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ON CERTAIN MULTIDIMENSIONAL NONLINEAR INTEGRALS

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Abstract. The aim of the paper is to obtain generalized convergence results for nonlinear multidimensional integrals of the form:

$$L_\eta(\omega;x)=\frac{\eta^n}{\Omega_{n-1}}\int\limits_D K(\eta\,|t-x|\,,\omega(t))dt.$$

We will prove some theorems concerning pointwise convergence of the family $L_\eta(\omega;x)$ as $\eta\to\infty$ at a fixed point $x\in D$ which represents any generalized Lebesgue point of the function $\omega\in L_1(D)$, where D is an open bounded subset of R^n. Moreover, we will consider the case $D=R^n$.

1. Introduction

The studies so far showed that Musielak [14] was the first researcher who investigated the approximation characteristics of convolution type nonlinear integral operators of the form:

$$T_w(f;s)=\int\limits_a^b K_w(x-s,f(x))dx, \tag{1.1}$$

where $s\in(a,b)\subset(-\infty,\infty)$, $w\in\mathcal{I}$ and $\mathcal{I}$ is a non-empty index set. His research was an intriguing contribution to literature related to this kind of nonlinear integral operators. Later, Swiderski and Wachnicki [19] studied the pointwise convergence of the operators of type (1.1). Extensive knowledge concerning this theory can be found in the monograph by Bardaro *et al.* [7]. Later on, multidimensional counterparts of the operators of type (1.1) were studied by Angeloni and Vinti [6] in some function spaces. Then, Jackson-type generalization of the operators defined

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in [6] were considered by Yilmaz [22]. Some studies on nonlinear operators in different settings can be found in [5,9,13,21]. Also, results and applications in wide range concerning linear operators can be found in [1,4,8,12,17,20]. Some weighted approximation results concerning well-known Gauss-Weierstrass and Picard integral operators can be found in the recent articles [23] and [24], respectively. In [16], a class of summation-integral-type operators covering many well-known ones was considered.

In the year 2016, Almali and Gadjiev [3] considered the following certain nonlinear integrals:

$$L_\eta(\omega;x)=\frac{\eta^n}{\Omega_{n-1}}\int\limits_D K(\eta\left|t-x\right|,\omega(t))dt, \tag{1.2}$$

where $D=R^n,\ t,x\in R^n,\ |t-x|=\sqrt{\sum\limits_{k=1}^{n}(t_k-x_k)^2}$ and Ω_{n-1} is the surface area of unit sphere $S^{n-1}=\{x\in R^n:|x|=1\}$ in R^n. Here, R^n denotes usual $n-$dimensional Euclidean space. Also, a real number η is considered as a positive parameter. They obtained pointwise convergence result for Lebesgue points of integrable functions. In the same article, exponential nonlinear integrals were also introduced. Some related works can be given as [2,11].

The aim of the current manuscript is to obtain convergence results for the operators of type (1.2) in two different settings via assigning two different definitions to a symbol D. We will prove pointwise convergence of the family $L_\eta(\omega;x)$ as $\eta\to\infty$ at a fixed point $x\in D$ which represents any generalized Lebesgue point of function $\omega\in L_1(D)$, where D is an open bounded subset of R^n, and $\omega\in L_1(R^n)$, separately. The space $L_1(D)$ consists of the measurable functions satisfying $\int\limits_D|\omega(t)|\,dt<\infty$. The norm formula in this space is given as follows: $\|\omega\|_{L_1(D)}=\int\limits_D|\omega(t)|\,dt$. The definition of the space $L_1(R^n)$ is analogous. Our results generalize and improve Theorem 2.2 in [3] in two different directions in view of the usages of generalized domain of integration and generalized characteristic point, respectively. Now, we consider the kernel function of the operators of type (1.2). Since $\eta\left|.\right|\in R_0^+$, for simplicity, we may denote $\eta\left|.\right|$ by $\eta\nu$. Therefore, $K(\eta\left|.\right|,\omega(t))=:K(\eta\nu,\omega)$, where $\nu\in R_0^+$ and $\omega:R^n\to R$.

The conditions on the kernel function to be given below are revised versions of the conditions used by Almali and Gadjiev [3].

We assume that real-valued kernel function $K(\eta\nu,\omega)$, where $\eta\nu\in R_0^+$ and $\omega:R^n\to R$, satisfies the following conditions:

a: For every $\nu\in R_0^+$ and $\eta\in R^+$, $K(\eta\nu,0)=0$ and $K(\eta\nu,\omega)$ is analytic at $\omega=0$ with radius of analyticity $\mathfrak{R}=\infty$ for all values of its first variable, that is, its Maclaurin series converges for all $\omega \in R$ and for all values of its first variable.

b: $\left.\frac{\partial^m K(\eta\nu,\omega)}{\partial\omega^m}\right|_{\omega=0}$ is a non-negative and non-increasing function with respect to ν on R_0^+ for any $m = 1, \ldots$ and for all values of $\eta \in R^+$.

c: The first partial derivative $\left.\frac{\partial K(\eta\nu,\omega)}{\partial\omega}\right|_{\omega=0}$ is a majorant function for all remaining derivatives, that is, $\left.\frac{\partial^m K(\eta\nu,\omega)}{\partial\omega^m}\right|_{\omega=0} \leq \left.\frac{\partial K(\eta\nu,\omega)}{\partial\omega}\right|_{\omega=0}$, where $m = 1, \ldots$, for all values of $\nu \in R_0^+$ and $\eta \in R^+$.

d: $\frac{\eta^n}{\Omega_{n-1}} \int\limits_{R^n} \left.\frac{\partial^m K(\eta|t|,\omega)}{\partial\omega^m}\right|_{\omega=0} dt = A_m < \infty$, where A_m with $m = 1, \ldots$ are certain positive constants which are independent of η and

$$\lim_{\eta\to\infty} \frac{\eta^n}{\Omega_{n-1}} \int\limits_{\zeta<|t|<\infty} \left.\frac{\partial^m K(\eta\,|t|\,,\omega)}{\partial\omega^m}\right|_{\omega=0} dt = 0$$

for all $\zeta > 0$ and $m = 1, \ldots$.

Definition 1. *A point $x \in R^n$ at which the following relation holds:*

$$\lim_{r\to 0^+} \frac{1}{r^{n(\alpha+1)}} \int\limits_{0<|t|\leq r} |\omega(t+x) - \omega(x)|\, dt = 0,$$

where $0 \leq \alpha < 1$, is called a generalized Lebesgue point of function $\omega \in L_1(R^n)$ (or for any function ω which is integrable on sufficiently large domain).

Definition of one-dimensional version of this point can be found in some recent papers, such as [13] and [5]. Definition of $d-$point analogue of this point in one-dimensional case was also considered by Gadjiev [12].

2. Main Theorems

Theorem 1. *Suppose that $K(.,\omega)$ satisfies conditions (a)-(d). If $x \in R^n$ is a generalized Lebesgue point of function $\omega \in L_1(R^n)$ and ω is a bounded function on R^n, that is, there exists a number $M > 0$ which depends on only ω such that $|\omega| \leq M$, then for the operators $L_\eta(\omega; x)$ which are defined in (1.2), we have*

$$\lim_{\eta\to\infty} L_\eta(\omega; x) = \sum_{m=1}^{\infty} \frac{A_m}{m!} [\omega(x)]^m$$

provided that the function

$$\eta^n \int\limits_{0<r<\infty} \left\{r^{n(\alpha+1)}\right\}_r' \left.\frac{\partial K(\eta r,\omega)}{\partial\omega}\right|_{\omega=0} dr,$$

where $r = |t|$, is bounded as $\eta \to \infty$.

Proof. By definition of generalized Lebesgue point, for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that

$$\int\limits_{0<|t|\leq r} |\omega(t+x) - \omega(x)|\, dt < \varepsilon r^{n(\alpha+1)}$$

holds provided that $r \leq \delta$ and $0 \leq \alpha < 1$.

Denoting the surface of unit sphere $\{t' \in R^n : |t'| = 1\}$ in R^n by S^{n-1}, we define

$$\int\limits_{S^{n-1}} |\omega(rt' + x) - \omega(x)|\, dt' =: u\,(r)\,,$$

where dt' is the surface element on S^{n-1} (see p.14 in [18]). For further details about polar coordinates transformation, we refer the reader to [10]. Therefore, we define the auxiliary function as

$$f(r) := \int\limits_0^r u\,(\rho)\, \rho^{n-1} d\rho \tag{2.1}$$

for which there holds:

$$f(r) \leq \varepsilon r^{n(\alpha+1)} \tag{2.2}$$

provided that $r \leq \delta$ and $0 \leq \alpha < 1$.

Following [3], we write the Maclaurin expansion of the function $K(.,\omega)$ with respect to ω as follows:

$$\begin{aligned} K(\eta\,|t-x|\,, \omega(t)) &= \sum_{m=0}^{\infty} \frac{1}{m!} K_\omega^{(m)}(\eta\,|t-x|\,,0)\,[\omega(t)]^m \\ &= \sum_{m=1}^{\infty} \frac{1}{m!} K_\omega^{(m)}(\eta\,|t-x|\,,0)\,[\omega(t)]^m\,, \end{aligned}$$

where $K_\omega^{(m)}(\eta\,|t-x|\,,0) := \left.\frac{\partial^m K(\eta|t-x|,\omega)}{\partial \omega^m}\right|_{\omega=0}$ and for every $\nu \in R_0^+$ and $\eta \in R^+$ with $K(\eta\nu, 0) = 0$. Since the conditions of Lebesgue dominated converge theorem (see, for example, [15]) are fulfilled, we can change the order of summation and integration. Since R^n is a locally compact abelian group, using change of variables and binomial representation of $[\omega(t+x)]^m$, we have

$$\begin{aligned} L_\eta(\omega; x) &= \frac{\eta^n}{\Omega_{n-1}} \sum_{m=1}^{\infty} \int\limits_{R^n} \frac{1}{m!} K_\omega^{(m)}(\eta\,|t|\,,0) \sum_{k=0}^{m-1} \binom{m}{k} [\omega(t+x) - \omega(x)]^{m-k}\, [\omega(x)]^k\, dt \\ &\quad + \frac{\eta^n}{\Omega_{n-1}} \sum_{m=1}^{\infty} \int\limits_{R^n} \frac{1}{m!} K_\omega^{(m)}(\eta\,|t|\,,0)\,[\omega(x)]^m\, dt. \end{aligned}$$

Let

$$I=\sum_{k=0}^{m-1}\binom{m}{k}\left[\omega(t+x)-\omega(x)\right]^{m-k}\left[\omega(x)\right]^{k}.$$

Now, without loss of generality, we consider the case ω is not identically zero on R^n. Since ω is bounded by a certain positive number M such that $|\omega(z)|\leq M$ for all $z\in R^n$, there holds

$$\begin{aligned}
|I| &= \left|\sum_{k=0}^{m-1}\binom{m}{k}\left[\omega(t+x)-\omega(x)\right]^{m-k}\left[\omega(x)\right]^{k}\right| \\
&\leq \sum_{k=0}^{m-1}\binom{m}{k}\left|\omega(t+x)-\omega(x)\right|^{m-k-1}\left|\omega(t+x)-\omega(x)\right|\left|\omega(x)\right|^{k} \\
&\leq \left|\omega(t+x)-\omega(x)\right|\sum_{k=0}^{m-1}\binom{m}{k}(2M)^{m-k-1}(M)^{k} \\
&\leq \left|\omega(t+x)-\omega(x)\right|\frac{1}{2M}\sum_{k=0}^{m-1}\binom{m}{k}(2M)^{m-k}(M)^{k} \\
&\quad +\left|\omega(t+x)-\omega(x)\right|\frac{1}{2M}\binom{m}{m}(2M)^{m-m}(M)^{m} \\
&= \left|\omega(t+x)-\omega(x)\right|\frac{1}{2M}\sum_{k=0}^{m}\binom{m}{k}(2M)^{m-k}(M)^{k} \\
&= \left|\omega(t+x)-\omega(x)\right|\frac{1}{2M}(3M)^{m}.
\end{aligned}$$

Therefore, using condition (c), we can write

$$\begin{aligned}
&\left|L_{\eta}(\omega;x)-\sum_{m=1}^{\infty}\frac{1}{m!}A_m\left[\omega(x)\right]^{m}\right| \\
&\leq \frac{\eta^n}{\Omega_{n-1}}\sum_{m=1}^{\infty}\int_{R^n}\frac{1}{m!}K_{\omega}^{(m)}(\eta\left|t\right|,0)\left|\omega(t+x)-\omega(x)\right|\frac{(3M)^{m}}{2M}dt \\
&\leq \frac{\eta^n}{\Omega_{n-1}}\frac{1}{2M}\sum_{m=1}^{\infty}\frac{(3M)^{m}}{m!}\int_{R^n}K_{\omega}^{(1)}(\eta\left|t\right|,0)\left|\omega(t+x)-\omega(x)\right|dt.
\end{aligned}$$

Fixing $\delta>0$, we have the following inequality:

$$\left|L_{\eta}(\omega;x)-\sum_{m=1}^{\infty}\frac{1}{m!}A_m\left[\omega(x)\right]^{m}\right|$$

$$\begin{aligned} &\leq \frac{\eta^n}{\Omega_{n-1}}\frac{\left(e^{3M}-1\right)}{2M}\left\{\int\limits_{|t|>\delta}+\int\limits_{|t|\leq\delta}\right\}K_\omega^{(1)}(\eta\,|t|\,,0)\,|\omega(t+x)-\omega(x)|\,dt \\ &=: \frac{1}{\Omega_{n-1}}\frac{\left(e^{3M}-1\right)}{2M}\left\{\eta^n I_\eta'+\eta^n I_\eta''\right\}. \end{aligned}$$

Let us show that $\eta^n I_\eta^{'} \to 0$ as $\eta \to \infty$. The following deductions are the natural consequences of conditions satisfied by our kernel function. Since

$$0 \leq \Omega_{n-1} K_\omega^{(1)}(\eta r,0)\frac{1}{n}r^n\left(1-\frac{1}{2^n}\right) \leq \int\limits_{\frac{r}{2}\leq|t|\leq r} K_\omega^{(1)}(\eta\,|t|\,,0)dt,$$

by (d) and well-known squeeze theorem, we see that $r^n K_\omega^{(1)}(\eta r,0) \to 0$ as $r\to\infty$ and $r\to 0$. In particular, this observation leads to $K_\omega^{(1)}(\eta r,0)\to 0$ as $r\to\infty$ and $r\to 0$. This type analysis is also performed in [3, 18]. For $\eta^n I_\eta'$, we obtain

$$\begin{aligned} \eta^n I_\eta' &\leq \eta^n K_\omega^{(1)}(\eta\delta,0)\int\limits_\delta^\infty\int\limits_{S^{n-1}} |\omega(rt'+x)|\,r^{n-1}dt'dr \\ &\quad +\eta^n\,|\omega(x)|\int\limits_{\delta<|t|<\infty} K_\omega^{(1)}(\eta\,|t|\,,0)dt \\ &\leq \eta^n K_\omega^{(1)}(\eta\delta,0)\,\|\omega\|_{L_1(R^n)}+\eta^n\,|\omega(x)|\int\limits_{\delta<|t|<\infty} K_\omega^{(1)}(\eta\,|t|\,,0)dt. \end{aligned}$$

The terms on the right-hand side tend to zero as $\eta\to\infty$ by overall hypotheses discussed previously. Hence, $\lim\limits_{\eta\to\infty}\eta^n I_\eta'=0$.

Now, we consider $\eta^n I_\eta''$. By relation (2.1), we can write

$$\begin{aligned} \eta^n I_\eta'' &= \eta^n\int\limits_0^\delta\int\limits_{S^{n-1}} |\omega(rt'+x)-\omega(x)|\,K_\omega^{(1)}(\eta r,0)r^{n-1}dt'dr \\ &= \eta^n\int\limits_0^\delta u\,(r)\,K_\omega^{(1)}(\eta r,0)r^{n-1}dr \\ &= \eta^n\int\limits_0^\delta K_\omega^{(1)}(\eta r,0)df(r). \end{aligned}$$

Using integration by parts for Stieltjes integrals and relation (2.2), we get the following inequality:

$$\eta^n I''_\eta \leq \varepsilon \eta^n \int_0^\infty \left\{ r^{n(\alpha+1)} \right\}'_r K^{(1)}_\omega(\eta r, 0) dr.$$

Since $\varepsilon > 0$ is arbitrarily small and the following expression:

$$\eta^n \int_0^\infty \left\{ r^{n(\alpha+1)} \right\}'_r K^{(1)}_\omega(\eta r, 0) dr$$

remains bounded as $\eta \to \infty$, we have

$$\lim_{\eta\to\infty} \eta^n I''_\eta = 0.$$

Combining all results gives

$$\lim_{\eta\to\infty} L_\eta(\omega; x) = \sum_{m=1}^\infty \frac{1}{m!} A_m \left[\omega(x)\right]^m .$$

Thus, the proof is completed. $\square$

In the second theorem, we give a local approximation result for nonlinear multi-dimensional integrals of the form:

$$T_\eta(\omega; x) = \frac{\eta^n}{\Omega_{n-1}} \int_D K(\eta \left| t - x \right|, \omega(t)) dt, \tag{2.3}$$

where $x \in D$ and D is any bounded open subset of R^n. We replaced R^n by D compared to operators of type (1.2).

Theorem 2. *Suppose that $K(.,\omega)$ satisfies conditions (a)-(d). If $x \in D$ is a generalized Lebesgue point of function $\omega \in L_1(D)$ with $\omega : R^n \to R$ and ω is a bounded function on D, that is, there exists a number $P > 0$ which depends on only ω such that $|\omega| \leq P$, then for the operators $T_\eta(\omega; x)$ which are defined in (2.3), we have*

$$\lim_{\eta\to\infty} T_\eta(\omega; x) = \sum_{m=1}^\infty \frac{A_m}{m!} \left[\omega(x)\right]^m$$

provided that the function

$$\eta^n \int_{0<r\leq\delta} \left\{ r^{n(\alpha+1)} \right\}'_r \left. \frac{\partial K(\eta r, \omega)}{\partial \omega} \right|_{\omega=0} dr,$$

where $r = |t|$ and $\delta > 0$ is a number chosen to ensure the existence of the integral, is bounded as $\eta \to \infty$.

Proof. We follow mainly the proof steps of previous theorem with some additional considerations.

By definition of generalized Lebesgue point, for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that

$$\int\limits_{0<|t|\leq r} |\omega(t+x) - \omega(x)|\, dt < \varepsilon r^{n(\alpha+1)}$$

holds provided that $r \leq \delta$ and $0 \leq \alpha < 1$.

Denoting the surface of unit sphere $\{t' \in R^n : |t'| = 1\}$ in R^n by S^{n-1}, we define

$$\int\limits_{S^{n-1}} |\omega(rt' + x) - \omega(x)|\, dt' =: \widetilde{u}\,(r)\,,$$

where dt' is the surface element on S^{n-1} (see p. 14 in [18]). Therefore, we define the new function as

$$\widetilde{f}(r) := \int\limits_0^r \widetilde{u}\,(\rho)\,\rho^{n-1} d\rho$$

for which there holds:

$$\widetilde{f}\,(r) \leq \varepsilon r^{n(\alpha+1)}$$

provided that $r \leq \delta$ and $0 \leq \alpha < 1$.

Now, we define the auxiliary function g by

$$g(t) := \begin{cases} \omega(t), & t \in D, \\ 0, & t \in R^n \backslash D. \end{cases} \tag{2.4}$$

We recall the Maclaurin series of $K(.,\omega)$ at $\omega = 0$ expressed as

$$K(\eta\,|t-x|\,, \omega(t)) = \sum_{m=1}^{\infty} \frac{1}{m!} K_{\omega}^{(m)}(\eta\,|t-x|\,, 0)\,[\omega(t)]^m\,,$$

where $K_{\omega}^{(m)}(\eta\,|t-x|\,,0) := \left.\frac{\partial^m K(\eta|t-x|,\omega)}{\partial\omega^m}\right|_{\omega=0}$ and for every $\nu \in R_0^+$ and $\eta \in R^+$ with $K(\eta\nu, 0) = 0$. In view of this, we infer that

$$\begin{aligned} & T_\eta(\omega; x) \\ = \quad & \frac{\eta^n}{\Omega_{n-1}} \sum_{m=1}^{\infty} \int\limits_D \frac{1}{m!} K_{\omega}^{(m)}(\eta\,|t-x|\,,0) \sum_{k=0}^{m} \begin{pmatrix} m \\ k \end{pmatrix} [\omega(t) - \omega(x)]^{m-k} [\omega(x)]^k\, dt \\ & + \frac{\eta^n}{\Omega_{n-1}} \sum_{m=1}^{\infty} \int\limits_{R^n} \frac{1}{m!} K_{\omega}^{(m)}(\eta\,|t-x|\,,0)\,[\omega(x)]^m\, dt \\ & - \frac{\eta^n}{\Omega_{n-1}} \sum_{m=1}^{\infty} \int\limits_{R^n} \frac{1}{m!} K_{\omega}^{(m)}(\eta\,|t-x|\,,0)\,[\omega(x)]^m\, dt. \end{aligned}$$

Let

$$I=\sum_{k=0}^{m-1}\binom{m}{k}\left[\omega(t)-\omega(x)\right]^{m-k}\left[\omega(x)\right]^{k}.$$

Without loss of generality, we consider the case ω is not identically zero on D. Since ω is bounded by a certain positive number P such that $|\omega(z)|\leq P$ for all $z\in D$, there holds

$$\begin{aligned}
|I| &= \left|\sum_{k=0}^{m-1}\binom{m}{k}\left[\omega(t)-\omega(x)\right]^{m-k}\left[\omega(x)\right]^{k}\right| \\
&\leq \sum_{k=0}^{m-1}\binom{m}{k}\left|\omega(t)-\omega(x)\right|^{m-k-1}\left|\omega(t)-\omega(x)\right|\left|\omega(x)\right|^{k} \\
&\leq \left|\omega(t)-\omega(x)\right|\frac{1}{2P}\left(3P\right)^{m}.
\end{aligned}$$

Therefore, in view of (2.4) and using condition (c), we obtain the following inequality:

$$\begin{aligned}
&\left|T_{\eta}(\omega;x)-\sum_{m=1}^{\infty}\frac{1}{m!}A_{m}\left[\omega(x)\right]^{m}\right| \\
\leq\ & \frac{\eta^{n}}{\Omega_{n-1}}\sum_{m=1}^{\infty}\int_{D}\frac{1}{m!}K_{\omega}^{(m)}(\eta\left|t-x\right|,0)\left|\omega(t)-\omega(x)\right|\frac{(3P)^{m}}{2P}dt \\
&+\left|\frac{\eta^{n}}{\Omega_{n-1}}\sum_{m=1}^{\infty}\int_{R^{n}\setminus D}\frac{1}{m!}K_{\omega}^{(m)}(\eta\left|t-x\right|,0)\sum_{k=0}^{m-1}\binom{m}{k}\left[-\omega(x)\right]^{m-k}\left[\omega(x)\right]^{k}dt\right| \\
\leq\ & \frac{\eta^{n}}{\Omega_{n-1}}\frac{1}{2P}\sum_{m=1}^{\infty}\frac{(3P)^{m}}{m!}\int_{D}K_{\omega}^{(1)}(\eta\left|t-x\right|,0)\left|\omega(t)-\omega(x)\right|dt \\
&+\left|\frac{\eta^{n}}{\Omega_{n-1}}\sum_{m=1}^{\infty}\int_{R^{n}\setminus D}\frac{1}{m!}K_{\omega}^{(1)}(\eta\left|t-x\right|,0)\sum_{k=0}^{m-1}\binom{m}{k}\left[-\omega(x)\right]^{m-k}\left[\omega(x)\right]^{k}dt\right| \\
=\ & :I_{1}+I_{2}.
\end{aligned}$$

Fixing $\delta>0$, we define $B_{\delta}:=\{t,x\in D:|t-x|\leq\delta\}\subset D$. Therefore, we have the following inequality:

$$I_{1}\leq\frac{\eta^{n}}{\Omega_{n-1}}\frac{\left(e^{3P}-1\right)}{2P}\int_{t\in D\setminus B_{\delta}}K_{\omega}^{(1)}(\eta\left|t-x\right|,0)\left|\omega(t)-\omega(x)\right|dt$$

$$
\begin{aligned}
&+\frac{\eta^n}{\Omega_{n-1}}\frac{\left(e^{3P}-1\right)}{2P}\int\limits_{|t|\le\delta}K_\omega^{(1)}(\eta\,|t|\,,0)\,|\omega(t+x)-\omega(x)|\,dt \\
= &:\frac{1}{\Omega_{n-1}}\frac{\left(e^{3P}-1\right)}{2P}\left\{\eta^n I'_\eta+\eta^n I''_\eta\right\}.
\end{aligned}
$$

Let us show that $\eta^n I'_\eta \to 0$ as $\eta\to\infty$. For $\eta^n I'_\eta$, we obtain

$$
\eta^n I'_\eta \le \eta^n K_\omega^{(1)}(\eta\delta,0)\,\|\omega\|_{L_1(D)}+\eta^n\,|\omega(x)|\int\limits_{\delta<|t|<\infty}K_\omega^{(1)}(\eta\,|t|\,,0)dt.
$$

The terms on the right-hand side tend to zero as $\eta\to\infty$ by (d). Hence, $\lim\limits_{\eta\to\infty}\eta^n I'_\eta = 0$.

It is easy to see that I_2 tends to zero as $\eta\to\infty$. The remaining part is analogous to proof of the preceding theorem. Hence

$$
\lim_{\eta\to\infty}T_\eta(\omega;x)=\sum_{m=1}^{\infty}\frac{1}{m!}A_m\,[\omega(x)]^m\,.
$$

Thus, the proof is completed. □

Example 1. *In [3], the authors considered the following kernel function satisfying the hypotheses:*

$$
K\left(\eta v,\omega\right)=\frac{1}{\sqrt{2\pi}}\left[\exp\left(e^{-(\eta v)^2}\omega\right)-1\right].
$$

Inspiring from the kernel given above and also Picard kernel, we consider the following kernel function without scaling:

$$
K\left(\eta v,\omega\right)=\exp\left(e^{-\eta v}\omega\right)-1,
$$

where for $\eta v\in R_0^+$, $K\left(\eta v,0\right)=0$ *and* $K_\omega^{(m)}(\eta v,0)=e^{-m\eta v}$ *with* $m=1,\dots$. *Clearly, this function is non-negative and non-increasing with respect to* ν *on* R_0^+ *for any* $m=1,\dots$ *and for all values of* $\eta\in R^+$, *and the first partial derivative majorizes the remaining derivatives. Lastly, in view of the well-known identity related to gamma function*

$$
\int\limits_0^\infty e^{-m\lambda}\lambda^{n-1}d\lambda=\frac{(n-1)!}{m^n},
$$

where $\lambda=\eta v$, *we see that the condition* (d) *easily holds there.*

3. Final Comments

Some theorems which are analogous to Theorem (3.3) and Theorem (3.5) in the article by Almali and Gadjiev [3] can be stated and proved using similar arguments. Also, more general theorems with respect to other characteristic points, such as $\mu-$generalized Lebesgue point, can also be proved.

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