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THE $\chi_{\lambda}^{2F}-$ SUMMABLE SEQUENCES OF STRONGLY FUZZY NUMBERS

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ABSTRACT. We introduce the classes of $\chi_{\lambda}^{2F}(A,p)-$ summable sequences of strongly fuzzy numbers and give some relations between these classes. We also give a natural relationship between $\chi_{\lambda}^{2F}-$ summable sequences of strongly fuzzy numbers and strongly $\chi_{\lambda}^{2F}(A)-$ statistical convergence of sequences of fuzzy numbers.

Keywords: Fuzzy numbers, de la Vallee-Poussin mean, statistical convergence,analytic sequence, gai sequence.

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1. INTRODUCTION

Throughout the paper, a double sequence is denoted by $X = (X_{mn})$ of fuzzy numbers and denote $w^2(F)$ all double sequences of fuzzy numbers. Nanda [6] studied single sequence of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers form a complete metric space. In [2], Savas studied the concept double convergent sequences of fuzzy numbers. Savas [1] studied the classes of difference sequences of fuzzy numbers $c(\Delta,F)$ and $\ell_{\infty}(\Delta,F)$. In [3] Savas studied the concepts of strongly double $\left[V,\overline{\lambda}\right]-$ summable and double $S_{\overline{\lambda}}-$ convergent sequences for double sequences of fuzzy numbers. Recently, Esi [5] studied the concepts of strongly double $\overline{\lambda}(\Delta,F)-$ summable and $S^2_{\overline{\lambda}}(\Delta)-$ convergence for double sequences of fuzzy numbers.

Right through this paper w,χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write w^2 for the set of all complex sequences (x_{mn}), where $m,n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate-wise addition and scalar multiplication.

Some initial work on double sequence spaces is found in Bromwich [8]. Later on, they were investigated by Hardy [11], Moricz [15], Moricz and Rhoades [16], Basarir and

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Solankan [9], Tripathy [23], Turkmenoglu [25], and many others.

Let us define the following sets of double sequences:

$$\begin{aligned}
\mathcal{M}_u(t) &:= \left\{(x_{mn}) \in w^2 : sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty\right\}, \\
\mathcal{C}_p(t) &:= \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn} - l|^{t_{mn}} = 1 \text{ for some } l \in \mathbb{C}\right\}, \\
\mathcal{C}_{0p}(t) &:= \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1\right\}, \\
\mathcal{L}_u(t) &:= \left\{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty\right\}, \\
\mathcal{C}_{bp}(t) &:= \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t);
\end{aligned}$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denote the limit in the Pringsheim's sense. In that case, $t_{mn} = 1$ for all $m, n \in \mathbb{N}$; $\mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively.

Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [27,28] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her Ph.D thesis, Zelter [29] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [30] have recently introduced the statistical convergence and Cauchy the double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Next, Mursaleen [31] and Mursaleen and Edely [32] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M-$core for double sequences and determined those four dimensional matrices transforming every bounded double sequence $x = (x_{jk})$ into one whose core is a subset of the $M-$core of x. More recently, Altay and Basar [33] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also have examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Quite recently Basar and Sever [34] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and have examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [35] have studied the space $\chi_M^2(p, q, u)$ of double sequences and have given some inclusion relations, of late.

Spaces are strongly summable sequences and is discussed by Kuttner [37], Maddox [38], and others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [14] as an extension of the definition of strongly Cesàro summable sequences. Connor [39] further extended this definition to a definition of strong $A-$ summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ summability with respect to a modulus, and $A-$ statistical convergence. In [40] the

notion of convergence of double sequences is presented by a Pringsheim. Also, in [41]-[44], and [45] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Robison and Hamilton. In their work and throughout this paper, the four dimensional matrices and double sequences have real-valued entries unless specified otherwise. In this paper we extend a few results known in the literature for ordinary(single) sequence spaces to multiply sequence spaces.
The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see[7]).

A sequence $x = (x_{mn})$is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)! |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. Double gai sequences are denoted by χ^2. Let $\phi = \{all\, finite\, sequences\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \Im_{ij}$ for all $m, n \in \mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn}) (m, n \in \mathbb{N})$ are also continuous.

Orlicz[19] uses the idea of Orlicz function to construct the space (L^M). Lindenstrauss and Tzafriri [13] investigates Orlicz sequence spaces in more detail, and they prove that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p \, (1 \le p < \infty)$. Subsequently, different classes of sequence spaces are defined by Parashar and Choudhary [20], Mursaleen et al. [17], Bektas and Altin [10], Tripathy et al. [24], Rao and Subramanian [21], and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [12].

Recalling [19] and [12], an Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing, and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by subadditivity of M, then this function is called modulus function, defined by Nakano [18] and further discussed by Ruckle [22] and Maddox [14], and many others.

An Orlicz function M is said to satisfy the Δ_2- condition for all values of u if there exists a constant $K > 0$ such that $M(2u) \le KM(u) \, (u \ge 0)$. The Δ_2- condition is equivalent to $M(\ell u) \le K\ell M(u)$, for all values of u and for $\ell > 1$.

Lindenstrauss and Tzafriri [13] use the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \, for\, some\, \rho > 0 \right\},$$

The space ℓ_M with the norm

$$\|x\| = inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p \, (1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.
If X is a sequence space, we give the following definitions:

(i)X'= the continuous dual of X;

(ii)$X^{\alpha} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \, for\, each\, x \in X \right\};$

(iii)$X^{\beta} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn}\, is\, convegent, \, for\, each\, x \in X \right\};$

(iv)$X^{\gamma} = \left\{ a = (a_{mn}) : sup_{mn} \geq 1 \left| \sum_{m,n=1}^{M,N} a_{mn}x_{mn} \right| < \infty, for\, each\, x \in X \right\};$

(v)$let\, X\, be\, an FK - space \supset \phi; \, then\, X^f = \left\{ f(\Im_{mn}) : f \in X' \right\};$

(vi)$X^{\delta} = \left\{ a = (a_{mn}) : sup_{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty, \, for\, each\, x \in X \right\};$

$X^{\alpha}.X^{\beta}, X^{\gamma}$ are called $\alpha - (or Köthe - Toeplitz)$dual of $X, \beta - (or\, generalized - Köthe - Toeplitz)\, dual\, of\, X, \gamma - dual\, of\, X, \, \delta - dual\, of\, X\, respectively. X^{\alpha}$ is defined by Gupta and Kamptan [26]. It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) is introduced by Kizmaz [36] as follows

$$Z(\Delta) = \{ x = (x_k) \in w : (\Delta x_k) \in Z \}$$

for $Z = c, c_0$ and ℓ_{∞}, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.
Here c, c_0 and ℓ_{∞} denote the classes of convergent,null and bounded sclar valued single sequences respectively. The difference space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by Başar and Altay in [48] and in the case $0 < p < 1$ by Altay and Başar in [49]. The spaces $c(\Delta), c_0(\Delta), \ell_{\infty}(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k \geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}, (1 \leq p < \infty).$$

Later on the notion has been further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{ x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z \right\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$

2. Definition and Preliminaries

Let D be the set of all bounded intervals $A = \left[\underline{A}, \overline{A}\right]$ on the real line R. For $A, B \in D$, define $A \leq B$ if and only if $\underline{A} \leq \underline{B}$ and $\overline{A} \leq \overline{B}, d(A, B) = max\left\{\underline{A} - \underline{B}, \overline{A} - \overline{B}\right\}$.

Then it can be easily see that d defines a metric on D and (D, d) is complete metric space.

A fuzzy number is fuzzy subset of the real line $\mathbb{R}$ which is bounded, convex and normal. Let $L(\mathbb{R})$ denote the set of all fuzzy numbers which are upper semi continuous and have compact support, i.e if $X \in L(\mathbb{R})$ then for any $\alpha \in [0, 1], X^{\alpha}$ is compact where

$$X^{(\alpha)} = \begin{cases} t : X(t) \geq \alpha & \text{if } 0 < \alpha \leq 1, \\ t : X(t) > 0, & \text{if} \alpha = 0 \end{cases}$$

For each $0 < \alpha \leq 1$, the $\alpha-$ level set X^{α} is a nonempty compact subset of $\mathbb{R}$. The linear structure of $L(\mathbb{R})$ includes addition $X + Y$ and scalar multiplication λX, (λ a scalar)in terms of $\alpha-$ level sets, by $[X + Y]^{\alpha} = [X]^{\alpha} + [Y]^{\alpha}$ and $[\lambda X]^{\alpha} = \lambda [X]^{\alpha}$. for each $0 \leq \alpha \leq 1$.

The additive identity andmultiplicative identity of $L(\mathbb{R})$ are denoted by $\overline{0}$ and $\overline{1}$ respectively. The zero sequence of fuzzy numbers is denoted by $\overline{\theta}$.

Define a map $\overline{d} : L(\mathbb{R}) \times L(\mathbb{R}) \to \mathbb{R}$ by $\overline{d}(X, Y) = sup_{0 \leq \alpha \leq 1} d(X^{\alpha}, Y^{\alpha})$.

For $X, Y \in L(\mathbb{R})$ define $X \leq Y$ if and only if $X^{\alpha} \leq Y^{\alpha}$ for any $\alpha \in [0, 1]$. It is known that $\left(L(\mathbb{R}), \overline{d}\right)$ is a complete metric space.

A sequence $X = (X_{mn})$ of fuzzy numbers is a function X from the set $\mathbb{N}$ of natural numbers into $L(\mathbb{R})$. The fuzzy number X_{mn} denotes the value of the function at $m, n \in \mathbb{N}$ and is called the $(m, n)^{th}$ term of the sequence.

A metric on $L(R^n)$ is said to be translation invariant if $d(X + Z, Y + Z) = d(X, Y)$ for all $X, Y, Z \in L(R^n)$

A real sequence $X = (X_{mn})$ is said to be statistically convergent to 0 if,

$$\lim_{pq} \frac{1}{pq} \left| \left\{ m, n \leq p, q : ((m + n)! \, |X_{mn}|)^{1/m+n}, \bar{0} \right\} \right| = 0,$$

where the vertical bars denote the cardinality of the set which they enclose, in which case we write $S - limX = 0$.

Let $\lambda = (\lambda_{pq})$ be a nondecreasing sequence of positive real numbers tending to infinity with $\lambda_{11} = 1, \lambda_{p+1,q+1} \leq \lambda_{pq} + 1$.

The generalized de la Vallee-Poussin mean is defined by

$$t_{rs}(X) = \frac{1}{\lambda_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} X_{mn}$$

where $I_{rs} = [r, s - \lambda_{rs} + 1, r, s]$. A real sequence $X = (X_{mn})$ is said to be $\left(\chi^2_{\lambda}\right)$ − summable to a number 0 if $t_{rs}(X) \to 0$ as $r, s \to \infty$.

3. The summable sequences of strongly:

Let $A = (a^{mn}_{k\ell})$ be an infinite four dimensional matrix of fuzzy numbers and $p = (p_{mn})$ be a double analytic sequence of positive real numbers, i.e.,

$$0 < h < inf_{mn} p_{mn} \leq p_{mn} \leq sup_{mn} p_{mn} = H < \infty,$$

and let $X = X_{mn}$ be a sequence of fuzzy numbers. Then we write

$$A_{mn}(X)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_{k\ell}^{mn}\ \left((m+n)!\,|X_{mn}|\right)^{1/m+n}$$

if the series converges for each $m,n\in\mathbb{N}$. we now define

$$\chi_{\lambda}^{2F}(A,p)=\left\{X=(X_{mn})\in w^{2F}:\lim_{rs}\frac{1}{\lambda_{rs}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}d\left(A_{mn}(X),\bar{0}\right)^{p_{mn}}=0\right\},$$

$$\Lambda_{\lambda}^{2F}(A,p)=\left\{X=(X_{mn})\in w^{2F}:\sup_{rs}\ lim_{rs}\frac{1}{\lambda_{rs}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}d\left(A_{mn}(X),\bar{0}\right)^{p_{mn}}<\infty\right\}.$$

A sequence $X=(X_{mn})$ of fuzzy numbers is said to be strongly $\chi_{\lambda}^{2F}(A,p)-$ convergent to a fuzzy number 0 if there is a fuzzy number such that $X=(X_{mn})\in\chi_{\lambda}^{2F}(A,p)$. In this case we write $X_{mn}\to 0\left(\chi_{\lambda}^{2F}(A,p)\right)$.

If $p_{mn}=1$ for $m,n\in\mathbb{N}$, then we write the classes $\chi_{\lambda}^{2F}(A)$ and $\Lambda_{\lambda}^{2F}(A)$ in place of the classes $\chi_{\lambda}^{2F}(A,p)$ and $\Lambda_{\lambda}^{2F}(A,p)$, respectively.

In this section we examine some topological properties of these classes of sequence of fuzzy numbers and investigate some inclusion relations between them.

3.1. **Theorem.** (i) $\chi_{\lambda}^{2F}(A,p)\subset\Lambda_{\lambda}^{2F}(A,p)$ (ii) $\chi_{\lambda}^{2F}(A,p)$ and $\Lambda_{\lambda}^{2F}(A,p)$ are closed under the operations of addition and scalar multiplication if d is a translation invariant four dimensional metric.

Proof: (i) For $\chi_{\lambda}^{2F}(A,p)\subset\chi_{\lambda}^{2F}(A,p)$, we use the triangle inequality

$$d\left(A_{mn}(X),\bar{0}\right)^{p_{mn}}\leq\left[d\left(A_{mn}(X),0\right)+d(0,\bar{0})\right]^{p_{mn}}$$
$$\leq K\,d\left(A_{mn}(X),0\right)^{p_{mn}}+K\,max\left(1,|0|\right),$$

where $K=max\left(1,sup_{mn}p_{mn}<\infty\right)$. So, $X=(X_{mn})\in\Lambda_{\lambda}^{2F}(A,p)$.

Proof: (ii) We consider only $\chi_{\lambda}^{2F}(A,p)$. The others can be treated similarly.

Suppose that $X=(X_{mn}),Y=(Y_{mn})\in\chi_{\lambda}^{2F}(A,p)$. By combining Minknowski's inequality with property

$$d(X+Y,Z+W)\leq d(X,Z)+d(Y,W) \tag{1}$$

and

$$d(cX,cY)=|c|\,d(X,Y) \tag{2}$$

of the metric d we derive that

$$d\left(A_{mn}(X)+A_{mn}(Y),\bar{0}+\bar{0}\right)\leq d\left(A_{mn}(X),\bar{0}\right)+d\left(A_{mn}(Y),\bar{0}\right).$$

Therefore,

$$d\left(A_{mn}(X)+A_{mn}(Y),\bar{0}+\bar{0}\right)^{p_{mn}}\leq K\,d\left(A_{mn}(X),\bar{0}\right)^{p_{mn}}+K\,d\left(A_{mn}(Y),\bar{0}\right)^{p_{mn}}$$

where $K=max\left(1,sup_{mn}p_{mn}<\infty\right)$. This implies that

$$X+Y=(X_{mn})+(Y_{mn})\in\chi_{\lambda}^{2F}(A,p).$$

Let $\alpha=(\alpha_{mn})\in\chi_{\lambda}^{2F}(A,p)$ and $\alpha\in\mathbb{R}$. Then by taking into account properties (1) and (2) of the metric d,

$$d\left(A_{mn}(\alpha\,X),\alpha\,\bar{0}\right)^{p_{mn}}\leq|\alpha|^{p_{mn}}\,d\left(A_{mn}(\alpha),\bar{0}\right)^{p_{mn}}\leq max\left(1,|\alpha|\right)d\left(A_{mn}(X),\bar{0}\right)^{p_{mn}}$$

since $|\alpha|^{p_{mn}} \leq max\,(1,|\alpha|)$. Hence $\alpha\, X \in \chi^{2F}_{\lambda}(A,p)$. This completes the proof.

3.2. **Theorem.** The space $\chi^{2F}_{\lambda}(A,p)$ is a complete metric space with the metric $\rho(X,Y) = sup_{r,s,m,n}\left(\frac{1}{\lambda_{rs}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}} d\left(A_{mn}(X)-A_{mn}(Y)\right)^{p_{mn}}\right)^{1/p_{mn}}, 1 \leq p_{mn} < \infty.$

Proof: Let (X^{uv}) be a Cauchy sequence in $\chi^{2F}_{\lambda}(A,p)$, where

$$X^{uv} = (X^{uv}_{mn})_{mn} = \begin{pmatrix} x^{uv}_{11} & x^{uv}_{12} & x^{uv}_{13}\cdots & x^{uv}_{1n} & 0 \\ x^{uv}_{21} & x^{uv}_{22} & x^{uv}_{23}\cdots & x^{uv}_{2n} & 0 \\ \vdots & & & & \\ x^{uv}_{m1} & x^{uv}_{m2} & x^{uv}_{m3}\cdots & x^{uv}_{mn} & 0 \\ 0 & 0 & 0\cdots & 0 & 0 \end{pmatrix} \in \chi^{2F}_{\lambda}(A,p)\ for\, each\, u,v\in\mathbb{N}.$$ Then

$\rho\left(X^{uv},X^{st}\right) = sup_{r,s,m,n}\left(\frac{1}{\lambda_{rs}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}} d\left(A_{mn}\left(X^{uv}\right)-A_{mn}\left(X^{st}\right)\right)^{p_{mn}}\right)^{1/p_{mn}} \to 0$ as $u,v,s,t\to\infty$. Hence

$d\left(A_{mn}\left(X^{uv}\right)-A_{mn}\left(X^{st}\right)\right) = d\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a^{mn}_{k\ell}\left((m+n)!\left(X^{uv}_{mn}-X^{st}_{mn}\right)\right)^{1/m+n}\right) \to 0$ as $u,v,s,t\to\infty$ for each $m,n\in\mathbb{N}$.

$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a^{mn}_{k\ell}\left(lim_{uvst} d\left((m+n)!\left(X^{uv}_{mn}-X^{st}_{mn}\right)\right)^{1/m+n}\right) = 0$, for each $m,n\in\mathbb{N}$. Hence $lim_{uvst} d\left(X^{uv}_{mn}-X^{st}_{mn}\right) = 0$, for each $m,n\in\mathbb{N}$.

Therefore $\left(X^{uv}\right)_{uv}$ is a Cauchy sequence in $L(\mathbb{R})$. Since $L(\mathbb{R})$ is complete, it is convergent, $lim_{uv} X^{uv}_{mn} = X_{mn}$ (say), for each $m,n\in\mathbb{N}$. Since $\left(X^{uv}\right)_{uv}$ is a Cauchy sequence, for each $\epsilon > 0$, there exists $p_0q_0 = p_0q_0(\epsilon)$ such that $\rho\left(X^{uv},X^{st}\right) < \epsilon$ for all $u,v,s,t \geq p_0q_0$. So, we have

$lim_{s,t\to\infty} d\left[\left(A_{mn}\left(X^{uv}\right)-A_{mn}\left(X^{st}\right)\right)^{p_{mn}}\right] = d\left[\left(A_{mn}\left(X^{uv}\right)-A_{mn}\left(X^{st}\right)\right)\right]^{p_{mn}} < \epsilon^{p_{mn}}$, for all$u,v \geq p_0q_0$.

This implies that $\rho\left(X^{uv},X^{st}\right) < \epsilon$, for all $u,v \geq p_0q_0$, that is $X^{uv} \to X$ as $u,v\to\infty$, where $X = (X_{mn})$. Since

$\frac{1}{\lambda_{mn}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}} d\left[A_{mn}(X),\bar{0}\right]^{p_{mn}} \leq$

$2^{p_{mn}}\frac{1}{\lambda_{mn}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}}\left\{d\left[A_{mn}\left(X^{p_0q_0}\right),\bar{0}\right]^{p_{mn}} + d\left[A_{mn}\left(X^{p_0q_0}\right)-A_{mn}(X)\right]^{p_{mn}}\right\} \to 0$ as $m,n\to\infty$.

So, we obtain $X = (X_{mn}) \in \chi^{2F}_{\lambda}(A,p)$. Therefore $\chi^{2F}_{\lambda}(A,p)$ is a complete metric space.

It can be also shown that $\chi^{2F}_{\lambda}(A,p)$ is a complete metric space with the metric

$\rho^1(X,Y) = sup_{r,s,m,n}\left(\frac{1}{\lambda_{rs}}\sum_{m\in I_{rs}}\sum_{n\in I_{rs}} d\left(A_{mn}(X)-A_{mn}(Y)\right)^{p_{mn}}\right)$,

$0 < p_{mn} < 1$. This completes the proof.

3.3. **Theorem.** Let $0 < p_{mn} \leq q_{mn}$ and $\left(\frac{p_{mn}}{q_{mn}}\right)$ be double analytic. Then $\chi^{2F}_{\lambda}(A,q) \subset \chi^{2F}_{\lambda}(A,p)$.

Proof: Let $X = (X_{mn}) \in \chi^{2F}_{\lambda}(A,q)$ and $w_{mn} = d\left[A_{mn}(X),\bar{0}\right]^{q_{mn}}$ and $\gamma_{mn} = \left(\frac{p_{mn}}{q_{mn}}\right)$ for all $m,n\in\mathbb{N}$. Then $0 < \gamma_{mn} \leq 1$ for all $m,n\in\mathbb{N}$. Let $0 < \gamma \leq \gamma_{mn} \leq 1$ for all $m,n\in\mathbb{N}$. We define the sequences (u_{mn}) and (v_{mn}) as follows: For $w_{mn} \geq 1$, let $u_{mn} = w_{mn}$ and $v_{mn} = 0$ and for $w_{mn} < 1$, let $u_{mn} = 0$ and $v_{mn} = w_{mn}$. Then it is clear that for all

$m,n \in \mathbb{N}$, we have $w_{mn} = u_{mn} + v_{mn}$ and $w_{mn}^{\gamma_{mn}} = u_{mn}^{\gamma_{mn}} + v_{mn}^{\gamma_{mn}}$. Now it follows that $u_{mn}^{\gamma_{mn}} \leq u_{mn} \leq w_{mn}$ and $v_{mn}^{\gamma_{mn}} \leq v_{mn}^{\gamma}$. Therefore
$\frac{1}{\lambda_{rs}} \sum_{m\in I_{rs}} \sum_{n\in I_{rs}} w_{mn}^{\gamma_{mn}} = \frac{1}{\lambda_{rs}} \sum_{m\in I_{rs}} \sum_{n\in I_{rs}} (u_{mn}+v_{mn})^{\gamma_{mn}}$
$\leq \frac{1}{\lambda_{rs}} \sum_{m\in I_{rs}} \sum_{n\in I_{rs}} w_{mn} + \frac{1}{\lambda_{rs}} \sum_{m\in I_{rs}} \sum_{n\in I_{rs}} v_{mn}^{\gamma}$.
Now for each rs,

$$\begin{aligned} \frac{1}{\lambda_{rs}} \sum_{m\in I_{rs}} \sum_{n\in I_{rs}} v_{mn}^{\gamma} &= \sum_{m\in I_{rs}} \sum_{n\in I_{rs}} \left(\frac{1}{\lambda_{rs}} v_{mn}^{\gamma}\right)^{\gamma} \left(\frac{1}{\lambda_{rs}}\right)^{1-\gamma} \\ &\leq \left[\sum_{m\in I_{rs}} \sum_{n\in I_{rs}} \left(\left(\frac{1}{\lambda_{rs}} v_{mn}^{\gamma}\right)^{\gamma}\right)^{1/\gamma}\right] \left[\sum_{m\in I_{rs}} \sum_{n\in I_{rs}} \left(\left(\frac{1}{\lambda_{rs}}\right)^{1-\gamma}\right)^{1/1-\gamma}\right]^{1/1-\gamma} \\ &= \left(\frac{1}{I_{rs}} v_{mn}\right)^{\gamma} \end{aligned}$$

and so
$\frac{1}{\lambda_{rs}} \sum_{m\in I_{rs}} \sum_{n\in I_{rs}} w_{mn}^{\gamma_{mn}} \leq \frac{1}{\lambda_{rs}} \sum_{m\in I_{rs}} \sum_{n\in I_{rs}} w_{mn} + \left(\frac{1}{\lambda_{rs}} \sum_{m\in I_{rs}} \sum_{n\in I_{rs}} v_{mn}\right)^{\gamma}$.
Hence $X = (X_{mn}) \in \chi_{\lambda}^{2F}(A,p)$, i.e., $\chi_{\lambda}^{2F}(A,q) \subset \chi_{\lambda}^{2F}(A,p)$. This completes the proof.

4. χ^2- Statistical convergence:

A sequence $X = (X_{mn})$ is said to be statistically convergent to 0 if,

$$lim_{pq} \frac{1}{pq} \left| \left\{ (mn) \leq (pq) : d\left(((m+n)! \, |X_{mn}|)^{1/m+n}, \bar{0} \right) \right\} \right| = 0,$$

The set of all statistically convergence sequences of fuzzy numbers is denoted by χ_{S}^{2F}.

4.1. **Definition.** A sequence $X = (X_{mn})$ of fuzzy numbers is said to be $\chi_{\lambda S}^{2F}(A)-$ statistically convergent to a fuzzy number 0,

$$lim_{rs} \frac{1}{rs} \left| \{ (mn) \in I_{rs} : d(A_{mn}(X), \bar{0}) \} \right| = 0$$

where $A_{mn}(X) = ((m+n)! \, |X_{mn}|)^{1/m+n}$.

The set of all $\chi_{\lambda s}^{2F}(A)-$ statistically χ^2- convergent sequences of fuzzy numbers is denoted by $\chi_{\lambda s}^{2F}(A)$ In this case we write $X_{mn} \to 0 \left(\chi_{\lambda s}^{2F}(A)\right)$

Now we give the relations between $\chi_{\lambda s}^{2F}(A)-$ statistical convergence and strongly $\chi_{\lambda}^{2F}(A,p)-$ convergence.

4.2. **Theorem.** The following statement are valid: (i) $\chi_{\lambda}^{2F}(A,p) \subset \chi_{\lambda s}^{2F}(A)$, (ii) If $X = (X_{mn}) \in \Lambda^{2F} \bigcap \chi_{\lambda s}^{2F}(A)$, then $X = (X_{mn}) \in \chi_{\lambda}^{2F}(A,p)$, (iii) $\Lambda^{2F}(A) \bigcap \chi_{\lambda s}^{2F}(A) = \chi^{2F}(A) \bigcap \chi_{\lambda}^{2F}(A,p)$, where

$$\Lambda^{2F}(A) = \left\{ X = (X_{mn}) \in w^{2F} : sup_{mn} d\left[A_{mn}(X), \bar{0}\right] < \infty \right\}$$

Proof: (i) Let $X = (X_{mn}) \in \chi_{\lambda}^{2F}(A,p)$. Then we have

$\frac{1}{\lambda_{rs}} \sum_{m\in I_{rs}} \sum_{n\in I_{rs}} d\left[A_{mn}(X), \bar{0}\right]^{p_{mn}} \geq \frac{1}{\lambda_{rs}} \left| \{ (mn) \in I_{rs} : d(A_{mn}(X), \bar{0}) \geq \epsilon \} \right| \cdot min\left(\epsilon^{h}, \epsilon^{H}\right)$.
Hence $x = (X_{mn}) \in \chi_{\lambda}^{2F}(A)$.

(ii) Suppose that $X = (X_{mn}) \in \Lambda^{2F} \bigcap \chi_{\lambda}^{2F}(A)$. Since $X = (X_{mn}) \in \Lambda^{2F}$, we can write $d(A_{mn}(X), \bar{0}) \leq T$ for all $m,n \in \mathbb{N}$. Given $\epsilon > 0$, let

$G_{rs} = \{(mn) \in I_{rs} : d\left(A_{mn}\left(X\right), \bar{0}\right) \geq \epsilon\}$ and $H_{rs} = \{(mn) \in I_{rs} : d\left(A_{mn}\left(X\right), \bar{0}\right) < \epsilon\}$.
Then we have

$$\begin{aligned}\frac{1}{\lambda_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} d\left[A_{mn}\left(X\right), \bar{0}\right]^{p_{mn}} &= \frac{1}{\lambda_{rs}} \sum_{m \in G_{rs}} \sum_{n \in G_{rs}} d\left[A_{mn}\left(X\right), \bar{0}\right]^{p_{mn}} \\ &\quad + \frac{1}{\lambda_{rs}} \sum_{m \in H_{rs}} \sum_{n \in H_{rs}} d\left[A_{mn}\left(X\right), \bar{0}\right]^{p_{mn}} \\ &\leq max\left(T^h, T^H\right) \frac{1}{\lambda_{rs}} \left|G_{rs}\right| + max\left(\epsilon^h . \epsilon^H\right)\end{aligned}$$

Taking the limit as $\epsilon \to 0$ and $r, s \to \infty$, it follows that $X = (X_{mn}) \in \chi_{\lambda}^{2f}(A, p)$.

(iii) Follows from (i) and (ii). This completes the proof.

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