_1 represents $O(\epsilon)$ correction term to the wave speed. As a matter of fact, this result could be obtained by setting the coefficient of sech$^2\zeta$ in the right hand side of equation (29) equal to zero. In other words, the existence of sech$^2\zeta$ term in the right hand side of equation (29) leads to secularity, and in order to remove the secularity the coefficient of sech$^2\zeta$ should be set equal to zero. Here the constant d_1 remains undetermined. In order consider the effect of $O(\epsilon)$ term to higher order perturbation expansion one may set d_1 equal to zero. Thus, the final form of the solution for U_2 takes the following form

$$U_2 = \frac{a^2}{6}\mathrm{sech}^2\zeta. \tag{32}$$

Finally, inserting (23),(26) and (32) into equation (21) we have

$$-\beta U_3' + \frac{1}{2}(U_1U_3)' + \frac{1}{2}\alpha^2 U_3''' = T_2', \tag{33}$$

where T_2 is defined by

$$T_2 = \beta g_1 U_2 + \beta g_2 U_1 - \frac{1}{4}U_2^2 - \beta S_1 - \frac{3}{4}\alpha^2\beta U_2''$$

$$-\frac{g_1\alpha^2}{2}\beta U_1'' - \frac{\beta}{4}U_1U_2 - \frac{1}{2}\alpha^2\beta^2 U_1''. \tag{34}$$

Integrating (33) with respect to ζ and utilizing the localization condition the following equation is obtained

$$U_3'' + 4(3\mathrm{sech}^2\zeta - 1)U_2 = a(4g_2 - \frac{5a^2}{18})\mathrm{sech}^2\zeta + \frac{a^4}{4}\mathrm{sech}^4\zeta. \tag{35}$$

Again, in order to remove the secularities we must have

$$4g_2 - \frac{5a^2}{18} = 0, \quad \text{or} \quad g_2 = \frac{5a^2}{72}. \tag{36}$$

Here g_2 corresponds to $O(\epsilon^3)$ correction term to to the wave speed. Then, the particular integral for this order takes the following form

$$u_3 = \frac{a^4}{24}\mathrm{sech}^2\zeta. \tag{37}$$

Thus, the total solution up to and including $O(\epsilon^3)$ terms is given by

$$u = (\epsilon a + \epsilon^2\frac{a^2}{6} + \epsilon^3\frac{a^3}{24})\mathrm{sech}^2\zeta \tag{38}$$

where, in terms of real physical quantities, the phase function ζ is defined by

$$\zeta = \epsilon^{1/2}(\frac{a}{12})^{1/2}[x - t - \epsilon\frac{a}{6}t - \epsilon^2\frac{a^2}{24}t - \epsilon^3\frac{5a^3}{432}t]. \tag{39}$$

2.3. **Comparison of the result with exact solution.** In this sub-section we shall compare the result obtained here through the modified reductive perturbation method with the exact solution of the symmetric regularized-long-wave equation with quadratic nonlinearity. For that purpose, we shall seek a progressive wave solution to the equation (2) of the form

$$U = U(\eta), \quad \eta = p(x - vt), \tag{40}$$

where v is the speed of propagation and p is a constant to be determined from the solution. Introducing (40) into (2) we have

$$(v^2 - 1)U'' - \frac{v}{2}(U^2)'' - v^2p^2U^{(iv)} = 0. \tag{41}$$

Here a prime denotes the differentiation of the corresponding quantity with respect to η. Integrating (41) with respect to η and utilizing the localization condition, i.e. U and its various order derivatives vanish as $\eta \to \pm\infty$, we obtain

$$(v^2-1)U - \frac{v}{2}U^2 - v^2p^2U'' = 0. \tag{42}$$

This equation admits a solitary wave solution of the form

$$U = A\ \mathrm{sech}^2\eta, \tag{43}$$

where

$$p = (\frac{v^2-1}{4v^2})^{1/2}, \quad A = \frac{3(v^2-1)}{v}. \tag{44}$$

Motivated with the progressive wave solution obtained in the previous section, we shall set $p = \epsilon^{1/2}(a/12)^{1/2}$. Here ϵ is a small parameter measuring the weakness of nonlinearity and a is the amplitude of the lowest order weak wave. Expanding v and A into a power series in terms ϵ we have

$$v = 1 + \epsilon\frac{a}{6} + \epsilon^2\frac{a^2}{24} + \epsilon^3\frac{5a^3}{432} + ...$$

$$A = \epsilon a + \epsilon^2\frac{a^2}{6} + \epsilon^3\frac{a^3}{24} + ... \quad . \tag{45}$$

Inserting these approximations in (43) we have

$$U = (\epsilon a + \epsilon^2\frac{a^2}{6} + \epsilon^3\frac{a^3}{24})\mathrm{sech}^2\eta,$$

$$\eta = \epsilon^{1/2}(\frac{a}{12})[x - t - \epsilon\frac{a}{6}t - \frac{\epsilon^2a^2}{24}t - \frac{\epsilon^3 5a^3}{432}t]. \tag{46}$$

The approximation given in (46) is exactly the same with those given in (37) and (38). This result shows that the modified reductive perturbation method can be effectively used im examining the higher order terms in the perturbation expansion of the field quantities.

3. Concluding Remarks

The study of effects of higher order terms in the perturbation expansion of the field quantities through the use of the classical perturbation expansion method leads to some secularities. To remove such secularities various methods, like re-normalization method of Kodama and Tanuiti [4], multiple scale expansion method of Kraenkel et al [6], have been presented in the current literature. The result of the present report and those given in [8] and [9] proved that the " modified reductive perturbation method " presented by us is the most simplest and effective one. By use of this method, any order of the correction term may be obtained without any serious difficulties.

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