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Conformable Derivatives and Integrals for the Functions of Two Variables

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Abstract

In this paper, we introduce conformable derivatives and integrals for the functions of two variables. This class of new fractional operators includes many definitions in the literature, such as Riemann-Liouville Fractional Derivatives and Integrals [6, 7], Conformable Calculus [8, 9], etc. In addition, some basic definitions and theorems have been obtained for these operators.

Keywords: *Fractional derivatives and integrals, Conformable integrals.*
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1. Introduction

In recent years, Fractional Derivatives and Integrals have been studied extensively by many authors. These definitions are the most well-known definitions of Riemann-Liouville, Caputo and Grünwald-Letnikov in the literature. Very recently, scientists have also studied some new ideas known as conformable derivatives and integral definitions. Some of these authors; Such as Abdeljawad, Khalil, Katugampola, Almeida and Akkurt. For all this, please see [1], [2], [5], [7], [10], [13].
Now, let's give the classical Riemann-Liouville fractional integral and fractional derivative definitions which are widely known in the literature.
The left-sided and right-sided Riemann-Liouville fractional integrals of order $\alpha > 0$, respectively, by [7]

$$\left(I_{a^+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \; x > a, \tag{1.1}$$

and

$$\left(I_{b^-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \; x < b. \tag{1.2}$$

The left-sided and right-sided Riemann-Liouville fractional derivatives of order $\alpha > 0$, respectively, by [7]

$$D_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)dt}{(x-t)^{\alpha}}, \; x > a, \tag{1.3}$$

and

$$D_{b^-}^{\alpha} f(x) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b \frac{f(t)dt}{(t-x)^{\alpha}} \; x < b. \tag{1.4}$$

In [9], Khalil et al. introduced limit definition of the derivative of a function as follows,

$$T_{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}. \tag{1.5}$$

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In [3], Almeida et al. introduced limit definition of the derivative of a function as follows,

$$f^{(\alpha)}(t)=\lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon k(t)^{1-\alpha}\right)-f(t)}{\varepsilon}. \tag{1.6}$$

Recently, in [5] Katugampola introduced the idea of fractional derivative,

$$D_{\alpha}(f)(t)=\lim_{\varepsilon\to 0}\frac{f\left(te^{\varepsilon t^{-\alpha}}\right)-f(t)}{\varepsilon}. \tag{1.7}$$

Very recently, in [1] Akkurt et al. gave the following conformable derivative definition:

$$D^{\alpha}(f)(t):=\lim_{\varepsilon\to 0}\frac{f\left(t-k(t)+k(t)e^{\varepsilon\frac{(k(t))^{-\alpha}}{k'(t)}}\right)-f(t)}{\varepsilon}. \tag{1.8}$$

Anderson [4] gave the definition of $\alpha-$fractional integral as follows:

$$\int_a^b f(x)\,d_{\alpha}x:=\int_a^b\frac{f(x)}{x^{1-\alpha}}dx.$$

In [1], Akkurt et al. gave the following generalized conformable integral definition:

$$I^{\alpha}(f)(t)=\int_a^b\frac{k'(x)f(x)}{(k(x))^{1-\alpha}}dx.$$

The left and right fractional conformable integrals of order $\mathrm{Re}(\beta)>0$ are defined by [14],

$${}_{a}^{\beta}J^{\alpha}f(x)=\frac{1}{\Gamma(\beta)}\int_a^x\left[\frac{(x-a)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right]^{\beta-1}\frac{f(t)dt}{(t-a)^{1-\alpha}}$$

and

$${}^{\beta}J_{b}^{\alpha}f(x)=\frac{1}{\Gamma(\beta)}\int_x^b\left[\frac{(b-x)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right]^{\beta-1}\frac{f(t)dt}{(b-t)^{1-\alpha}},$$

respectively.

Definition 1.1. *For $a<b,\ c<d$ and $1\leq p<\infty$, a function $f(x,y)$ is said to be in the $L_p[(a,b)\times(c,d)]$ space if ([11])*

$$L_p[(a,b)\times(c,d)]=\left\{f:\|f\|_{L_p}=\left(\int_a^b\int_c^d|f(x,y)|^p\,dxdy\right)^{\frac{1}{p}}<\infty\right\}, \tag{1.9}$$

and for the case $p=\infty$

$$\|f\|_{\infty}=ess\sup_{\substack{a\leq x\leq b\\ c\leq y\leq c}}|f(x,y)|. \tag{1.10}$$

2. Conformable Integrals for the functions of two variables

Theorem 2.1. *Let $f\in L_1[(a,b)\times(c,d)]$ and $\gamma_1\neq 0,\ \gamma_2\neq 0$. The conformable integral $I_{a^+,c^+}^{m,n}$ of order $m,\ n$, with $a,\ c\geq 0$, $x>a$ and $y>c$ is defined by;*

$$\begin{aligned}&\left({}^{\gamma_1,\gamma_2}I_{a^+,c^+}^{m,n}f\right)(x,y)\\ =\;&\frac{1}{\Gamma(m)\Gamma(n)}\int_a^x\int_c^y\left[\frac{(x-a)^{\gamma_1}-(s-a)^{\gamma_1}}{\gamma_1}\right]^{m-1}\\ &\times\left[\frac{(y-c)^{\gamma_2}-(t-c)^{\gamma_2}}{\gamma_2}\right]^{n-1}\frac{f(s,t)dtds}{(s-a)^{1-\gamma_1}(t-c)^{1-\gamma_2}},\end{aligned}$$

here, m and n are positive integers.

Proof. Let us denote the following $n-$fold integral

$$\int_a^x \frac{1}{(s_1-a)^{1-\gamma_1}} \int_a^{s_1} \frac{1}{(s_2-a)^{1-\gamma_1}} \cdots \int_a^{s_n-1} \frac{1}{(s_n-a)^{1-\gamma_1}}$$
$$\times \left[\int_c^y \frac{1}{(t_1-c)^{1-\gamma_2}} \left(\int_c^{t_1} \frac{1}{(t_2-c)^{1-\gamma_2}} \left(\cdots \int_c^{t_{n-1}} \frac{1}{(t_n-c)^{1-\gamma_2}} f(s_n,t_n)dt_n \ldots \right) dt_2 \right) dt_1 \right]$$
$$\times ds_n \ldots ds_2 ds_1. \tag{2.1}$$

First, let's calculate the following integral.

$$\int_c^y \frac{1}{(t_1-c)^{1-\gamma_2}} \left(\int_c^{t_1} \frac{1}{(t_2-c)^{1-\gamma_2}} \cdots \left(\int_c^{t_{n-1}} \frac{f(s_n,t_n)}{(t_n-c)^{1-\gamma_2}} dt_n \right) \ldots dt_2 \right) dt_1.$$

Here, let's change the order of integration to calculate the integral. Thus, we can write the limits of integration region as follows;

$$c < t_n < t_{n-1} < \ldots < t_3 < t_2 < t_1 < y. \tag{2.2}$$

Thus, the integral can be written as follows;

$$\int_c^y \frac{1}{(t_1-c)^{1-\gamma_2}} \left(\int_c^{t_1} \frac{1}{(t_2-c)^{1-\gamma_2}} \cdots \left(\int_c^{t_{n-1}} \frac{f(s_n,t_n)}{(t_n-c)^{1-\gamma_2}} dt_n \right) \ldots dt_2 \right) dt_1$$
$$= \int_c^y \frac{f(s_n,t_n)}{(t_n-c)^{1-\gamma_2}} \left(\int_{t_n}^y \frac{1}{(t_{n-1}-c)^{1-\gamma_2}} \left(\cdots \left(\int_{t_2}^y \frac{dt_1}{(t_1-c)^{1-\gamma_2}} \right) \cdots \right) dt_{n-1} \right) dt_n. \tag{2.3}$$

If the integral (2.3) is calculated step by step, then the first two steps are as follows.

$$\int_{t_2}^y \frac{dt_1}{(t_1-c)^{1-\gamma_2}} = \frac{(y-c)^{\gamma_2}-(t_2-c)^{\gamma_2}}{\gamma_2},$$

$$\int_{t_3}^y \frac{(y-c)^{\gamma_2}-(t_2-c)^{\gamma_2}}{\gamma_2} \frac{1}{(t_2-c)^{1-\gamma_2}} dt_2 = \frac{1}{2!} \left[\frac{(y-c)^{\gamma_2}-(t_3-c)^{\gamma_2}}{\gamma_2} \right]^2,$$

and

$$\frac{1}{2} \int_{t_4}^y \left[\frac{(y-c)^{\gamma_2}-(t_3-c)^{\gamma_2}}{\gamma_2} \right]^2 \frac{1}{(t_3-c)^{1-\gamma_2}} dt_3 = \frac{1}{3!} \left[\frac{(y-c)^{\gamma_2}-(t_3-c)^{\gamma_2}}{\gamma_2} \right]^3.$$

If we continue to calculate in this way consecutively, then we get,

$$\int_c^y \frac{1}{(t_1-c)^{1-\gamma_2}} \int_c^{t_1} \frac{1}{(t_2-c)^{1-\gamma_2}} \cdots \int_c^{t_{n-1}} \frac{1}{(t_n-c)^{1-\gamma_2}} f(s_n,t_n) dt_n \ldots dt_2 dt_1$$
$$= \frac{1}{(n-1)!} \int_c^y \left[\frac{(y-c)^{\gamma_2}-(t_n-c)^{\gamma_2}}{\gamma_2} \right]^{n-1} \frac{f(s_n,t_n)}{(t_n-c)^{1-\gamma_2}} dt_n.$$

This equality is written in the eqnarray (2.1), we have

$$\int_a^x \frac{1}{(s_1-a)^{1-\gamma_1}} \int_a^{s_1} \frac{1}{(s_2-a)^{1-\gamma_1}} \cdots \int_a^{s_n-1} \frac{1}{(s_n-a)^{1-\gamma_1}}$$
$$\times \left(\frac{1}{(n-1)!} \int_c^y \left[\frac{(y-c)^{\gamma_2}-(t_n-c)^{\gamma_2}}{\gamma_2} \right]^{n-1} \frac{f(s_n,t_n)}{(t_n-c)^{1-\gamma_2}} dt_n \right) ds_n \ldots ds_2 ds_1.$$

Thus, we write

$$\frac{1}{(n-1)!} \int_c^y \left[\frac{(y-c)^{\gamma_2}-(t_n-c)^{\gamma_2}}{\gamma_2} \right]^{n-1} \frac{1}{(t_n-c)^{1-\gamma_2}}$$
$$\times \left(\int_a^x \frac{1}{(s_1-a)^{1-\gamma_1}} \int_a^{s_1} \frac{1}{(s_2-a)^{1-\gamma_1}} \cdots \int_a^{s_n-1} \frac{f(s_n,t_n)}{(s_n-a)^{1-\gamma_1}} ds_n \ldots ds_2 ds_1 \right) dt_n. \tag{2.4}$$

Similarly, let's calculate the following integral:

$$\int_a^x \frac{1}{(s_1-a)^{1-\gamma_1}} \int_a^{s_1} \frac{1}{(s_2-a)^{1-\gamma_1}} \cdots \int_a^{s_n-1} \frac{f(s_n,t_n)}{(s_n-a)^{1-\gamma_1}} ds_n \ldots ds_2 ds_1.$$

Here again, let's change the order of integration to calculate the integral. Thus, we can write the boundaries of the integration region as follows;

$$a < s_n < s_{n-1} < \ldots < s_3 < s_2 < s_1 < x.$$

Thus, the integral can be written as follows;

$$\int_a^x \frac{f(s_n,t_n)}{(s_n-a)^{1-\gamma_1}} \left(\int_{s_n}^x \frac{1}{(s_{n-1}-a)^{1-\gamma_1}} \left(\ldots \int_{s_3}^x \frac{1}{(s_2-a)^{1-\gamma_1}} \left(\int_{s_2}^x \frac{1}{(s_1-a)^{1-\gamma_1}} ds_1 \right) ds_2 \ldots \right) ds_{n-1} \right) ds_n.$$

$$\int_{s_2}^x \frac{1}{(s_1-a)^{1-\gamma_1}} ds_1 = \frac{(x-a)^{\gamma_1}-(s_2-a)^{\gamma_1}}{\gamma_1},$$

$$\int_{s_3}^x \frac{(x-a)^{\gamma_1}-(s_2-a)^{\gamma_1}}{\gamma_1} \frac{ds_2}{(s_2-a)^{1-\gamma_1}} = \frac{1}{2}\left[\frac{(x-a)^{\gamma_1}-(s_3-a)^{\gamma_1}}{\gamma_1}\right]^2,$$

and, so

$$\int_a^x \frac{1}{(s_1-a)^{1-\gamma_1}} \int_a^{s_1} \frac{1}{(s_2-a)^{1-\gamma_1}} \cdots \left(\int_a^{s_n-1} \frac{f(s_n,t_n)}{(s_n-a)^{1-\gamma_1}} ds_n \right) \ldots ds_2 ds_1 = \frac{1}{(m-1)!} \int_a^x \left[\frac{(x-a)^{\gamma_1}-(s_2-a)^{\gamma_1}}{\gamma_1}\right]^{m-1} \frac{f(s_n,t_n)}{(s_n-a)^{1-\gamma_1}} ds_n. \tag{2.5}$$

Finally, from (2.4) and (2.5), we get

$$ {}^{\gamma_1,\gamma_2}I^{m,n}_{a^+,c^+} f(x,y) = \frac{1}{(m-1)!(n-1)!} \int_a^x \int_c^y \left[\frac{(x-a)^{\gamma_1}-(s_n-a)^{\gamma_1}}{\gamma_1}\right]^{m-1} \times \left[\frac{(y-c)^{\gamma_2}-(t_n-c)^{\gamma_2}}{\gamma_2}\right]^{n-1} \frac{f(s_n,t_n)dt_n ds_n}{(s_n-a)^{1-\gamma_1}(t_n-c)^{1-\gamma_2}}. \tag{2.6}$$

Thus, the desired result is obtained and proof is completed.

$$ {}^{\gamma_1,\gamma_2}I^{m,n}_{a^+,c^+} f(x,y) = \frac{1}{\Gamma(m)\Gamma(n)} \int_a^x \int_c^y \left[\frac{(x-a)^{\gamma_1}-(s_n-a)^{\gamma_1}}{\gamma_1}\right]^{m-1} \times \left[\frac{(y-c)^{\gamma_2}-(t_n-c)^{\gamma_2}}{\gamma_2}\right]^{n-1} \frac{f(s_n,t_n)dt_n ds_n}{(s_n-a)^{1-\gamma_1}(t_n-c)^{1-\gamma_2}}.$$

□

Here $\gamma_1 \neq 0$ and $\gamma_2 \neq 0$ are real number. In this (2.6) eqnarray, m and n are positive integers. From the definition of the gamma function, if n and m are not an integer, the following definitions can be given.

Definition 2.2. *Let $f \in L_1[(a,b)\times(c,d)]$ and let $\gamma_1 \neq 0,\ \gamma_2 \neq 0,\ \alpha,\ \beta \in \mathbb{C},\ Re(\alpha) > 0$ and $Re(\beta) > 0$. The generalized conformable integral of order $\alpha,\ \beta$ of $f(x,y)$ is defined by;*

$$\left({}^{\gamma_1,\gamma_2}I^{\alpha,\beta}_{a^+,c^+} f\right)(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y \left(\frac{(x-a)^{\gamma_1}-(s-a)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1} \times \left(\frac{(y-c)^{\gamma_2}-(t-c)^{\gamma_2}}{\gamma_2}\right)^{\beta-1} \frac{f(s,t)dtds}{(s-a)^{1-\gamma_1}(t-c)^{1-\gamma_2}}, \tag{2.7}$$

$$\left({}^{\gamma_1,\gamma_2}I^{\alpha,\beta}_{b^-,c^+} f\right)(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_c^y \left(\frac{(s-b)^{\gamma_1}-(x-b)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1} \times \left(\frac{(y-c)^{\gamma_2}-(t-c)^{\gamma_2}}{\gamma_2}\right)^{\beta-1} \frac{f(s,t)dtds}{(s-b)^{1-\gamma_1}(t-c)^{1-\gamma_2}}, \tag{2.8}$$

$$\left({}^{\gamma_1,\gamma_2}I^{\alpha,\beta}_{a^+,d^-} f\right)(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_y^d \left(\frac{(x-a)^{\gamma_1}-(s-a)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1} \times \left(\frac{(t-d)^{\gamma_2}-(y-d)^{\gamma_2}}{\gamma_2}\right)^{\beta-1} \frac{f(s,t)dtds}{(s-a)^{1-\gamma_1}(t-d)^{1-\gamma_2}}, \tag{2.9}$$

and

$$\left({}^{\gamma_1,\gamma_2}I^{\alpha,\beta}_{b^-,d^-} f\right)(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_y^d \left(\frac{(s-b)^{\gamma_1}-(x-b)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1} \times \left(\frac{(t-d)^{\gamma_2}-(y-d)^{\gamma_2}}{\gamma_2}\right)^{\beta-1} \frac{f(s,t)dtds}{(s-b)^{1-\gamma_1}(t-d)^{1-\gamma_2}} \tag{2.10}$$

the Generalized Conformable integrals.

Remark 2.3. *If $\gamma_1=\gamma_2=1$ in (2.7), (2.8), (2.9) and (2.10), we have*

$$\left(I^{\alpha,\beta}_{a^+,c^+}f\right)(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y(x-t)^{\alpha-1}(y-s)^{\beta-1}f(s,t)dtds,$$

$$\left(I^{\alpha,\beta}_{b^-,c^+}f\right)(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_c^y(s-x)^{\alpha-1}(y-t)^{\beta-1}f(s,t)dtds,$$

$$\left(I^{\alpha,\beta}_{a^+,d^-}f\right)(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_y^d(x-s)^{\alpha-1}(t-y)^{\beta-1}f(s,t)dtds,$$

and

$$\left(I^{\alpha,\beta}_{b^-,d^-}f\right)(x,y)=\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_x^b\int_y^d(x-s)^{\alpha-1}(t-y)^{\beta-1}f(s,t)dtds,$$

the Fractional integrals of the functions of two variables [12].

Remark 2.4. *If we consider $\alpha=\beta=1$ in (2.7), (2.8), (2.9) and (2.10), we have*

$$\left(I^{1,1}_{a^+,c^+}f\right)(x,y)=\int_a^x\int_c^y\frac{f(s,t)dtds}{(s-a)^{1-\gamma_1}(t-c)^{1-\gamma_2}}$$

$$\left(I^{1,1}_{b^-,c^+}f\right)(x,y)=\int_x^b\int_c^y\frac{f(s,t)dtds}{(s-b)^{1-\gamma_1}(t-c)^{1-\gamma_2}}$$

$$\left(I^{1,1}_{a^+,d^-}f\right)(x,y)=\int_a^x\int_y^d\frac{f(s,t)dtds}{(s-a)^{1-\gamma_1}(t-d)^{1-\gamma_2}}$$

$$\left(I^{1,1}_{b^-,d^-}f\right)(x,y)=\int_x^b\int_y^d\frac{f(s,t)dtds}{(s-b)^{1-\gamma_1}(t-d)^{1-\gamma_2}}$$

the Conformable fractional integrals for Double Integrals.

Remark 2.5. *If we consider $\alpha=\beta=1$ in (2.7), we have*

$$\left(I^{1,1}_{0^+,0^+}f\right)(x,y)=\int_0^x\int_0^y\frac{f(s,t)dtds}{s^{1-\gamma_1}t^{1-\gamma_2}}$$

the Conformable fractional integrals for Double Integrals.

Now, we consider some properties of the conformable integrals.

Theorem 2.6. *Let $f\in L_1(a,b)$. Let $\gamma_1\neq 0,\ \gamma_2\neq 0$. Then the following semi group propert holds for generalized conformable integrals,*

1. $\left({}^{\gamma_1,\gamma_2}I^{\alpha_1,\beta_1}_{a^+,c^+}\left[{}^{\gamma_1,\gamma_2}I^{\alpha_2,\beta_2}_{a^+,c^+}f(x,y)\right]\right)=\left({}^{\gamma_1,\gamma_2}I^{\alpha_1+\alpha_2,\beta_1+\beta_2}_{a^+,c^+}\right)f(x,y),$

2. $\left({}^{\gamma_1,\gamma_2}I^{\alpha_1,\beta_1}_{b^-,c^+}\left[{}^{\gamma_1,\gamma_2}I^{\alpha_2,\beta_2}_{b^-,c^+}f(x,y)\right]\right)=\left({}^{\gamma_1,\gamma_2}I^{\alpha_1+\alpha_2,\beta_1+\beta_2}_{b^-,c^+}\right)f(x,y),$

3. $\left({}^{\gamma_1,\gamma_2}I^{\alpha_1,\beta_1}_{a^+,d^-}\left[{}^{\gamma_1,\gamma_2}I^{\alpha_2,\beta_2}_{a^+,d^-}f(x,y)\right]\right)=\left({}^{\gamma_1,\gamma_2}I^{\alpha_1+\alpha_2,\beta_1+\beta_2}_{a^+,d^-}\right)f(x,y),$

and

4. $\left({}^{\gamma_1,\gamma_2}I^{\alpha_1,\beta_1}_{b^-,d^-}\left[{}^{\gamma_1,\gamma_2}I^{\alpha_2,\beta_2}_{b^-,d^-}f(x,y)\right]\right)=\left({}^{\gamma_1,\gamma_2}I^{\alpha_1+\alpha_2,\beta_1+\beta_2}_{b^-,d^-}\right)f(x,y).$

Proof. Note that,

$$\begin{aligned}
&\left({}^{\gamma_1,\gamma_2}I^{\alpha_1,\beta_1}_{a^+,c^+}\left[{}^{\gamma_1,\gamma_2}I^{\alpha_2,\beta_2}_{a^+,c^+}f(x,y)\right]\right)\\
=\ & \frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_a^x\int_c^y\left(\frac{(x-a)^{\gamma_1}-(t-a)^{\gamma_1}}{\gamma_1}\right)^{\alpha_1-1}\\
&\times\left(\frac{(y-c)^{\gamma_2}-(s-c)^{\gamma_2}}{\gamma_2}\right)^{\beta_1-1}\frac{\left[{}^{\gamma_1,\gamma_2}I^{\alpha_2,\beta_2}_{a^+,c^+}f(t,s)\right]dsdt}{(s-c)^{1-\gamma_2}(t-a)^{1-\gamma_1}}.
\end{aligned}$$

So,

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\int_a^x\int_c^y\left(\frac{(x-a)^{\gamma_1}-(t-a)^{\gamma_1}}{\gamma_1}\right)^{\alpha_1-1}\left(\frac{(y-c)^{\gamma_2}-(s-c)^{\gamma_2}}{\gamma_2}\right)^{\beta_1-1}\\
&\times\left[\frac{1}{\Gamma(\alpha_2)\Gamma(\beta_2)}\int_a^t\int_c^s\left(\frac{(t-a)^{\gamma_1}-(u-a)^{\gamma_1}}{\gamma_1}\right)^{\alpha_2-1}\left(\frac{(s-c)^{\gamma_2}-(v-c)^{\gamma_2}}{\gamma_2}\right)^{\beta_2-1}\right.\\
&\left.\times\frac{f(u,v)dvdu}{(u-a)^{1-\gamma_1}(v-c)^{1-\gamma_2}}\right]\frac{dsdt}{(s-c)^{1-\gamma_2}(t-a)^{1-\gamma_1}},
\end{aligned}$$

Here, let's change the order of integration to calculate the integral.

$$\begin{aligned}
&\left({}^{\gamma_1,\gamma_2}I^{\alpha_1,\beta_1}_{a^+,c^+}\left[{}^{\gamma_1,\gamma_2}I^{\alpha_2,\beta_2}_{a^+,c^+}f(x,y)\right]\right)\\
=\ & \frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\frac{1}{\Gamma(\alpha_2)\Gamma(\beta_2)}\int_a^x\int_c^y f(v,u)\frac{du}{(u-c)^{1-\gamma_2}}\frac{dv}{(v-a)^{1-\gamma_1}}\\
&\times\left[\int_u^x\int_v^y\left(\frac{(x-a)^{\gamma_1}-(t-a)^{\gamma_1}}{\gamma_1}\right)^{\alpha_1-1}\left(\frac{(y-c)^{\gamma_2}-(s-c)^{\gamma_2}}{\gamma_2}\right)^{\beta_1-1}\right.\\
&\left.\left(\frac{(t-a)^{\gamma_1}-(u-a)^{\gamma_1}}{\gamma_1}\right)^{\alpha_2-1}\left(\frac{(s-c)^{\gamma_2}-(v-c)^{\gamma_2}}{\gamma_2}\right)^{\beta_2-1}\frac{ds}{(s-c)^{1-\gamma_2}}\frac{dt}{(t-a)^{1-\gamma_1}}\right].
\end{aligned}$$

Here, we have used the change of variable,

$$(t-a)^{\gamma_1}=(u-a)^{\gamma_1}+\left[(x-a)^{\gamma_1}-(u-a)^{\gamma_1}\right]\theta,$$

and

$$(s-c)^{\gamma_2}=(v-c)^{\gamma_2}+\left[(y-c)^{\gamma_2}-(v-c)^{\gamma_2}\right]\phi,$$

we obtain

$$\begin{aligned}
&\left({}^{\gamma_1,\gamma_2}I^{\alpha_1,\beta_1}_{a^+,c^+}\left[{}^{\gamma_1,\gamma_2}I^{\alpha_2,\beta_2}_{a^+,c^+}f(x,y)\right]\right)\\
=\ & \frac{1}{\Gamma(\alpha_1)\Gamma(\beta_1)}\frac{1}{\Gamma(\alpha_2)\Gamma(\beta_2)}\int_a^x\int_c^y\left(\frac{(x-a)^{\gamma_1}-(u-a)^{\gamma_1}}{\gamma_1}\right)^{\alpha_1+\alpha_2-1}\\
&\times\left(\frac{(y-c)^{\gamma_2}-(v-c)^{\gamma_2}}{\gamma_2}\right)^{\beta_1+\beta_2-1}\left[\int_0^1\int_0^1(1-\theta)^{\alpha_1-1}\theta^{\alpha_2-1}(1-\phi)^{\beta_1-1}\phi^{\beta_2-1}d\phi d\theta\right]\\
&\times f(v,u)\frac{du}{(u-c)^{1-\gamma_2}}\frac{dv}{(v-a)^{1-\gamma_1}}.
\end{aligned}$$

From $\int_0^1(1-y)^{\beta-1}y^{\gamma-1}dy=\frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)}$, we have;

$$\begin{aligned}
&\left({}^{\gamma_1,\gamma_2}I^{\alpha_1,\beta_1}_{a^+,c^+}\left[{}^{\gamma_1,\gamma_2}I^{\alpha_2,\beta_2}_{a^+,c^+}f(x,y)\right]\right)\\
=\ & \frac{1}{\Gamma(\alpha_1+\alpha_2)\Gamma(\beta_1+\beta_2)}\int_a^x\int_c^y\left(\frac{(x-a)^{\gamma_1}-(v-a)^{\gamma_1}}{\gamma_1}\right)^{\alpha_1+\alpha_2-1}\\
&\times\left(\frac{(y-c)^{\gamma_2}-(u-c)^{\gamma_2}}{\gamma_2}\right)^{\beta_1+\beta_2-1}f(v,u)\frac{du}{(u-c)^{1-\gamma_2}}\frac{dv}{(v-a)^{1-\gamma_1}}\\
=\ & \left({}^{\gamma_1,\gamma_2}I^{\alpha_1+\alpha_2,\beta_1+\beta_2}_{a^+,c^+}\right)f(x,y)
\end{aligned}$$

Thus, the first identity is proved. The other formulas can be proved in a similar way. Thus, the proof of this theorem is complete. □

3. Conformable Derivatives for the functions of two variables

In this section we will give generalized conformable derivatives.

Definition 3.1. *Let $f \in L_1([a,b]\times[c,d])$ and $\gamma_1 \neq 0,\ \gamma_2 \neq 0$. The Conformable Derivatives $D^{\alpha,\beta}_{a^+,c^+}$, $D^{\alpha,\beta}_{a^+,d^-}$, $D^{\alpha,\beta}_{b^-,d^-}$ and $D^{\alpha,\beta}_{b^-,c^+}$ of order $Re(\alpha)>0$ and $Re(\beta)>0$, with $a,c\geq 0$ are defined by;*

$$\begin{aligned}\varphi(x,y) &= \frac{{}_aT^{\gamma_1}f(x){}_cT^{\gamma_2}f(y)}{\Gamma(1-\alpha)\Gamma(1-\beta)}\int_a^x\int_c^y\left[\frac{(x-a)^{\gamma_1}-(t-a)^{\gamma_1}}{\gamma_1}\right]^{-\alpha} \\ &\quad\times\left[\frac{(y-c)^{\gamma_2}-(u-c)^{\gamma_2}}{\gamma_2}\right]^{-\beta}\frac{f(t,u)dudt}{(t-a)^{1-\gamma_1}(u-c)^{1-\gamma_2}},\end{aligned}$$

where

$${}_aT^{\gamma_1}f(x)=(x-a)^{1-\gamma_1}\frac{\partial}{\partial x}$$

and

$${}_cT^{\gamma_2}f(y)=(y-c)^{1-\gamma_2}\frac{\partial}{\partial y}.$$

Proof. For $0<\alpha\leq 1$ and $0<\beta\leq 1$, consider the following eqnarray,

$$\begin{aligned}f(x,y) &= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(\beta)}\int_a^x\int_c^y\left[\frac{(x-a)^{\gamma_1}-(t-a)^{\gamma_1}}{\gamma_1}\right]^{\alpha-1} \\ &\quad\times\left[\frac{(y-c)^{\gamma_2}-(u-c)^{\gamma_2}}{\gamma_2}\right]^{\beta-1}\frac{\varphi(t,u)dudt}{(t-a)^{1-\gamma_1}(u-c)^{1-\gamma_2}}.\end{aligned}$$

By editing this eqnarray, we write

$$\begin{aligned}f(t,u) &= \frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(\beta)}\int_a^t\int_c^u\left[\frac{(t-a)^{\gamma_1}-(s-a)^{\gamma_1}}{\gamma_1}\right]^{\alpha-1} \\ &\quad\times\left[\frac{(u-c)^{\gamma_2}-(z-c)^{\gamma_2}}{\gamma_2}\right]^{\beta-1}\frac{\varphi(s,z)dzds}{(s-a)^{1-\gamma_1}(z-c)^{1-\gamma_2}}.\end{aligned}$$

This eqnarray multiplied by $\left[\frac{(x-a)^{\gamma_1}-(t-a)^{\gamma_1}}{\gamma_1}\right]^{-\alpha}\left[\frac{(y-c)^{\gamma_2}-(u-c)^{\gamma_2}}{\gamma_2}\right]^{-\beta}\frac{1}{(t-a)^{1-\gamma_1}}\frac{1}{(u-c)^{1-\gamma_2}}$ to get the integral from a to x and c to y, we get,

$$\begin{aligned}&\int_a^x\int_c^y\left[\frac{(x-a)^{\gamma_1}-(t-a)^{\gamma_1}}{\gamma_1}\right]^{-\alpha}\left[\frac{(y-c)^{\gamma_2}-(u-c)^{\gamma_2}}{\gamma_2}\right]^{-\beta}\frac{f(t,u)dudt}{(t-a)^{1-\gamma_1}(u-c)^{1-\gamma_2}} \\ =&\int_a^x\int_c^y\left[\frac{(x-a)^{\gamma_1}-(t-a)^{\gamma_1}}{\gamma_1}\right]^{-\alpha}\left[\frac{(y-c)^{\gamma_2}-(u-c)^{\gamma_2}}{\gamma_2}\right]^{-\beta}\frac{1}{(t-a)^{1-\gamma_1}(u-c)^{1-\gamma_2}} \\ &\times\left(\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(\beta)}\int_a^t\int_c^u\left[\frac{(t-a)^{\gamma_1}-(s-a)^{\gamma_1}}{\gamma_1}\right]^{\alpha-1}\left[\frac{(u-c)^{\gamma_2}-(z-c)^{\gamma_2}}{\gamma_2}\right]^{\beta-1}\right. \\ &\left.\frac{\varphi(s,z)dzds}{(s-a)^{1-\gamma_1}(z-c)^{1-\gamma_2}}\right)dudt.\end{aligned}\tag{3.1}$$

Here, if the Leibniz rule is applied, the following expression is obtained.

$$\begin{aligned}a &< t<x\longrightarrow a<s<t\Longrightarrow a<s<t<x \\ c &< u<y\longrightarrow c<z<u\Longrightarrow c<z<u<y\end{aligned}$$

Thus, the eqnarray (3.1) can be written as follows.

$$\begin{aligned}&\int_a^x\int_c^y\left[\frac{(x-a)^{\gamma_1}-(t-a)^{\gamma_1}}{\gamma_1}\right]^{-\alpha}\left[\frac{(y-c)^{\gamma_2}-(u-c)^{\gamma_2}}{\gamma_2}\right]^{-\beta}\frac{f(t,u)dudt}{(t-a)^{1-\gamma_1}(u-c)^{1-\gamma_2}} \\ =&\ \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y\frac{\varphi(s,z)}{(s-a)^{1-\gamma_1}(z-c)^{1-\gamma_2}} \\ &\times\left(\int_s^x\int_z^y\left[\frac{(x-a)^{\gamma_1}-(t-a)^{\gamma_1}}{\gamma_1}\right]^{-\alpha}\left[\frac{(y-c)^{\gamma_2}-(u-c)^{\gamma_2}}{\gamma_2}\right]^{-\beta}\right. \\ &\times\left.\left[\frac{(t-a)^{\gamma_1}-(s-a)^{\gamma_1}}{\gamma_1}\right]^{\alpha-1}\left[\frac{(u-c)^{\gamma_2}-(z-c)^{\gamma_2}}{\gamma_2}\right]^{\beta-1}\frac{dudt}{(t-a)^{1-\gamma_1}(u-c)^{1-\gamma_2}}\right)dzds\end{aligned}\tag{3.2}$$

Now let's calculate the inner integral. Thus,

$$\begin{aligned}
&\int_s^x\int_z^y \left[\frac{(x-a)^{\gamma_1}-(t-a)^{\gamma_1}}{\gamma_1}\right]^{-\alpha}\left[\frac{(y-c)^{\gamma_2}-(u-c)^{\gamma_2}}{\gamma_2}\right]^{-\beta}\\
&\times\left[\frac{(t-a)^{\gamma_1}-(s-a)^{\gamma_1}}{\gamma_1}\right]^{\alpha-1}\left[\frac{(u-c)^{\gamma_2}-(z-c)^{\gamma_2}}{\gamma_2}\right]^{\beta-1}\frac{dudt}{(t-a)^{1-\gamma_1}(u-c)^{1-\gamma_2}}\\
=\;&\int_s^x \left[\frac{(x-a)^{\gamma_1}-(t-a)^{\gamma_1}}{\gamma_1}\right]^{-\alpha}\left[\frac{(t-a)^{\gamma_1}-(s-a)^{\gamma_1}}{\gamma_1}\right]^{\alpha-1}\frac{dt}{(t-a)^{1-\gamma_1}}\\
&\times\left(\int_z^y \left[\frac{(y-c)^{\gamma_2}-(u-c)^{\gamma_2}}{\gamma_2}\right]^{-\beta}\left[\frac{(u-c)^{\gamma_2}-(z-c)^{\gamma_2}}{\gamma_2}\right]^{\beta-1}\frac{du}{(u-c)^{1-\gamma_2}}\right).
\end{aligned}$$

First, consider the following integral.

$$\int_z^y \left[\frac{(y-c)^{\gamma_2}-(u-c)^{\gamma_2}}{\gamma_2}\right]^{-\beta}\left[\frac{(u-c)^{\gamma_2}-(z-c)^{\gamma_2}}{\gamma_2}\right]^{\beta-1}\frac{du}{(u-c)^{1-\gamma_2}}$$

If we apply the formula from right to left and make the substitution,

$$(u-c)^{\gamma_2}=(z-c)^{\gamma_2}+\left[(y-c)^{\gamma_2}-(z-c)^{\gamma_2}\right]\theta$$

we obtain

$$\gamma_2(u-c)^{\gamma_2-1}du=\left[(y-c)^{\gamma_2}-(z-c)^{\gamma_2}\right]d\theta.$$

Thus, the integral becomes,

$$\begin{aligned}
&\int_z^y \left[\frac{(y-c)^{\gamma_2}-(u-c)^{\gamma_2}}{\gamma_2}\right]^{-\beta}\left[\frac{(u-c)^{\gamma_2}-(z-c)^{\gamma_2}}{\gamma_2}\right]^{\beta-1}\frac{du}{(u-c)^{1-\gamma_2}}\\
=\;&\int_0^1 \left[\frac{(y-c)^{\gamma_2}-(z-c)^{\gamma_2}-\left[(y-c)^{\gamma_2}-(z-c)^{\gamma_2}\right]\theta}{\gamma_2}\right]^{-\beta}\\
&\times\left[\frac{(z-c)^{\gamma_2}+\left[(y-c)^{\gamma_2}-(z-c)^{\gamma_2}\right]\theta-(z-c)^{\gamma_2}}{\gamma_2}\right]^{\beta-1}\frac{(y-c)^{\gamma_2}-(z-c)^{\gamma_2}}{\gamma_2}d\theta\\
=\;&\int_0^1 \theta^{-\beta}(1-\theta)^{\beta-1}d\theta\\
=\;&\Gamma(1-\beta)\Gamma(\beta).
\end{aligned}$$

Similarly, we have the following integral for this $(t-a)^{\gamma_1}=(s-a)^{\gamma_1}+\left[(x-a)^{\gamma_1}-(s-a)^{\gamma_1}\right]\phi$ change of variables,

$$\int_s^x \left[\frac{(x-a)^{\gamma_1}-(t-a)^{\gamma_1}}{\gamma_1}\right]^{-\alpha}\left[\frac{(t-a)^{\gamma_1}-(s-a)^{\gamma_1}}{\gamma_1}\right]^{\alpha-1}\frac{dt}{(t-a)^{1-\gamma_1}}=\Gamma(1-\alpha)\Gamma(\alpha).$$

If these results are written in their place in eqnarray (3.2), we get the following eqnarray.

$$\begin{aligned}
&\int_a^x\int_c^y \left[\frac{(x-a)^{\gamma_1}-(t-a)^{\gamma_1}}{\gamma_1}\right]^{-\alpha}\left[\frac{(y-c)^{\gamma_2}-(u-c)^{\gamma_2}}{\gamma_2}\right]^{-\beta}\frac{f(t,u)dudt}{(t-a)^{1-\gamma_1}(u-c)^{1-\gamma_2}}\\
=\;&\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(1-\alpha)\Gamma(1-\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y\frac{\varphi(s,z)dzds}{(s-a)^{1-\gamma_1}(z-c)^{1-\gamma_2}},
\end{aligned}$$

and so,

$$\begin{aligned}
&\int_a^x\int_c^y \left[\frac{(x-a)^{\gamma_1}-(t-a)^{\gamma_1}}{\gamma_1}\right]^{-\alpha}\left[\frac{(y-c)^{\gamma_2}-(u-c)^{\gamma_2}}{\gamma_2}\right]^{-\beta}\frac{f(t,u)dudt}{(t-a)^{1-\gamma_1}(u-c)^{1-\gamma_2}}\\
=\;&\Gamma(1-\alpha)\Gamma(1-\beta)\int_a^x\int_c^y\frac{\varphi(s,z)dzds}{(s-a)^{1-\gamma_1}(z-c)^{1-\gamma_2}}
\end{aligned}$$

Using the fundamental theorem of calculus to find the derivative (with respect to x and y) of the last integral, we get

$$\begin{aligned}
\frac{\varphi(x,y)}{(x-a)^{1-\gamma_1}(y-c)^{1-\gamma_2}} &= \frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\int_a^x\int_c^y\left[\frac{(x-a)^{\gamma_1}-(t-a)^{\gamma_1}}{\gamma_1}\right]^{-\alpha}\\
&\times\left[\frac{(y-c)^{\gamma_2}-(u-c)^{\gamma_2}}{\gamma_2}\right]^{-\beta}\frac{f(t,u)dudt}{(t-a)^{1-\gamma_1}(u-c)^{1-\gamma_2}}.
\end{aligned}$$

So,

$$\varphi(x,y) = \frac{(x-a)^{1-\gamma_1}(y-c)^{1-\gamma_2}}{\Gamma(1-\alpha)\Gamma(1-\beta)} \frac{\partial}{\partial x}\frac{\partial}{\partial y}\int_a^x\int_c^y \left[\frac{(x-a)^{\gamma_1}-(t-a)^{\gamma_1}}{\gamma_1}\right]^{-\alpha} \times \left[\frac{(y-c)^{\gamma_2}-(u-c)^{\gamma_2}}{\gamma_2}\right]^{-\beta} \frac{f(t,u)dudt}{(t-a)^{1-\gamma_1}(u-c)^{1-\gamma_2}}.$$

The desired result is obtained. With the help of this result, the following definition can be given: □

Definition 3.2. *Let $f \in L_1([a,b]\times[c,d])$ and $\gamma_1 \neq 0,\ \gamma_2 \neq 0$. The Conformable Derivatives $D^{\alpha,\beta}_{a^+,c^+}$, $D^{\alpha,\beta}_{a^+,d^-}$, $D^{\alpha,\beta}_{b^-,d^-}$ and $D^{\alpha,\beta}_{b^-,c^+}$ of order $0<\alpha\leq 1$ and $0<\beta\leq 1$, with $a,c\geq 0$ are defined by*

$$D^{\alpha,\beta}_{a^+,c^+}f(x,y) = \frac{(x-a)^{1-\gamma_1}(y-c)^{1-\gamma_2}}{\Gamma(1-\alpha)\Gamma(1-\beta)} \frac{\partial}{\partial x}\frac{\partial}{\partial y}\int_a^x\int_c^y \left[\frac{(x-a)^{\gamma_1}-(t-a)^{\gamma_1}}{\gamma_1}\right]^{-\alpha} \times \left[\frac{(y-c)^{\gamma_2}-(u-c)^{\gamma_2}}{\gamma_2}\right]^{-\beta} \frac{f(t,u)dudt}{(t-a)^{1-\gamma_1}(u-c)^{1-\gamma_2}}, \tag{3.3}$$

$$D^{\alpha,\beta}_{a^+,d^-}f(x,y) = -\frac{(x-a)^{1-\gamma_1}(y-d)^{1-\gamma_2}}{\Gamma(1-\alpha)\Gamma(1-\beta)} \frac{\partial}{\partial x}\frac{\partial}{\partial y}\int_a^x\int_y^d \left[\frac{(x-a)^{\gamma_1}-(t-a)^{\gamma_1}}{\gamma_1}\right]^{-\alpha} \times \left[\frac{(u-d)^{\gamma_2}-(y-d)^{\gamma_2}}{\gamma_2}\right]^{-\beta} \frac{f(t,u)dudt}{(t-a)^{1-\gamma_1}(u-d)^{1-\gamma_2}}, \tag{3.4}$$

$$D^{\alpha,\beta}_{b^-,d^-}f(x,y) = \frac{(x-b)^{1-\gamma_1}(y-d)^{1-\gamma_2}}{\Gamma(1-\alpha)\Gamma(1-\beta)} \frac{\partial}{\partial x}\frac{\partial}{\partial y}\int_x^b\int_y^d \left[\frac{(t-b)^{\gamma_1}-(x-b)^{\gamma_1}}{\gamma_1}\right]^{-\alpha} \times \left[\frac{(u-d)^{\gamma_2}-(y-d)^{\gamma_2}}{\gamma_2}\right]^{-\beta} \frac{f(t,u)dudt}{(t-b)^{1-\gamma_1}(u-d)^{1-\gamma_2}}, \tag{3.5}$$

and

$$D^{\alpha,\beta}_{b^-,c^+}f(x,y) = -\frac{(x-b)^{1-\gamma_1}(y-c)^{1-\gamma_2}}{\Gamma(1-\alpha)\Gamma(1-\beta)} \frac{\partial}{\partial x}\frac{\partial}{\partial y}\int_x^b\int_y^d \left[\frac{(t-b)^{\gamma_1}-(x-b)^{\gamma_1}}{\gamma_1}\right]^{-\alpha} \times \left[\frac{(y-c)^{\gamma_2}-(u-c)^{\gamma_2}}{\gamma_2}\right]^{-\beta} \frac{f(t,u)dudt}{(t-b)^{1-\gamma_1}(u-c)^{1-\gamma_2}}, \tag{3.6}$$

respectively. Here, Γ is the Gamma function,

If we choose $\gamma_1=\gamma_2=1$ in (3.3), (3.4), (3.5) and (3.6), then we obtain the Riemann-Liouville Fractional derivatives for the functions of two variables.

$$D^{\alpha,\beta}_{a^+,c^+}f(x,y) = \frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)} \frac{\partial}{\partial x}\frac{\partial}{\partial y}\int_a^x\int_c^y \frac{f(t,u)dudt}{(x-t)^{\alpha}(y-u)^{\beta}},$$

$$D^{\alpha,\beta}_{a^+,d^-}f(x,y) = \frac{-1}{\Gamma(1-\alpha)\Gamma(1-\beta)} \frac{\partial}{\partial x}\frac{\partial}{\partial y}\int_a^x\int_y^d \frac{f(t,u)dudt}{(x-t)^{\alpha}(u-y)^{\beta}},$$

$$D^{\alpha,\beta}_{b^-,d^-}f(x,y) = \frac{1}{\Gamma(1-\alpha)\Gamma(1-\beta)} \frac{\partial}{\partial x}\frac{\partial}{\partial y}\int_x^b\int_y^d \frac{f(t,u)dudt}{(t-x)^{\alpha}(u-y)^{\beta}},$$

and

$$D^{\alpha,\beta}_{b^-,c^+}f(x,y) = \frac{-1}{\Gamma(1-\alpha)\Gamma(1-\beta)} \frac{\partial}{\partial x}\frac{\partial}{\partial y}\int_x^b\int_c^y \frac{f(t,u)dudt}{(t-x)^{\alpha}(y-u)^{\beta}}.$$

Theorem 3.3. *(Inverse property): Let* $0<\beta<1$ *and* $f\in X_c^p(a,b)$*. Let* $\rho>0$ *and* $a>0$*, we have;*

1. $\left[D_{a^+,c^+}^{\alpha,\beta}\left(I_{a^+,c^+}^{\alpha,\beta}\right)\right]f(x,y)=f(x,y),$

2. $\left[D_{a^+,d^-}^{\alpha,\beta}\left(I_{a^+,d^-}^{\alpha,\beta}\right)\right]f(x,y)=f(x,y),$

3. $\left[D_{b^-,d^-}^{\alpha,\beta}\left(I_{b^-,d^-}^{\alpha,\beta}\right)\right]f(x,y)=f(x,y),$

and

4. $\left[D_{b^-,c^+}^{\alpha,\beta}\left(I_{b^-,c^+}^{\alpha,\beta}\right)\right]f(x,y)=f(x,y).$

Proof. It is enough to prove the first equality. Other eqnarrays can be proved similarly. For this, considering the following eqnarray,

$$\begin{aligned}
D_{a^+,c^+}^{\alpha,\beta}f(x,y) &= \frac{{}_aT^{\gamma_1}f(x)_cT^{\gamma_2}f(y)}{\Gamma(1-\alpha)\Gamma(1-\beta)}\int_a^x\int_c^y\left[\frac{(x-a)^{\gamma_1}-(t-a)^{\gamma_1}}{\gamma_1}\right]^{-\alpha}\times\left[\frac{(y-c)^{\gamma_2}-(u-c)^{\gamma_2}}{\gamma_2}\right]^{-\beta}\left(I_{a^+,c^+}^{\alpha,\beta}f(t,u)\right)\frac{dudt}{(t-a)^{1-\gamma_1}(u-c)^{1-\gamma_2}}\\
&= \frac{{}_aT^{\gamma_1}f(x)_cT^{\gamma_2}f(y)}{\Gamma(1-\alpha)\Gamma(1-\beta)}\int_a^x\int_c^y\left[\frac{(x-a)^{\gamma_1}-(t-a)^{\gamma_1}}{\gamma_1}\right]^{-\alpha}\times\left[\frac{(y-c)^{\gamma_2}-(u-c)^{\gamma_2}}{\gamma_2}\right]^{-\beta}\frac{dudt}{(t-a)^{1-\gamma_1}(u-c)^{1-\gamma_2}}\\
&\quad\times\left[\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(\beta)}\int_a^t\int_c^u\left(\frac{(t-a)^{\gamma_1}-(z-a)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\times\left(\frac{(u-c)^{\gamma_2}-(s-c)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}f(z,s)\frac{ds}{(s-c)^{1-\gamma_2}}\frac{dz}{(z-a)^{1-\gamma_1}}\right].
\end{aligned}$$

Here, if the Leibniz rule is applied, the following expression is obtained.

$$\begin{aligned}
D_{a^+,c^+}^{\alpha,\beta}f(x,y) &= \frac{{}_aT^{\gamma_1}f(x)_cT^{\gamma_2}f(y)}{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y\frac{f(z,s)}{(s-c)^{1-\gamma_2}(z-a)^{1-\gamma_1}}\times\left[\int_z^x\int_s^y\left(\frac{(x-a)^{\gamma_1}-(t-a)^{\gamma_1}}{\gamma_1}\right)^{-\alpha}\left(\frac{(y-c)^{\gamma_2}-(u-c)^{\gamma_2}}{\gamma_2}\right)^{-\beta}\right.\\
&\quad\left.\left(\frac{(t-a)^{\gamma_1}-(z-a)^{\gamma_1}}{\gamma_1}\right)^{\alpha-1}\left(\frac{(u-c)^{\gamma_2}-(s-c)^{\gamma_2}}{\gamma_2}\right)^{\beta-1}\frac{dudt}{(t-a)^{1-\gamma_1}(u-c)^{1-\gamma_2}}\right]dsdz
\end{aligned}$$

If the following change of variables in the last integral,

$$(t-a)^{\gamma_1}=(z-a)^{\gamma_1}+\left[(x-a)^{\gamma_1}-(z-a)^{\gamma_1}\right]\theta$$

and

$$(u-c)^{\gamma_2}=(s-c)^{\gamma_2}+\left[(y-c)^{\gamma_2}-(s-c)^{\gamma_2}\right]\phi,$$

so, the integral becomes,

$$D_{a^+,c^+}^{\alpha,\beta}f(x,y)=\frac{{}_aT^{\gamma_1}f(x)_cT^{\gamma_2}f(y)}{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(\alpha)\Gamma(\beta)}\int_a^x\int_c^y\frac{f(z,s)dsdz}{(s-c)^{1-\gamma_2}(z-a)^{1-\gamma_1}}\times\int_0^1\int_0^1(1-\theta)^{-\alpha}\theta^{\alpha-1}(1-\phi)^{-\beta}\phi^{\beta-1}d\phi d\theta.$$

Where,

$$\begin{aligned}
\int_0^1\int_0^1(1-\theta)^{-\alpha}\theta^{\alpha-1}(1-\phi)^{-\beta}\phi^{\beta-1}d\phi d\theta &= \frac{\Gamma(1-\alpha)\Gamma(\alpha)\Gamma(1-\beta)\Gamma(\beta)}{\Gamma(1-\alpha+\alpha)\Gamma(1-\beta+\beta)}\\
&= \Gamma(1-\alpha)\Gamma(\alpha)\Gamma(1-\beta)\Gamma(\beta).
\end{aligned}$$

Thus,

$$D_{a^+,c^+}^{\alpha,\beta}f(x,y)=\frac{\Gamma(1-\alpha)\Gamma(\alpha)\Gamma(\beta)\Gamma(1-\beta)}{\Gamma(\alpha)\Gamma(1-\alpha)\Gamma(\beta)\Gamma(1-\beta)}\,{}_aT^{\gamma_1}f(x)_cT^{\gamma_2}f(y)\int_a^x\int_c^y\frac{f(z,s)dsdz}{(s-c)^{1-\gamma_2}(z-a)^{1-\gamma_1}}\,{}_aT^{\gamma_1}f(x)_cT^{\gamma_2}f(y)\int_a^x\int_c^y\frac{f(z,s)}{(s-c)^{1-\gamma_2}(z-a)^{1-\gamma_1}}dsdz.$$

Considering the following eqnarrays,

$${}_aT^{\gamma_1}f(x)=(x-a)^{1-\gamma_1}\frac{\partial}{\partial x}$$

and

$${}_cT^{\gamma_2}f(y)=(y-c)^{1-\gamma_2}\frac{\partial}{\partial y}.$$

Thus,

$$D_{a^+,c^+}^{\alpha,\beta}f(x,y)=(x-a)^{1-\gamma_1}(y-c)^{1-\gamma_2}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\int_a^x\int_c^y\frac{f(z,s)}{(s-c)^{1-\gamma_2}(z-a)^{1-\gamma_1}}dsdz=f(x,y)$$

The proof of the theorem is completed. □

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