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Periodic, kink and bell shape wave solutions to the Caudrey-Dodd-Gibbon (CDG) equation and the Lax equation

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Abstract

This paper addresses the implementation of the new generalized (G'/G)- expansion method to the Caudrey-Dodd-Gibbon (CDG) equation and the Lax equation which are two special case of the fifth order KdV (fKdV) equation. The method works well to derive a new variety of travelling wave solutions with distinct physical structures such as soliton, singular soliton, kink, singular kink, bell-shaped soltion, anti-bell-shaped soliton, periodic, exact periodic and bell type solitary wave solutions. Solutions provided by this method are numerous comparing to other methods. To understand the physical aspects and importance of the method, solutions have been graphically simulated. Our results unquestionably disclose that new generalized (G'/G)-expansion method is incredibly influential mathematical tool to work out new solutions of various types of nonlinear partial differential equations arises in the fields of applied sciences and engineering.

Keywords: Caudrey-Dodd-Gibbon (CDG) equation, Lax equation, New generalized (G'/G)-expansion method.
2010 MSC: 35C07, 35C08, 35Q53, 37K10.

1. Introduction

It is well observed that almost every natural phenomena is nonlinear and mathematically which appears in the form of nonlinear evolution equations (NLEEs).The studies of NLEEs, a special type of nonlinear partial

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differential equations (NPDEs), becomes one of the most exciting and extremely active areas of research and investigation because several problems in various scientific and engineering fields, such as solid state physics, chemical physics, plasma physics, optics, biology, chemical kinematics, geochemistry, fluid mechanics and hydrodynamics are frequently describe by NLEEs. To understand the internal mechanism of these problems, finding the exact traveling wave solutions is becoming more and more fascinating day-by-day in nonlinear science. But there is not any integrated method which could be utilized to deal with all types of NLEEs. That is why a variety of efficient and reliable methods have been developed. For example, the Painleve expansion method [1], the inverse scattering method [2,3], the Darboux transformation method [4,5], the Cole-Hopf transformation method [6,7] the Jacobi elliptic function method [8,9], the HirotaâĂŹs bilinear transformation method [10,11], the Backlund transformation method [12,13], the sineâĂŞcosine function method [14], the tanh method [15-17], the improved F-expansion method [18], the tanh-coth method [19], the exp-function method [20], the exp the exp $(-\phi(\xi))$-expansion method [21], the modified simple equation method [22, 23], the (G'/G)-expansion method [24,25], the novel (G'/G)-expansion method [26], the improved (G'/G)-expansion method [27], the generalized (G'/G)-expansion method [28,29], the double $(G'/G, 1/G)$- expansion method [30] , the modified sineâĂŞcosine function method [31], the canonical transformation method [32], the compatible transform method [33], the new generalized ϕ^6-model expansion method [34], the homotopy analysis method [35] and so on.

1.the integrable nonlinear fifth order Caudrey-Dodd-Gibbon (CDG) equation is of the form [30,31]

$$u_t + u_{xxxxx} + 30uu_{xxx} + 30u_x u_{xx} + 180u^2 u_x = 0, \tag{1}$$

2. the integrable nonlinear fifth order Lax equation is of the form [32]

$$u_t + u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x = 0. \tag{2}$$

Although these two equations are being alike but they are being dissimilar in their coefficients of the derivatives. Moreover, these two equations have also been shown to be related to the integrable cases of the He ‘non-Heiles [39] system. However, Eq. (1) and Eq. (2) have been studied in a series of paper [34-48]. The investigation of different types of solutions of Eq. (1) and Eq. (2) has been performed by Wazwaz using several methods namely, the tanh method [40-42], the sine-cosine method [42], the extended tanh method [43], the tanh-coth method [38,44], Hirota's direct method combined with the simplified Hereman method [44] and Hirota's bilinear method [45,46] and the obtained solutions are periodic, soliton and multiple soliton etc. Moreover, exact travelling wave solutions of CDG equation and Lax equation acquired by Bilige and Chaolu applying the extended simplest equation method [47]. Furthermore, solutions of Eq. (1) examined by Salas [48] using the projective Riccati equation method, Xu *et al.* [49] employing the exp-function method, Go‘mez and Salas [50] utilizing the generalized tanh-coth method, Jin [51] applying the variational iteration method, Naher*et al.* [25] implementing the (G'/G)-expansion method and Bisaws*et al.* [52] using the modified F-expansion method, exp-function method as well as the (G'/G) method. Also, solutions of Eq. (2) investigated by Abbasbandy and Zakaria [53] utilizing the homotopy analysis method and Go‘mez [54] using the generalized extended tanh method. However, no one studied the solutions to the aforesaid equations through the generalized (G'/G)-expansion method.

In this paper we aim is to investigate Eq. (1) and Eq. (2) using the generalized (G'/G)-expansion method to explore more exact solutions which include new periodic, soliton and kink solutions.

This paper is organized as follows: In Section 2, we will review briefly the generalized (G'/G)-expansion method.In Section 3, we present the application of the methods to Eq. (1) and Eq. (2)and the obtained solutions. In Section 4, we give the physical and graphical presentation of the obtained results. Finally, In Section 5, conclusions are drawn.

2. Description of the New Generalized (G'/G)-expansion Method

Let us consider a general nonlinear PDE in the form

$$P\left(u,\, u_t,\, u_x,\, u_{tt},\, u_{tx},\, u_{xx}, \ldots\right) = 0, \tag{3}$$

where $u = u(x,t)$ is an unknown function, P is a polynomial in $u(x,t)$ and its partial derivatives in which the highest order derivatives and the nonlinear terms are involved.The main steps of the generalized (G'/G)expansion method are as follows:

Step 1: We suppose that the combination of real variables x and t by a variable ξ as follows:

$$u\,(x,\, t) = u(\xi), \xi = x - ct \tag{4}$$

where c is the speed of the traveling wave. The traveling wave transformation (4)allows us to reduce equation (3) to an ODE for $u = u(\xi)$ in the form

$$R\left(u,\, u^{'},\, u^{''}, u^{'''}, \ldots\right) = 0, \tag{5}$$

where R is a function of $u(\xi)$and the superscripts indicate the ordinary derivatives withrespect to ξ.

Step 2: In many instances, equation (5) can be integrated term by term one or more times, yielding constants of integration, which can be set equal to zero for straightforwardness.

Step 3: We assume that the traveling wave solution of equation (5) can be expressed as follows:

$$u\,(\xi) = \sum_{k=0}^{N} a_k (d+Y)^k + \sum_{k=1}^{N} b_k (d+Y)^{-k}, \tag{6}$$

where either $a_N$or $b_N$may be zero, but both $a_N$and b_N cannot be zero at a time, a_k $(k = 0,\ 1,\ 2,\ 3,\ \ldots N)$, $b_k\,(k = 1,\ 2,\ 3, \ldots N)$and d are arbitrary constants to be determined later and $Y(\xi)$ is given by

$$Y(\xi) = (G'/G), \tag{7}$$

where $G = G(\xi)$ satisfies the following auxiliary nonlinear ordinary differential equation

$$AGG^{''} - BGG^{'} - EG^2 - C\left(G^{'}\right)^2 = 0, \tag{8}$$

where prime indicates the derivative with respect to ξ and $A,\ BC,\ E$ are real parameters.

Step 4: The positive integer N can be determined by using the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in (5).

Step 5: Substituting equations (6) and (8) including equation (7) into equation (5) together with the value of N attained in Step 4, we reach a polynomial in $(d+Y)^N$, $(N = 0,\ 1,\ 2, \ldots)$ and $(d+Y)^{-N}$, $(N = 1,\ 2, \ldots)$. We set each coefficient of the resulting polynomial to zero, yield an over-determined set of algebraic equations for a_k ($k = 0,\ 1,\ 2, \ldots, N$),
$b_k\,(k = 1,\ 2, \ldots, N)$, d and c.

Step 6: We stare that the values of the constants can be determined by solving the algebraic equations achieved in Step 5. Since the general solution of (7) is in general known, inserting the value of $a_k\,(k = 0,\ 1,\ 2, \ldots, N)$, $b_k\,(k = 1,\ 2, \ldots, N)$, d and c into (6) yields the comprehensive and newly produced exact traveling wave solutions of the nonlinear partial differential equation (3).

Step 7: By using the general solution of equation (8), we admit the following solution of equation (7).

Family 1: When B= 0, ψ=A−C and Ω= ψE>0

$$Y(\xi) = \left(\frac{G'}{G}\right) = \frac{\sqrt{\Omega}}{\psi}\frac{r\sinh(\frac{\sqrt{\Omega}}{\psi}\ \xi)+s\cosh(\frac{\sqrt{\Omega}}{\psi}\ \xi)}{r\cosh(\frac{\sqrt{\Omega}}{\psi}\xi)+s\sinh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)}. \tag{9}$$

Family 2: When B= 0, ψ=A−C and Ω= ψE<0,

$$Y(\xi) = \left(\frac{G'}{G}\right) = \frac{\sqrt{-\Omega}}{\psi}\frac{-r\sin(\frac{\sqrt{-\Omega}}{\psi}\ \xi)+s\cos(\frac{\sqrt{-\Omega}}{\psi}\ \xi)}{r\cos(\frac{\sqrt{-\Omega}}{\psi}\xi)+s\sin\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)}. \tag{10}$$

Family 3: When B?0, ψ=A−C and Δ=B^2+4E (A−C) >0,

$$Y(\xi) = \left(\frac{G'}{G}\right) = \frac{B}{2\psi}+\frac{\sqrt{\Delta}}{2\psi}\frac{r\sinh(\frac{\sqrt{\Delta}}{2\psi}\ \xi)+s\cosh(\frac{\sqrt{\Delta}}{2\psi}\ \xi)}{r\cosh(\frac{\sqrt{\Delta}}{2\psi}\xi)+s\sinh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)}. \tag{11}$$

Family 4: When $B \neq 0$, ψ=A−C and Δ=B^2+4E $(A-C) < 0$,

$$Y(\xi) = \left(\frac{G'}{G}\right) = \frac{B}{2\psi}+\frac{\sqrt{-\Delta}}{2\psi}\frac{-r\sin(\frac{\sqrt{-\Delta}}{2\psi}\ \xi)+s\cos(\frac{\sqrt{-\Delta}}{2\psi}\ \xi)}{r\cos(\frac{\sqrt{-\Delta}}{2\psi}\xi)+s\sin\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)}. \tag{12}$$

Family 5: When $B \neq 0$, ψ=A−C and Δ=B^2+4E (A−C) = 0,

$$Y(\xi) = \left(\frac{G'}{G}\right) = \frac{B}{2\psi}+\frac{s}{r+s\xi}. \tag{13}$$

3. Applications

In this section, the new generalized (G'/G)-expansion method has been put in use to examine travelling wave solutions of the nonlinear fifth order Caudrey-Dodd-Gibbon (CDG) and Lax equations.

Example 3.1. *The CDG equation* (1) *can be rewritten as*

$$u_t + \frac{\partial}{\partial x}\left(u_{xxxx} + 30uu_{xx} + 60u^3\right) = 0. \tag{14}$$

Under the transformation $u(x,\, t) = u(\xi)$, $\xi = x - ct$, *Eq.* (14) *reduces to the ordinary differential equation*

$$-cu'^{+}\left(u^{(iv)} + 30uu'' + 60u^3\right)' = 0. \tag{15}$$

On integrating (15) *with respect to* ξ *once and letting the constant of integration to zero, we obtain*

$$-cu + \left(u^{(iv)} + 30uu'' + 60u^3\right) = 0. \tag{16}$$

According to the method described in Section 2, and after balancing we obtain $N = 2$. *Therefore, we seek solutions to* (16) *in the form*

$$u(\xi) = a_0 + a_1(d+Y) + a_2(d+Y)^2 + b_1(d+Y)^{-1} + b_2(d+Y)^{-2} \tag{17}$$

where $a_0,\ a_1, a_2, b_1, b_2$ *and* d *are arbitrary constants to be determined later.*

Now substituting Eq.(17) *into Eq.* (16) *and using* (7) *and* (8)*, the left hand side of Eq.* (16) *is translated into the polynomials in* $(d{+}Y)^N$, $(N{=}\ 0,\ 1,\ 2,\ \ldots)$ *and* $(d{+}Y)^{-N}$, $(N{=}\ 1,\ 2,\ \ldots)$*. Equating the coefficients of these polynomials to zero, we obtain an algebraic system (for simplicity, we leave out the displaying of the equations) with respect to* $a_0,\ a_1, a_2, b_1, b_2,\ c$ *and* d*.*

Solving the system of algebraic equations with the aid of the Maple 17 yields the following families of values of $a_0,\ a_1, a_2, b_1, b_2,\ c$ *and* d*.*

Case 1*:*

$$a_0 = -\frac{d^2\psi^2 + (Bd - E)\psi}{A^2}, a_1 = \frac{2d\psi^2 + B\psi}{A^2}, a_2 = -\frac{\psi^2}{A^2},$$
$$b_1 = 0, \qquad b_2 = 0, \qquad c = \frac{\left(B^2 + 4\Omega\right)^2}{A^4}, \qquad d = d, \tag{18}$$

where $\psi{=}A{-}C, \Omega = E\psi,\ d, A,\ B, C$ *and* E *are free parameters.*

Case 2:

$$a_0 = -\frac{d^2\psi^2 + (Bd - E)\psi}{A^2}, \qquad a_1 = 0, \qquad a_2 = 0,$$
$$b_1 = \frac{d\psi\left(2d^2\psi + 3Bd - 2E\right) + B(Bd - E)}{A^2},$$
$$b_2 = -\frac{d\psi\left(d^3\psi + 2d^2B - 2dE\right) + (Bd - E)^2}{A^2}, \tag{19}$$
$$c = \frac{\left(B^2 + 4\Omega\right)^2}{A^2}, \qquad d = d,$$

where $\psi{=}A{-}C, \Omega = E\psi,\ d, A,\ B, C$ *and* E *are free parameters.*

Case 3:

$$a_0 = \frac{-d\psi\left(d\psi + B\right) + \frac{1}{2}\left(1 - \frac{\sqrt{105}}{15}\right)\Omega - \frac{1}{8}\left(1 + \frac{\sqrt{105}}{15}\right)B^2}{A^2}, a_1 = 0, \quad a_2 = 0,$$
$$b_1 = \frac{d\psi\left(2d^2\psi + 3Bd - 2E\right) + B(Bd - E)}{A^2},$$
$$b_2 = -\frac{d\psi\left(d^3\psi + 2d^2B - 2dE\right) + (Bd - E)^2}{A^2}, \tag{20}$$
$$c = \frac{(11 + \sqrt{105})\left(B^2 + 4\Omega\right)^2}{8A^4}, \qquad d = d,$$

where $\psi{=}A{-}C, \Omega = E\psi,\ d, A,\ B, C$ *and* E *are free parameters.*

Case 4:

$$a_0 = \frac{-d\psi(d\psi{+}B){+}\frac{1}{2}\left(1{-}\frac{\sqrt{105}}{15}\right)\Omega{-}\frac{1}{8}\left(1{+}\frac{\sqrt{105}}{15}\right)B^2}{A^2}, a_1 = \frac{2d\psi^2{+}B\psi}{A^2}, a_2 = -\frac{\psi^2}{A^2},$$
$$b_1 = 0, \qquad b_2 = 0, \qquad c = \frac{(11 + \sqrt{105})\left(B^2 + 4\Omega\right)^2}{8A^4}, \quad d = d, \tag{21}$$

where $\psi{=}A{-}C, \Omega = E\psi,\ d, A,\ B, C$ *and* E *are free parameters.*

Case 5:

$$a_0=\frac{(B^2+4\Omega)}{2A^2},a_1=0,a_2=-\frac{\psi^2}{A^2},$$
$$b_1=0,\quad b_2=-\frac{(B^2+4\Omega)^2}{16A^2\psi^2},\quad c=\frac{16(B^2+4\Omega)^2}{A^4},\quad d=\ -\frac{B}{2\psi},\tag{22}$$

where $\psi=A-C,\Omega=E\psi$, *A*, *B*, *C and E are free parameters.*

Case 6:

$$a_0=\frac{\sqrt{105}(B^2+4\Omega)}{30A^2},a_1=0,a_2=-\frac{\psi^2}{A^2},$$
$$b_1=0,\quad b_2=-\frac{(B^2+4\Omega)^2}{16A^2\psi^2},\quad c=\frac{2(11-\sqrt{105})(B^2+4\Omega)^2}{A^4},$$
$$d=\ -\frac{B}{2\psi},\tag{23}$$

where $\psi=A-C,\Omega=E\psi$, *A*, *B*, *C and E are free parameters.*

For Case 1:

From Case 1 putting the values of constants into Eq. (17) *and combining with Eqs.* (9) *to* (12) *and simplifying, we attain following traveling wave solutions for* $r=0$ *but* $s\neq 0$ *respectively*

$$u1_1(\xi)=\alpha_1+\alpha_2\left[d+\frac{\sqrt{\Omega}}{\psi}\coth\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]-\frac{\psi^2}{A^2}\left[d+\frac{\sqrt{\Omega}}{\psi}\coth\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^2,$$
$$u1_2(\xi)=\alpha_1+\alpha_2\left[d+\frac{\sqrt{-\Omega}}{\psi}\cot\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]-\frac{\psi^2}{A^2}\left[d+\frac{\sqrt{-\Omega}}{\psi}\cot\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^2,$$
$$u1_3(\xi)=\alpha_1+\alpha_2\left[d+\frac{B}{2\psi}+\frac{\sqrt{\Delta}}{2\psi}\coth\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]-\frac{\psi^2}{A^2}\left[d+\frac{B}{2\psi}+\frac{\sqrt{\Delta}}{2\psi}\coth\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^2,$$
$$u1_4(\xi)=\alpha_1+\alpha_2\left[d+\frac{B}{2\psi}+\frac{\sqrt{-\Delta}}{2\psi}\cot\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]-\frac{\psi^2}{A^2}\left[d+\frac{B}{2\psi}+\frac{\sqrt{-\Delta}}{2\psi}\cot\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^2,$$

where $\alpha_1=-\frac{d^2\psi^2+(Bd-E)\psi}{A^2}$,$\alpha_2=\frac{2d\psi^2+B\psi}{A^2}$ *and* $\xi=x-\frac{(B^2+4\Omega)^2}{A^4}t$.

In similar fashion, substituting the values of the constants arranged in Eq. (18) *into Eq.* (17), *as well as* (9) *to* (12) *and simplifying, we attain following traveling wave solutions for* $s=0$ *but* $r\neq 0$ *respectively*

$$u1_5(\xi)=\alpha_1+\alpha_2\left[d+\frac{\sqrt{\Omega}}{\psi}\tanh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]-\frac{\psi^2}{A^2}\left[d+\frac{\sqrt{\Omega}}{\psi}\tanh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^2,$$
$$u1_6(\xi)=\alpha_1+\alpha_2\left[d-\frac{\sqrt{-\Omega}}{\psi}\tan\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]-\frac{\psi^2}{A^2}\left[d-\frac{\sqrt{-\Omega}}{\psi}\tan\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^2,$$
$$u1_7(\xi)=\alpha_1+\alpha_2\left[d+\frac{B}{2\psi}+\frac{\sqrt{\Delta}}{2\psi}\tanh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]-\frac{\psi^2}{A^2}\left[d+\frac{B}{2\psi}+\frac{\sqrt{\Delta}}{2\psi}\tanh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^2,$$
$$u1_8(\xi)=\alpha_1+\alpha_2\left[d+\frac{B}{2\psi}-\frac{\sqrt{-\Delta}}{2\psi}\tan\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]-\frac{\psi^2}{A^2}\left[d+\frac{B}{2\psi}-\frac{\sqrt{-\Delta}}{2\psi}\tan\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^2.$$

But in case of Eq. (13) *we didn't admit any kind (for $r = 0$ but $s \neq 0$ and $s = 0$ but $r \neq 0$) travelling wave solution because in this case travelling wave velocity became zero.*

For Case 2:
Proceeding as before, making use of the values of constants in Case 2 into Eq. (17) *along with Eqs.* (9) *to* (12) *we have the following traveling wave solutions for $r = 0$ but $s \neq 0$ respectively*

$$u4_1(\xi) = \beta_1 + \beta_2\left[d + \frac{\sqrt{\Omega}}{\psi}\coth\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-1} + \beta_3\left[d+\frac{\sqrt{\Omega}}{\psi}\coth\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-2},$$

$$u4_2(\xi) = \beta_1 + \beta_2\left[d+\frac{\sqrt{-\Omega}}{\psi}\cot\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-1} + \beta_3\left[d + \frac{\sqrt{-\Omega}}{\psi}\cot\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-2},$$

$$u4_3(\xi) = \beta_1 + \beta_2\left[d + \frac{B}{2\psi}+\frac{\sqrt{\Delta}}{2\psi}\coth\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-1} + \beta_3\left[d + \frac{B}{2\psi}+\frac{\sqrt{\Delta}}{2\psi}\coth\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-2},$$

$$u4_4(\xi) = \beta_1 + \beta_2\left[d + \frac{B}{2\psi}+\frac{\sqrt{-\Delta}}{2\psi}\cot\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-1} + \beta_3\left[d + \frac{B}{2\psi}+\frac{\sqrt{-\Delta}}{2\psi}\cot\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-2},$$

where $\beta_1 = -\frac{d^2\psi^2+(Bd-E)\psi}{A^2}$, $\beta_2 = \frac{2d^3\psi^2+3Bd^2\psi-2d\Omega+B(Bd-E)}{A^2}$, $\beta_3 = -\frac{d^4\psi^2+2Bd^3\psi-2Ed\Omega+(Bd-E)^2}{A^2}$ *and* $\xi = x - \frac{(B^2+4\Omega)^2}{A^4}t$.

Now, inserting (19) *into* (17) *and using* (9) *to* (12)*, respectively we get the traveling wave solutions as for $s = 0$ but $r \neq 0$*

$$u4_5(\xi) = \beta_1 + \beta_2\left[d + \frac{\sqrt{\Omega}}{\psi}\tanh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-1} + \beta_3\left[d + \frac{\sqrt{\Omega}}{\psi}\tanh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-2},$$

$$u4_6(\xi) = \beta_1 + \beta_2\left[d-\frac{\sqrt{-\Omega}}{\psi}\tan\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-1} + \beta_3\left[d-\frac{\sqrt{-\Omega}}{\psi}\tan\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-2},$$

$$u4_7(\xi) = \beta_1 + \beta_2\left[d + \frac{B}{2\psi}+\frac{\sqrt{\Delta}}{2\psi}\tanh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-1} + \beta_3\left[d + \frac{B}{2\psi}+\frac{\sqrt{\Delta}}{2\psi}\tanh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-2},$$

$$u4_8(\xi) = \beta_1 + \beta_2\left[d + \frac{B}{2\psi}-\frac{\sqrt{-\Delta}}{2\psi}\tan\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-1} + \beta_3\left[d + \frac{B}{2\psi}-\frac{\sqrt{-\Delta}}{2\psi}\tan\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-2}.$$

For Family 5 traveling wave solutions are not admit able because obtained velocity of the wave is zero.

For Case 3:
Also, from Case 3 placing the values of constants provided in Eq. (20) *into Eq.* (17) *accompanied with* (9) *to* (12) *and after simplification, respectively we find the following travelling solutions for $r = 0$ but $s \neq 0$*

$$u3_1(\xi) = \gamma_1 + \gamma_2\left[d + \frac{\sqrt{\Omega}}{\psi}\coth\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-1} + \gamma_3\left[d+\frac{\sqrt{\Omega}}{\psi}\coth\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-2},$$

$$u3_2(\xi) = \gamma_1 + \gamma_2\left[d+\frac{\sqrt{-\Omega}}{\psi}\cot\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-1} + \gamma_3\left[d + \frac{\sqrt{-\Omega}}{\psi}\cot\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-2},$$

$$u3_3(\xi)=\gamma_1+\gamma_2\left[d+\frac{B}{2\psi}+\frac{\sqrt{\Delta}}{2\psi}\coth\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-1}+\gamma_3\left[d+\frac{B}{2\psi}+\frac{\sqrt{\Delta}}{2\psi}\coth\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-2},$$

$$u3_4(\xi)=\gamma_1+\gamma_2\left[d+\frac{B}{2\psi}+\frac{\sqrt{-\Delta}}{2\psi}\cot\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-1}+\gamma_3\left[d+\frac{B}{2\psi}+\frac{\sqrt{-\Delta}}{2\psi}\cot\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-2},$$

where

$$\begin{aligned}\gamma_1&=\frac{-d^2\psi^2-Bd\psi+\frac{1}{2}\left(1-\frac{\sqrt{105}}{15}\right)\Omega-\frac{1}{8}\left(1+\frac{\sqrt{105}}{15}\right)B^2}{A^2},\\ \gamma_2&=\frac{2d^3\psi^2+3Bd^2\psi-2d\Omega+B(Bd-E)}{A^2},\\ \gamma_3&=-\frac{d^4\psi^2+2Bd^3\psi-2Ed\Omega+(Bd-E)^2}{A^2}\end{aligned}$$

and $\xi=x-\frac{\left(11+\sqrt{105}\right)\left(B^2+4\Omega\right)^2}{8A^4}t.$

Furthermore, substituting (20) *into Eq.* (17) *along with* (9) *to* (12) *and simplifying, respectively we find the following travelling solutions for* $s=0$ *but* $r\neq 0$

$$u3_5(\xi)=\gamma_1+\gamma_2\left[d+\frac{\sqrt{\Omega}}{\psi}\tanh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-1}+\gamma_3\left[d+\frac{\sqrt{\Omega}}{\psi}\tanh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-2},$$

$$u3_6(\xi)=\gamma_1+\gamma_2\left[d-\frac{\sqrt{-\Omega}}{\psi}\tan\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-1}+\gamma_3\left[d-\frac{\sqrt{-\Omega}}{\psi}\tan\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-2},$$

$$u3_7(\xi)=\gamma_1+\gamma_2\left[d+\frac{B}{2\psi}+\frac{\sqrt{\Delta}}{2\psi}\tanh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-1}+\gamma_3\left[d+\frac{B}{2\psi}+\frac{\sqrt{\Delta}}{2\psi}\tanh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-2},$$

$$u3_8(\xi)=\gamma_1+\gamma_2\left[d+\frac{B}{2\psi}-\frac{\sqrt{-\Delta}}{2\psi}\tan\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-1}+\gamma_3\left[d+\frac{B}{2\psi}-\frac{\sqrt{-\Delta}}{2\psi}\tan\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-2}.$$

Furthermore, when we combine with Eq. (13) *with Case 3 we find no traveling wave speed and that's why it became impossible to get any traveling wave solutions for Eq.* (13).

Similarly, Case 3, Case 4 and Case 5 exert traveling wave solutions of CDG equation for shake of simplicity which aren't reported here.

Example 3.2. *In this section we will examine the nonlinear fifth order Lax equation* (2).

The fifth order Lax equation (2) *can be rewritten as*

$$u_t+\frac{\partial}{\partial x}\left(u_{xxxx}+10uu_{xx}+5(u_x)^2+10u^3\right)=0. \tag{24}$$

Applying $\xi=x-ct$*, equation* (24) *converts into the following ODE for* $u(x,t)=v(\xi)$,

$$-cv'+\left(v^{(iv)}+10vv''+5\left(v'\right)^2+10v^3\right)'=0. \tag{25}$$

Integrating (25), *setting the constant of integration to zero, we obtain*

$$-cv+v^{(iv)}+10vv''+5\left(v'\right)^2+10v^3=0. \tag{26}$$

Balancing the highest order linear term $v^{(iv)}$ and nonlinear term of the highest order v^3 in equation (26)*, yields $N = 2$. Therefore, the solution of equation* (19) *appears in the following form:*

$$v(\xi) = a_0 + a_1(d+Y) + a_2(d+Y)^2 + b_1(d+Y)^{-1} + b_2(d+Y)^{-2} \tag{27}$$

where $a_0,\ a_1, a_2, b_1, b_2$ and d are arbitrary constants to be determined.

Substituting (27) *accompanied with* (7) *and* (8) *into* (26)*, the left-hand side is diverted into the polynomials in $(d{+}Y)^N$, $(N{=}\ 0,\ 1,\ 2,\ \ldots)$ and $(d{+}Y)^{-N}$, $(N{=}\ 1,\ 2,\ \ldots)$. We draw together each coefficient of this resulted polynomial and setting them to zero yields an over determined set of algebraic equations (for simplicity the equations are not presented here) for $a_0,\ a_1, a_2, b_1, b_2,\ c$ and d. Solving these algebraic equations with the help of symbolic computation software, such as, Maple 17, we obtain the following*

Case 1:

$$a_0 = -\frac{2\left(d^2\psi^2 + (Bd-E)\psi\right)}{A^2}, a_1 = 0, a_2 = 0, b_1 = \frac{2\left(d\psi\left(2d^2\psi + 3Bd - 2E\right) + B(Bd-E)\right)}{A^2},$$

$$b_2 = -\frac{2\left(d\psi\left(d^3\psi + 2Bd^2 - 2Ed\right) + (Bd-E)^2\right)}{A^2}, \qquad c = \frac{\left(B^2 + 4\Omega\right)^2}{A^2}, \qquad d = d, \tag{28}$$

where $\psi{=}A{-}C, \Omega = E\psi,\ d, A,\ B, C$ and E are free parameters.

Case 2:

$$\begin{gathered} a_0 = \frac{-2d\psi\left(d\psi + B\right) + \left(1 - \frac{1}{\sqrt{5}}\right)\Omega - \frac{1}{4}\left(1 + \frac{1}{\sqrt{5}}\right)B^2}{A^2}, a_1 = 0, a_2 = 0, \\ b_1 = \frac{2\left(d\psi\left(2d^2\psi + 3Bd - 2E\right) + B(Bd-E)\right)}{A^2}, \\ b_2 = -\frac{2\left(d\psi\left(d^3\psi + 2Bd^2 - 2Ed\right) + (Bd-E)^2\right)}{A^2}, \\ c = \frac{(3+\sqrt{5})\left(B^2 + 4\Omega\right)^2}{4A^4}, \qquad d = d, \end{gathered} \tag{29}$$

where $\psi{=}A{-}C, \Omega = E\psi,\ d, A,\ B, C$ and E are free parameters.

Case 3:

$$a_0 = \frac{-2d\psi\left(d\psi + B\right) + \left(1 - \frac{1}{\sqrt{5}}\right)\Omega - \frac{1}{4}\left(1 + \frac{1}{\sqrt{5}}\right)B^2}{A^2}, a_1 = \frac{2\left(2d\psi^2 + B\psi\right)}{A^2}, a_2 = -\frac{2\psi^2}{A^2},$$

$$b_1 = 0, \qquad b_2 = 0, \qquad c = \frac{(3+\sqrt{5})\left(B^2 + 4\Omega\right)^2}{4A^4}, \qquad d = d, \tag{30}$$

where $\psi{=}A{-}C, \Omega = E\psi,\ d, A,\ B, C$ and E are free parameters.

Case 4:

$$a_0 = -\frac{2\left(d^2\psi^2 + (Bd-E)\psi\right)}{A^2}, a_1 = \frac{2\left(2d\psi^2 + B\psi\right)}{A^2}, a_2 = -\frac{2\psi^2}{A^2},$$

$$b_1 = 0, \qquad b_2 = 0,\ c = \frac{\left(B^2 + 4\Omega\right)^2}{A^4}, \qquad d = d, \tag{31}$$

where ψ=A–C,$\Omega = E\psi$, d, A, B, C and E are free parameters.

Case 5:

$$a_0 = \frac{(B^2+4\Omega)}{A^2}, a_1 = 0, a_2 = -\frac{2\psi^2}{A^2},$$

$$b_1 = 0, \quad b_2 = -\frac{(B^2+4\Omega)^2}{8A^2\psi^2}, \quad c = \frac{16(B^2+4\Omega)^2}{A^4}, \quad d = -\frac{B}{2\psi}, \tag{32}$$

where ψ=A–C,$\Omega = E\psi$, A, B, C and E are free parameters.

Case 6:

$$a_0 = \frac{(B^2+4\Omega)}{\sqrt{5}A^2}, a_1 = 0, a_2 = -\frac{2\psi^2}{A^2},$$

$$b_1 = 0, \quad b_2 = -\frac{(B^2+4\Omega)^2}{8A^2\psi^2}, \quad c = \frac{4(3-\sqrt{5})(B^2+4\Omega)^2}{A^4}, \quad d = -\frac{B}{2\psi}, \tag{33}$$

where ψ=A–C,$\Omega = E\psi$, A, B, C and E are free parameters.

For Case 1:

By use of the values of constants from Case 1 into Eq. (27) *and combining with Eqs.* (9) *to* (12) *we obtain the following travelling wave solutions for $r = 0$ but $s \neq 0$ respectively*

$$v1_1(\xi) = \sigma_1 + \sigma_2\left[d + \frac{\sqrt{\Omega}}{\psi}\coth\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-1} + \sigma_3\left[d + \frac{\sqrt{\Omega}}{\psi}\coth\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-2},$$

$$v1_2(\xi) = \sigma_1 + \sigma_2\left[d + \frac{\sqrt{-\Omega}}{\psi}\cot\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-1} + \sigma_3\left[d + \frac{\sqrt{-\Omega}}{\psi}\cot\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-2},$$

$$v1_3(\xi) = \sigma_1 + \sigma_2\left[d + \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{2\psi}\coth\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-1} + \sigma_3\left[d + \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{2\psi}\coth\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-2},$$

$$v1_4(\xi) = \sigma_1 + \sigma_2\left[d + \frac{B}{2\psi} + \frac{\sqrt{-\Delta}}{2\psi}\cot\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-1} + \sigma_3\left[d + \frac{B}{2\psi} + \frac{\sqrt{-\Delta}}{2\psi}\cot\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-2},$$

where $\sigma_1 = -\frac{2\left(d^2\psi^2+(Bd-E)\psi\right)}{A^2}$, $\sigma_2 = \frac{2\left(d\psi\left(2d^2\psi+3Bd-2E\right)+B(Bd-E)\right)}{A^2}$, $\sigma_3 = -\frac{2\left(d\psi\left(d^3\psi+2Bd^2-2Ed\right)+(Bd-E)^2\right)}{A^2}$ *and* $\xi = x - \frac{(B^2+4\Omega)^2}{A^4}t$.

Again, substituting the values of the constants arranged in Eq. (28) *into Eq.* (27), *as well as Eqs.* (9) *to* (12) *and simplifying, we attain following traveling wave solutions for $s = 0$ but $r \neq 0$ respectively*

$$v1_5(\xi) = \sigma_1 + \sigma_2\left[d + \frac{\sqrt{\Omega}}{\psi}\tanh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-1} + \sigma_3\left[d + \frac{\sqrt{\Omega}}{\psi}\tanh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-2},$$

$$v1_6(\xi) = \sigma_1 + \sigma_2\left[d - \frac{\sqrt{-\Omega}}{\psi}\tan\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-1} + \sigma_3\left[d - \frac{\sqrt{-\Omega}}{\psi}\tan\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-2},$$

$$v1_7(\xi) = \sigma_1 + \sigma_2\left[d + \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{2\psi}\tanh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-1} + \sigma_3\left[d + \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{2\psi}\tanh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-2},$$

$$v1_8(\xi) = \sigma_1 + \sigma_2\left[d + \frac{B}{2\psi} - \frac{\sqrt{-\Delta}}{2\psi}\tan\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-1} + \sigma_3\left[d + \frac{B}{2\psi} - \frac{\sqrt{-\Delta}}{2\psi}\tan\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-2}.$$

Since wave speed become zero for Eq. (13) *when combine with* (27) *so traveling wave solution is not attainable.*

For Case 2:

In similar fashion, determined values of the constants, presenting in Case 2, putting into (27) *accompanied with* (9) *to* (12) *respectively we obtain the travelling wave solutions for* $r = 0$ *but* $s \neq 0$ *as follows*

$$v2_1(\xi) = \rho_1 + \rho_2\left[d + \frac{\sqrt{\Omega}}{\psi}\coth\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-1} + \rho_3\left[d + \frac{\sqrt{\Omega}}{\psi}\coth\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-2},$$

$$v2_2(\xi) = \rho_1 + \rho_2\left[d + \frac{\sqrt{-\Omega}}{\psi}\cot\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-1} + \rho_3\left[d + \frac{\sqrt{-\Omega}}{\psi}\cot\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-2},$$

$$v2_3(\xi) = \rho_1 + \rho_2\left[d + \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{2\psi}\coth\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-1} + \rho_3\left[d + \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{2\psi}\coth\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-2},$$

$$v2_4(\xi) = \rho_1 + \rho_2\left[d + \frac{B}{2\psi} + \frac{\sqrt{-\Delta}}{2\psi}\cot\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-1} + \rho_3\left[d + \frac{B}{2\psi} + \frac{\sqrt{-\Delta}}{2\psi}\cot\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-2},$$

where $\rho_1 = \frac{-2d\psi(d\psi+B)+\left(1-\frac{1}{\sqrt{5}}\right)\Omega-\frac{1}{4}\left(1+\frac{1}{\sqrt{5}}\right)B^2}{A^2}$, $\rho_2 = \frac{2\left(d\psi\left(2d^2\psi+3Bd-2E\right)+B(Bd-E)\right)}{A^2}$, $\rho_3 = -\frac{2\left(d\psi\left(d^3\psi+2Bd^2-2Ed\right)+(Bd-E)^2\right)}{A^2}$ *and* $\xi = x - \frac{(3+\sqrt{5})(B^2+4\Omega)^2}{4A^4}t.$

Again setting (29) *into Eq.* (27) *along with* (9) *to* (12) *and simplifying we get following traveling wave solutions for* $s = 0$ *but* $r \neq 0$ *respectively*

$$v2_5(\xi) = \rho_1 + \rho_2\left[d + \frac{\sqrt{\Omega}}{\psi}\tanh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-1} + \rho_3\left[d + \frac{\sqrt{\Omega}}{\psi}\tanh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^{-2},$$

$$v2_6(\xi) = \rho_1 + \rho_2\left[d - \frac{\sqrt{-\Omega}}{\psi}\tan\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-1} + \rho_3\left[d - \frac{\sqrt{-\Omega}}{\psi}\tan\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^{-2},$$

$$v2_7(\xi) = \rho_1 + \rho_2\left[d + \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{2\psi}\tanh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-1} + \rho_3\left[d + \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{2\psi}\tanh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^{-2},$$

$$v2_8(\xi) = \rho_1 + \rho_2\left[d + \frac{B}{2\psi} - \frac{\sqrt{-\Delta}}{2\psi}\tan\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-1} + \rho_3\left[d + \frac{B}{2\psi} - \frac{\sqrt{-\Delta}}{2\psi}\tan\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^{-2}.$$

Also using (29) *in to* (27) *together with* (13) *no traveling wave solution exist for both cases (when* $r = 0$ *but* $s \neq 0$ *and* $s = 0$ *but* $r \neq 0$*).*

For Case 3:

By means of the values of the constants contained in Eq. (30) *into* (27)*, together with Eqs.* (9) *to* (12) *and simplifying, we attain the traveling wave solutions as follows for* $r = 0$ *but* $s \neq 0$ *respectively*

$$v3_1(\xi) = \varepsilon_1 + \varepsilon_2\left[d + \frac{\sqrt{\Omega}}{\psi}\coth\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right] - \frac{2\psi^2}{A^2}\left[d + \frac{\sqrt{\Omega}}{\psi}\coth\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^2,$$

$$v3_2(\xi) = \varepsilon_1 + \varepsilon_2\left[d + \frac{\sqrt{-\Omega}}{\psi}\cot\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right] - \frac{2\psi^2}{A^2}\left[d + \frac{\sqrt{-\Omega}}{\psi}\cot\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^2,$$

$$v3_3(\xi)=\varepsilon_1+\varepsilon_2\left[d+\frac{B}{2\psi}+\frac{\sqrt{\Delta}}{2\psi}\coth\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]-\frac{2\psi^2}{A^2}\left[d+\frac{B}{2\psi}+\frac{\sqrt{\Delta}}{2\psi}\coth\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^2,$$

$$v3_4(\xi)=\varepsilon_1+\varepsilon_2\left[d+\frac{B}{2\psi}+\frac{\sqrt{-\Delta}}{2\psi}\cot\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]-\frac{2\psi^2}{A^2}\left[d+\frac{B}{2\psi}+\frac{\sqrt{-\Delta}}{2\psi}\cot\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^2,$$

where $\varepsilon_1=\frac{-2d\psi(d\psi+B)+\left(1-\frac{1}{\sqrt{5}}\right)\Omega-\frac{1}{4}\left(1+\frac{1}{\sqrt{5}}\right)B^2}{A^2}$ $,\varepsilon_2=\frac{2\left(2d\psi^2+B\psi\right)}{A^2}$ *and* $\xi=x-\frac{(3+\sqrt{5})(B^2+4\Omega)^2}{4A^4}t.$

Again, substituting (30) *into* (27)*, accompanied with Eqs.* (9) *to* (12) *and after simplification, we find the following traveling wave solutions for* $s=0$ *but* $r\neq 0$ *respectively*

$$v3_5(\xi)=\varepsilon_1+\varepsilon_2\left[d+\frac{\sqrt{\Omega}}{\psi}\tanh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]-\frac{2\psi^2}{A^2}\left[d+\frac{\sqrt{\Omega}}{\psi}\tanh\left(\frac{\sqrt{\Omega}}{\psi}\xi\right)\right]^2,$$

$$v3_6(\xi)=\varepsilon_1+\varepsilon_2\left[d-\frac{\sqrt{-\Omega}}{\psi}\tan\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]-\frac{2\psi^2}{A^2}\left[d-\frac{\sqrt{-\Omega}}{\psi}\tan\left(\frac{\sqrt{-\Omega}}{\psi}\xi\right)\right]^2,$$

$$v3_7(\xi)=\varepsilon_1+\varepsilon_2\left[d+\frac{B}{2\psi}+\frac{\sqrt{\Delta}}{2\psi}\tanh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]-\frac{2\psi^2}{A^2}\left[d+\frac{B}{2\psi}+\frac{\sqrt{\Delta}}{2\psi}\tanh\left(\frac{\sqrt{\Delta}}{2\psi}\xi\right)\right]^2,$$

$$v3_8(\xi)=\varepsilon_1+\varepsilon_2\left[d+\frac{B}{2\psi}-\frac{\sqrt{-\Delta}}{2\psi}\tan\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]-\frac{2\psi^2}{A^2}\left[d+\frac{B}{2\psi}-\frac{\sqrt{-\Delta}}{2\psi}\tan\left(\frac{\sqrt{-\Delta}}{2\psi}\xi\right)\right]^2.$$

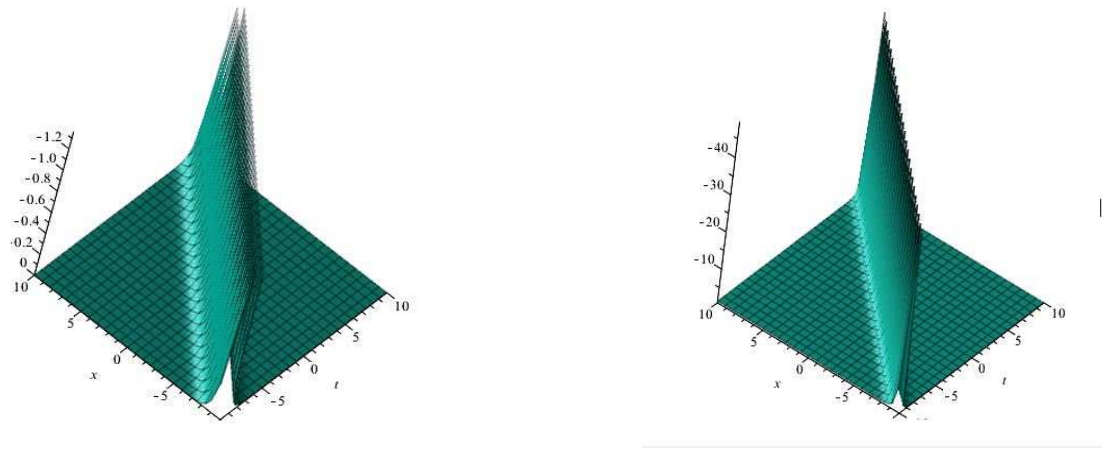
In this case also traveling wave solution for (13) *is not exist able.*

In similar way rest of the case also provide exact traveling wave solutions of Lax equation,to elude the pestering which aren't presented here.

4. Graphical and physical explanation of the acquired solutions

Herein, we put forth to represent some three dimensionals Figures of the modulus of the extracted solutions of CDG equation and Lax equation. Figures are constructed, by choosing suitable values of the parameters in order to understand the mechanism of the original equations (1) & (2), with the help of mathematical software *Maple 17.*

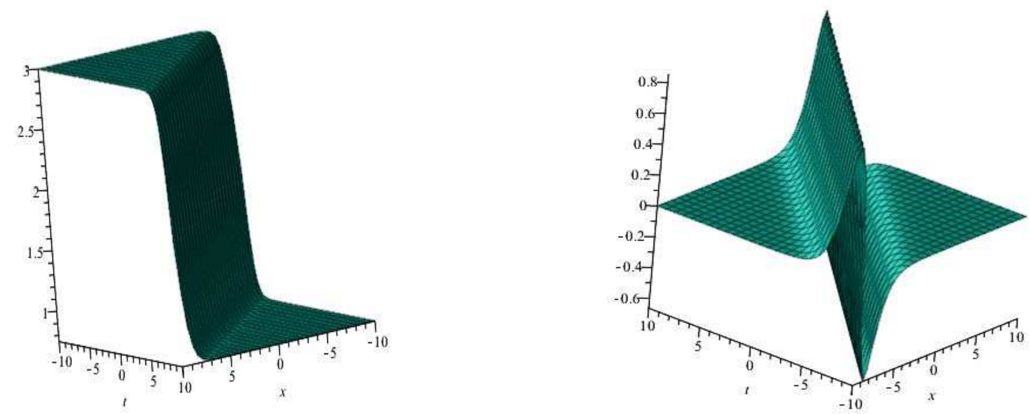
From our obtained solutions, we observe that solutions $u1_1(\xi), u1_2(\xi), u1_3(\xi), u1_4(\xi)$ and $u2_8(\xi)$ of CDG equation and $v1_8(\xi)$ of Lax equation are singular soliton. Fig. 1(a). shows the shape of the singular soliton solution$u1_1(\xi)$ for $d=1, A=2, B=0, C=1, E=1$ within the range $-10\leq x,t\leq 10$.Moreover, solutions $u2_1(\xi)$and $u2_2(\xi)$of CDG equation and $v1_1(\xi)$and $v1_3(\xi)$ of Lax equation represent soliton solutions. The modulus of the plot of soliton profile of $v1_1(\xi)$ for $d=-10, A=2, B=0, C=1, E=1$ within the interval $-10\leq x,t\leq 10$ are shown in Fig.1(b).



(a) **(b)**

Fig.1. Modulus plot of **(a)** singular soliton wave, shape of$u1_1(\xi)$and **(b)** soliton wave, shape of$v1_1(\xi)$.

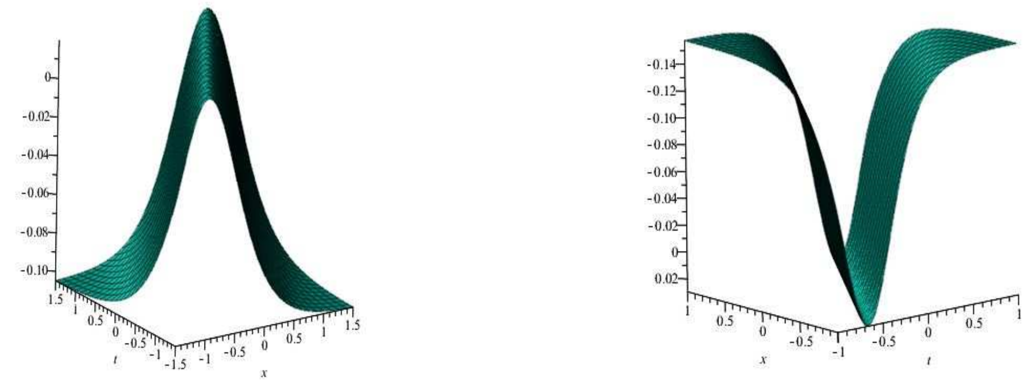
Also, solutions $v1_7(\xi), v2_7(\xi), v3_1(\xi), v3_3(\xi), v3_5(\xi)$ and $v3_7(\xi)$ illustrate the kink type wave solutions. Kink type wave solution of $v1_7(\xi)$ appeared in Fig. 2(a). by choosing the value of parameters $d = -1, A = 2, B = 4, C = 3, E = 3$ with range $-10 \leq x, t \leq 10$. Furthermore, the solution of CDG equation $u1_5(\xi)$ and the solutions of Lax equation $v2_1(\xi), v2_3(\xi)$and $v2_5(\xi)$ are singular kink traveling wave solutions. Fig.2 (b). shows the shape of singular kink traveling wave solution of $u1_5(\xi)$ for $d = 5, A = 2, B = 0, C = 1, E = 1$ within the limit$-10 \leq x, t \leq 10$.



(a) **(b)**

Fig.2. Modulus plot of **(a)** kink type wave, shape of$v1_7(\xi)$ and **(b)** singular kink type wave,shape of $u1_5(\xi)$.

Obtained solutions $u3_1(\xi), u3_3(\xi)$ and $u3_7(\xi)$ represent the bell-shaped soliton and $u3_5(\xi)$ represent anti-bell-shaped soliton. Fig. 3(a). is plotted for the bell-shaped soliton solution of $u3_3(\xi)$ with $d = -1, A = 4, B = 2, C = 3, E = 1$ within the interval$-1.5 \leq x, t \leq 1.5$ and Fig. 3(b). is plotted for the anti-bell-shaped soliton solution of $u3_5(\xi)$ for $d = -10, A = 4, B = 0, C = 3, E = 3$ with range$-1 \leq x, t \leq 1$.



(a) **(b)**

Fig.3. Modulus plot of **(a)** bell-shaped soliton wave, shape of $u3_3(\xi)$ and **(b)** anti-bell-shaped

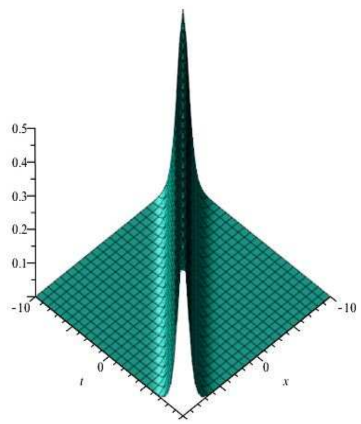
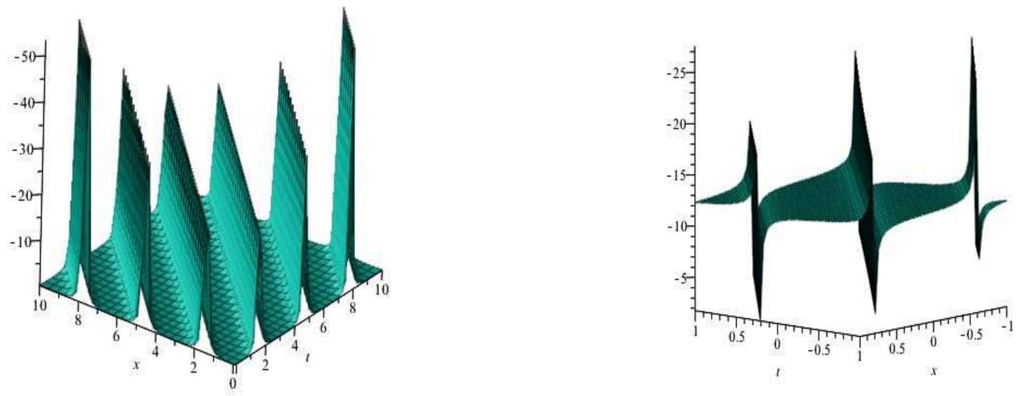


Fig.4. Modulus plot of bell-shaped sech^2 solitary traveling wave, shape of $v1_5(\xi)$.

The modulus of solutions $u1_6(\xi), u1_8(\xi), u2_2(\xi), u2_4(\xi), u2_6(\xi)$and $u3_4(\xi)$ of CDG equation and $v1_2(\xi), v1_4(\xi)$, $v1_6(\xi), v2_4(\xi), v2_8(\xi), v3_4(\xi)$ and $v3_8(\xi)$of Lax equation exude exact periodic traveling wave solutions. The graphical illustration of exact periodic traveling wave solutions of $u3_4(\xi)$ with $d = 1, A = 2, B = 2, C = 3, E = 2$ and $-.10 \le x, t \le 10$ is given in Fig. 5(a). And, solutions $v2_2(\xi), v2_6(\xi), v3_2(\xi)$, and $v3_6(\xi)$ are periodic traveling wave solutions. Fig. 5(b). illustrates the shape of the periodic traveling wave solution of $v3_2(\xi)$ for $d = 5, A = 2, B = 0, C = 3, E = 5$ within the range $-1 \le x, t \le 1$.



(a) **(b)**

Fig.5. Modulus plot of **(a)** exact periodic wave, shape of $u3_4(\xi)$ and **(b)** periodic wave, shape of $v3_2(\xi)$.

5. Conclusion

In this research, we succeeded in applying generalized (G'/G) expansion method on two specific fifth order KdV (fKdV) equations namely, CDG equation (1) and Lax equation (2). And we successfully obtained wider classes of exact travelling wave solutions with a variety of distinct physical structures such as soliton, singular soliton, kink, singular kink, bell-shaped soltion, anti-bell-shaped soliton, periodic, exact periodic and bell type solitary wave solutions which are shown in Fig. 1- Fig. 5. On comparing our results in this paper with the well-known results obtained in [43-57], most of the obtained solutions are exclusively new. The pivotal privilege of this implemented method against other methods is that the method provides more general and huger amount of new wave solutions which validate the superiority of this method.

References

[1] M.J. Ablowitz and P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge **149** (1991).

[2] R. Beals and R.R. Coifman, Scattering and inverse scattering for 1st order system, Commun. Pure. Appl. Math., **37** (1984) 39–90.

[3] V.B. Matveev and M.A. Salle, Darboux Transformations and Solitons, Springer-Verlag, Berlin,(1991).

[4] J. Weiss, M. Tabor and G. Carnevale, The Painleve property for partial differential equations, J. Math. Phys., **24** (1983) 522-526.

[5] H. Cai, S. Jing, T. De-Chen and H. Nian-Nin, Darboux Transformation Method for Solving the Sine-Gordon Equation in a Laboratory Reference, Chin. Phys. Lett., **19** (2002) 908–911.

[6] J.D. Cole, On a quasi-linear parabolic equation occurring in aerodynamics, Quart. Appl. Math., **9** (1951) 225-236.

[7] E. Hopf, The partial differential equation $u_t + uu_X = \mu u_{xx}$, Commun. Pure Appl. Math., **3** (1950) 201–230.

[8] S. Liu, Z. Fu, S. Liu and Q. Zhao, Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations, Phys. Lett. A., **289** (2001) 69–74.

[9] M. Alquran and A. Jarrah, Jacobi elliptic function solutions for a two-mode KdV equation, J. King Saud. Uni-Sci., **31(4)** (2019) 485–489.

[10] R. Hirota, Exact solution of the Korteweg-de-Vries equation for multiple collisions of solutions, Phys. Rev. Lett., **27** (1971) 1192–1194.

[11] A.M. Wazwaz, Multiple-soliton solutions and multiple-singular soliton solutions for two higher-dimensional shallow water wave equations, Appl. Math. Comput., **211** (2009) 495–510.

[12] M.R. Mimura, Bäcklund Transformation, Springer, Berlin, Germany (1978).

[13] S.M. Sayed, The Bäcklund Transformations, Exact Solutions, and Conservation Laws for the Compound Modified Korteweg-de Vries-Sine-Gordon Equations which Describe Pseudospherical Surfaces, J. Appl. Math., **2013** (2013), 7 pages.

[14] K.R. Raslan, T.S. EL-Danaf and K.K. Alia, New exact solution of coupled general equal width wave equation using sine-cosine function method, J. Egypt. Math. Scoi., **25** (2017) 350–354.

[15] W. Malfliet, Solitary wave solutions of nonlinear wave equations, Am. J. Phys., **60** (1992) 650–654.

[16] M.A. Abdelkawy, A.H. Bhrawy, E. Zerrad and A. Biswas, Application of Tanh Method to Complex Coupled Nonlinear Evolution Equations, Acta Phys. Pol. A., **129** (2016) 278–283.

[17] A.M. Wazwaz, Analytic study of the fifth order integrable nonlinear evolution equations by using the tanh method, Appl. Math. Comput., **174** (2006) 289–299.

[18] M.S. Islam, M.A. Akbar and K. Khan, The improved F-expansion method and its application to the MEE circular rod equation and the ZKBBM equation, Cogent. Math., **4(1)** (2017).

[19] Y.B. Chukkol, M.N. Mohamad and M.I. Muminov, Exact Solutions to the KDV-Burgers Equation with Forcing Term Using Tanh-Coth Method, AIP Conf. Proc., **1870(1)** (2017) 040024–1–040024–5.

[20] L.K. Ravi, S.S. Ray and S. Sahoo, New exact solutions of coupled Boussinesq–Burgers equations by Exp-function method, J.Ocean Engng Sci., **2** (2017) 34–46.

[21] N. Kadkhoda and H. Jafari, Analytical solutions of the Gerdjikov–Ivanov equation by using exp $(-\phi(\xi))$-expansion method, Optik Int. J. Light Electron Opt., **139** (2017) 72–76.

[22] A. Biswas, Y. Yildirim, E. Yasar, Q. Zhou, S.P. Moshokoa and M. Belic, Optical solitons for Lakshmanan-Porsezian-Daniel model by modified simple equation method, Optik., **160** (2018) 24–32.

[23] M.H. Bashar and S.M. R. Islam, Exact solutions to the (2 + 1)-Dimensional Heisenberg ferromagnetic spin chain equation by using modified simple equation and improve F-expansion methods, Open Phys., **5** (2020) article 100027.

[24] O. Yassin and M. Alquran, Constructing New Solutions for Some Types of Two-Mode Nonlinear Equations, Appl. Math. Inf. Sci., **12** (2018) 361–367.

[25] H. Naher, F.A. Abdullah and M.A. Akbar, The (G'/G)-Expansion Method for Abundant Traveling Wave Solutions of Caudrey-Dodd-Gibbon Equation, Math. Prob. Eng., **2011** (2011) 11 pages.

[26] M.G. Hafez, New traveling wave solutions of the (1+1)-dimensional cubic nonlinear Schrodinger equation using novel (G'/G)-expansion method, Beni-Seuf Univ. J. Appl. Sci., **5** (2016) 109–118.

[27] A. Hassaballa and T.M. Elzaki, Applications of the Improved (G'/G)-Expansion Method for solve Burgers-Fishers Equation, J. Comput. Theor. Nanosci., **14(10)** (2017) 4664–4668.

[28] H. Naher and F.A. Abdullah, New approach of (G'/G)-expansion method and new approach of generalized (G/G)-expansion method for nonlinear evolution equation, AIP Adv., **3** (2013) 032116–1–032116–14.

[29] H. Naher and F.A. Abdullah, New Generalized (G'/G)-expansion Method to the Zhiber-Shabat Equation and Liouville Equations, J. Phys. Conf. Series., **890** (2017) article 012018.

[30] M.M. Miah, H.M. S. Ali, M.A. Akbar and A.M. Wazwaz, Some applications of the $(G'/G, 1/G)$-expansion method to find new exact solutions of NLEEs, Eur. Phys. J. Plus., **132(6)** (2017) article 252.

[31] Q.M. Al-Mdallal and M.I. Syam, Sine–Cosine method for finding the soliton solutions of the generalized fifth-order nonlinear equation, Chaos Solitons Fractals, **33** (2007) 1610–1617.

[32] Q.M. Al-Mdallal, A new family of exact solutions to the unsteady Navier–Stokes equations using canonical transformation with complex coefficients, Appl. Math. Comput., **196** (2008) 303–308.

[33] Z.H. Khan, M. Qasim, R.U. Haq and Q.M. Al-Mdallald, Closed form dual nature solutions of fluid flow and heat transfer over a stretching/shrinking sheet in a porous medium, Chin. J. Phys., **55** (2017) 1284–1293.

[34] E.M.E. Zayed and A.G. Al-Nowehy, New generalized ϕ^6-model expansion method and its applications to the (3+1) dimensional resonant nonlinear Schrödinger equation with parabolic law nonlinearity, Optik., **214** (2020) article 164702.

[35] Y. El-Khatib, A Homotopy Analysis Method for the Option Pricing PDE in Post-Crash Markets, Math. Econ. Lett., **2** (2014) 1–6.

[36] P.J. Caudrey, R.K. Dodd and J.D. Gibbon, A new heirarchy of Korteweg-de Vries equations, Proc. Roy. Soc. Lond. A, **351** (1976) 407–422.

[37] R.K. Dodd and J.D. Gibbon, The prolongation structure of a higher order Korteweg-de Vries equations, Proc. Roy. Soc. Lond. A, **358** (1977) 287–300.

[38] J. Weiss, On classes of integrable systems and the Painleve' property, J. Math. Phys., **25** (1984) 13–14.

[39] M. Hénon and C. Heiles, The applicability of the third integral of motion: Some numerical experiments, Astron. J., **69** (1964) 73–79.

[40] A.M. Wazwaz, Analytic study of the fifth order integrable nonlinear evolution equations by using the tanh method, Appl. Math. Comput., **174** (2006) 289–299.

[41] A.M. Wazwaz, Abundant solitons solutions for several forms of the fifth-orderKdV equation by using the tanh method, Appl. Math. Comput., **182** (2006) 283–300.

[42] A.M. Wazwaz, Solitons and periodic solutions for the fifth-orderKdV equation, Appl. Math. Lett., **19** (2006) 1162–1167.

[43] A.M. Wazwaz, The extended tanh method for new solitons solutions for many forms of the fifth-orderKdV equations, Appl. Math. Comput., **184** (2006) 1002–1014.

[44] A.M. Wazwaz, Multiple-soliton solutions for the fifth order Caudrey-Dodd-Gibbon (CDG) equation, Appl. Math. Comput., **197** (2008) 719–724.

[45] A.M. Wazwaz, N-soliton solutions for the combined KdV-CDG equation and the KdV-Lax equation, Appl. Math. Comput., **203** (2008) 402–407.

[46] A.M. Wazwaz, Multiple soliton solutions for (2+1)-dimensional Sawada-Kotera and Caudrey-Dodd-Gibbon equations, Math. Meth. Appl. Sci., **34** (2011) 1580–1586.

[47] S. Bilige and T. Chaolu, An extended simplest equation method and its application to several forms of the fifth-orderKdV equation, Appl. Math. Comput., **216** (2010) 3146–3153.

[48] A.Salas, Some exact solutions for the Caudrey-Dodd-Gibbon equation, Math. Phys., (2008) arXiv: 0805.2969v2.
[49] Y.G. Xu, X.W. Zhou and L. Yao, Solving the fifth order Caudrey-Dodd-Gibbon (CDG) equation using the exp-function method, Appl. Math. Comput., **206** (2008) 70–73.
[50] C.A. Gomez and A.H. Salas, The generalized tanh-coth method to special types of the fifth-orderKdV equation, Appl. Math. Comput., **203** (2008) 873–880.
[51] L. Jin, Application of the variational iteration method for solving the fifth order Caudrey-Dodd-Gibbon equation, Int. Math. Forum, **5** (2010) 3259–3265.
[52] A. Biswas, G. Ebadi, H. Triki, A. Yildirim and N. Yousefzadeh, Topological Soliton and Other Exact Solutions to KdV-Caudrey-Dodd-Gibbon Equation, Results. Math., **63** (2013) 687–703.
[53] S. Abbasbandy and F.S. Zakaria, Soliton solutions for the fifth-orderKdV equation with the homotopy analysis method, Nonlinear Dyn., **51** (2008) 83–87.
[54] C.A. Gomez S., Special forms of the fifth-orderKdV equation with new periodic and soliton solutions, Appl. Math. Comput., **189(2)** (2007) 1066–1077.