Actions of Double Group-Groupoids and Covering Morphism

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Highlights
• Covering morphism of double groupoids derived by action double groupoid was considered.
• Action double group-groupoid on a group-groupoid was characterized.
• Covering morphism of double group-groupoids was obtained.
• A categorical equivalence was proved.

Abstract

The purpose of the paper is to consider the covering morphism and the action of a double groupoid on a groupoid; and then characterize these notions for double group- groupoids. We also give a categorical equivalence between the actions and covering morphisms of double group-groupoids.

1. INTRODUCTION

The concept of actions and coverings of groupoids have a significant role in the utilization of groupoids (see [1] and [2]). It is familiar that for a certain groupoid G, the category $GpdAct(G)$ of groupoid actions of G on sets and the category $GpdCov/G$ of covering morphisms of G are equivalent. Topological version of this equivalence was given in [3]. An associated categorical equivalence was proved in [4, Proposition 3.1] for the case G is a group-groupoid used under the name *2-group* [5] and G -*groupoids* [6]. For different generalizations of this result see [7-10].

Double groupoids defined by Ehresmann in [11, 12] as internal groupoids in the category of groupoids are useful to calculate the fundamental groupoids of topological spaces [13]. By [14] double groupoids are categorically equivalent to crossed modules defined by Whitehead in [15, 16] and this categorical equivalence was characterized for group-groupoids in [17]. Quotient and normal double groupoids were characterized in [18]. In [19, Theorem 1.7] it was showed that horizontal actions and horizontal action morphisms of a double Lie groupoid are equivalent.

In this paper, for the convenience of the presentation we introduce some preliminaries on double groupoids, action of a double groupoid on a groupoid, covering groupoid and some related results. Then we extend these notions to double group-groupoids. More precisely, we define the concept of covering morphism for double group-groupoids. Finally, we prove a categorical equivalence between horizontal actions and covering morphisms associated with horizontal actions of a double group-groupoi.

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2. PRELIMINARIES ON GROUPOIDS AND ACTIONS

A category whose all arrows are isomorphism is called a *groupoid.* (see [1] and [20] for more details on groupoids). So a groupoid G base on G_0 usually denoted by G or (G, G_0), has a set G of arrows or elements and a set G_0 of *objects* with *source* and *target* maps $d_0, d_1 : G \rightarrow G_0$ and *object inclusion* map $\varepsilon : G_0 \rightarrow G$ satisfying the property $d_0 \varepsilon = d_1 \varepsilon = 1_{G_0}$. For $x \in G_0$ the morphism $\varepsilon(x)$ acts as the identity and it is denoted by 1_x. An associative partial composition of G is $G * G \rightarrow G, (g, h) \mapsto g \circ h$, where $G * G$ is $\{(g, h) \in G * G | d_1(g) = d_0(h)\}$. The *composite* $g \circ h$ for $g, h \in G$ exists whenever $d_1(g) = d_0(h)$. Here $d_0(g \circ h) = d_0(g)$ and $d_1(g \circ h) = d_1(h)$. Each element g has an inverse g^{-1} with $d_0(g^{-1}) = d_1(g)$, $d_1(g^{-1}) = d_0(g)$, $g \circ g^{-1} = \varepsilon(d_0(g))$, $g^{-1} \circ g = \varepsilon(d_1(g))$. The *inversion* map is defined to be $G \rightarrow G, g \mapsto g^{-1}$. The source and target, the object inclusion, the inversion maps and the partial composition are all called *structural maps*.

For a groupoid G let $x, y \in G_0$. We will use $G(x, y)$ to denote the set of all arrows whose source x and target y and $G(x, x)$ denote the *object group* at x. As mentioned in [1], for $x \in G_0$ the set $\{a \in G | d_0(a) = x\}$ of arrows with source x is called *star* of x and we denote it by $St_G x$.

A morphism from the groupoid H to G consists of two maps $f_1 : H \rightarrow G$ and $f_0 : H_0 \rightarrow G_0$ with the property $f_1(h \circ h') = f_1(h) \circ f_1(h')$ when $(h, h') \in H * H$ subject to the conditions $f_0 d_1 = d_0 f_1$ and $f_0 d_0 = d_0 f_1$. We write $f : H \rightarrow G$ for a groupoid morphism.

If $p : H \rightarrow G$ is a morphism and for each $y \in H_0$, the restriction of p, $St_H y \rightarrow St_G p(y)$, is bijective, then p is named as a *covering morphism* and H a *covering groupoid* of G.

We remind groupoid action on a set as follows [1, p.373].

Definition 2.1. An *action of a groupoid G on a set X* includes two functions $\omega : X \rightarrow G_0$ and $\varphi : X * G \rightarrow X, (x, g) \mapsto x \bullet g$ where $X * G$ is the pullback of d_0 and ω and the following properties hold

   i. $\omega(x \bullet g) = d_1(g)$ for $(x, g) \in X * G$,

   ii. $x \bullet (g_1 \circ g_2) = (x \bullet g_1) \bullet g_2$ for $(g_1, g_2) \in G * G$ and $(x, g) \in X * G$,

   iii. $x \bullet \varepsilon(\omega(x)) = x$ for $x \in X$.

Write (X, ω) for an action. A morphism $f : (X, \omega) \rightarrow (X', \omega')$ is a function $f : X \rightarrow X'$ with $\omega' f = \omega$ and $f(x \bullet g) = f(x) \bullet g$. So we have a category $GpdAct(G)$ such that objects are actions and arrows are morphisms.

Let's given such an action. According to [1, p.374], we have a groupoid $G \ltimes \omega$ base on X which is called *semidirect product groupoid* or *action groupoid*. For $x, y \in X$, arrows of $(G \ltimes \omega)(x, y)$ are (g, x) where $g \in G(\omega(x), \omega(y))$ and $x \bullet g = y$. The composition is defined by

$$(g_1, x) \circ (g_2, y) = (g_1 \circ g_2, x)$$

under the condition $y = x \bullet g_1$.

The projection map $p : G \ltimes \omega \to G$ defined on base by $\omega : X \to G_0$ and on arrows by $(g, s) \to g$ is a covering morphism. We call this *action covering morphism*. Hence by this assignment we have a categorical equivalence

$$GpdAct(G) \simeq GpdCov / G$$

For a topological treatment of this equivalence we refer the readers to [3].

A groupoid whose base and arrow sets have both group structures and the structural maps are group-homomorphism is called *group-groupoid*. Here $g \circ h$ is the composition of arrows in groupoid while $g + h$ is the addition in group and write g^{-1} for the inverse of g in groupoid and $-g$ for the one in group. Note that the groupoid composition is a group homomorphism if and only if the following *interchange rule* for $g, h, k, l \in G$ hold whenever one side composition is well defined

$$(g + h) \circ (k + l) = (g \circ k) + (h \circ l).$$

A morphism of groupoids preserving group addition is a morphism of group-groupoids. So group-groupoids and morphisms between them form a category *GpGpd* .

In the action of a group-groupoid G on group X we have a group homomorphism $\omega : X \to G_0$ and an action of groupoid G on the set X by ω providing that

$$(x \bullet g) + (x' \bullet g') = (x + x') \bullet (g + g').$$

A morphism of group-groupoid actions from (X, ω) to (X', ω') is a group homomorphism $f : X \to X'$ which is a morphism on underlying actions of G . This constitutes a category *GpGpdAct(G)* . For this kind of action, we have semidirect product groupoid $G \ltimes \omega$ which is a group-groupoid with group composition

$$(g, x) + (g', x') = (g + g', x + x').$$

The projection map $p : G \ltimes \omega \to G$ is an object of *GpGpdCov / G* . Then by [4, Proposition 3.1], we have the categorical equivalence stated in the next theorem.

Theorem 2.2. The categories *GpGpdCov / G* and *GpGpdAct(G)* are equivalent.

3. ACTIONS OF DOUBLE GROUPOIDS ON GROUPOIDS

In this part we consider the covering morphism of double groupoids derived by action double groupoid following [19].

A double groupoid has four related groupoid structures with compatible structural maps. Two of these groupoids are H and V with base P . Other two are on S which are vertical groupoid base on H denoted by S_V and horizontal groupoid base on V denoted by S_H . From now on the set of quadruple $(S; H, V; P)$ denote a double groupoid.

For a double groupoid we will use multiplicative notation for groupoid compositions in H and 1_b for the identity element in H for $b \in P$. The source and target maps, object inclusion, multiplication for H are $d_0^H, d_1^H : H \to P, \varepsilon^H : P \to H$ and $m^H : H * H \to H$ respectively and similar notations will be used for V.

The horizontal groupoid structure S_H on S with base V have source and target $d_0^h, d_1^h : S \to V$, object inclusion $\varepsilon^h : V \to S$ and partial composition $\circ_h : S * S \to S, (s_1, s_2) \mapsto s_1 \circ_h s_2$. Inverse is denoted by s^{-h} for $h \in H$. By a similar argument, vertical groupoid S_V on S with base H have source and target $d_0^v, d_1^v : S \to H$, object inclusion $\varepsilon^v : H \to S$ and partial composition $\circ_v : S * S \to S$, $(s_1, s_2) \mapsto s_1 \circ_v s_2$. Inverse is denoted by s^{-v}.

Elements of S are squares with boundaries as follows.

$$
\begin{array}{c}
d_0^v s \\
d_0^h s \quad \boxed{\quad s \quad} \quad d_1^h s \\
d_1^v s
\end{array}
$$

A double groupoid have the following interchange rule

$$(s_1 \circ_h s_2) \circ_v (s_3 \circ_h s_4) = (s_1 \circ_v s_3) \circ_h (s_2 \circ_v s_4)$$

for $s_1, s_2, s_3, s_4 \in S$. Let $(S; H, V; P)$ and $(S'; H', V'; P')$ be two double groupoids. A morphism

$$\phi = (\phi_s, \phi_h, \phi_v, \phi_p) : (S'; H', V'; P') \to (S; H, V; P)$$

consists of four maps which commute with structural maps. These constitute a category $DGpd$ of double groupoids.

We state following construction which is originally given for Lie case appears in [19, Theorem1.7].

Definition 3.1. For a quadruple $S = (S; H, V; P)$ and a groupoid G, let $\omega : G \to V$ and $\omega_0 : G_0 \to P$ be morphisms. A horizontal action of double groupoid S on (ω, ω_0) includes actions of S_H on $\omega : G \to V$ and H on $\omega_0 : G_0 \to P$ in the sense of ordinary groupoid actions such that

   i. $d_1^G(g \bullet s) = d_1^G(g) \bullet d_1^v(s)$ and $d_0^G(g \bullet s) = d_0^G(g) \bullet d_0^v(s)$ for all $s \in S, g \in G$ satisfying
       $d_0^h(s) = \omega(g)$,

   ii.   For $s_1, s_2 \in S$ and $g_1, g_2 \in G$ we have
$$(g_1 g_2) \bullet (s_1 \circ_v s_2) = (g_1 \bullet s_1) \circ_v (g_2 \bullet s_2),$$

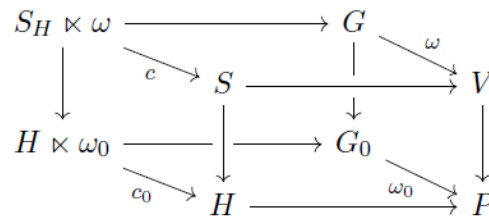   iii. For all $h \in H$ and $x \in G_0$ with $d_0^H(h) = \omega_0(x)$ we have $1_x \bullet \varepsilon^v(h) = 1_{hx}$.

There is also similar vertical action of double groupoid that consists of actions of S_V on $\omega: G \to H$ and of V on $\omega_0 : G_0 \to P$.

Denote such a horizontal action by $(G, (\omega, \omega_0))$ and define $f : (G, (\omega, \omega_0)) \to (G', (\omega', \omega'_0))$ between such actions to be consisting of groupoid homomorphisms $f : G \to G'$ and $f_0 : G_0 \to G'_0$ with the properties $f(g \bullet s) = f(g) \bullet s$ and $f_0(x \bullet h) = f_0(x) \bullet h$ such that the equalities $\omega' f = \omega$ and $\omega'_0 f_0 = \omega_0$ hold. Hence for given a double groupoid S we have a category denoted by $DGpdAct_H(S)$.

Consider given a horizontal action of S on (ω, ω_0). Then we have a quadruple $(S_H \ltimes \omega; H \ltimes \omega_0, G; G_0)$ denoted by $S \ltimes (\omega, \omega_0)$ and called *action double groupoid* with respect to horizontal action of S on (ω, ω_0).

Remark that we have projections $(c, \omega) : S \ltimes \omega \to G$ and $(c_0, \omega_0) : H \ltimes \omega_0 \to G_0$ which are covering morphisms of ordinary groupoids.

$$
\begin{array}{ccc}
S_H \ltimes \omega \longrightarrow & G & \\
\Big\downarrow \quad \overset{c}{\searrow} \; S \longrightarrow & V & \overset{\omega}{\nearrow} \\
H \ltimes \omega_0 \longrightarrow & G_0 & \\
\quad \overset{c_0}{\searrow} \; H \longrightarrow & P & \overset{\omega_0}{\nearrow}
\end{array}
$$

Elements of $S \ltimes (\omega, \omega_0)$ are the squares
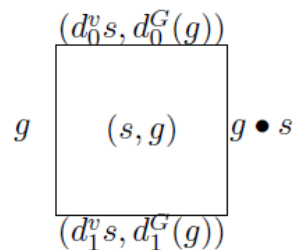
with compositions

$$(s_1, g_1) \circ_h (s_2, g_2) = (s_1 \circ_h s_2, g_1)$$

$$(s_1, g_1) \circ_v (s_2, g_2) = (s_1 \circ_v s_2, g_1 g_2).$$

Definition 3.2. A morphism of double groupoids $\phi = (\phi_s, \phi_h, \phi_v, \phi_p) : (S'; H', V'; P') \to (S; H, V; P)$ is called covering morphism associated with the horizontal action if (ϕ_s, ϕ_v) and (ϕ_h, ϕ_p) are covering morphisms of ordinary groupoids.

Let *CovDGpd / S* denote the category of covering morphisms of double groupoids obtained from horizontal action. The following theorem which is originally given for Lie double groupoids in [19, Theorem 1.7] expresses the categorical equivalence of action double groupoids and corresponding covering morphisms.

Theorem 3.3. The following equivalence of the categories holds for a double groupoid S

$$CovDGpd \, / \, S \simeq DGpdAct_H(S) \, .$$

4. CATEGORICAL EQUIVALENCE BETWEEN ACTIONS AND COVERING MORPHISMS OF DOUBLE GROUP-GROUPOIDS

In this section we characterize action of a double group-groupoid on a group-groupoid and then obtain covering morphism of double group-groupoids corresponding to the horizontal action. By this way for a given double group-groupoid S, we obtain a categorical equivalence between actions and covering morphisms of the double group-groupoid S.

A *double group-groupoid* is defined in [17] to be an internal groupoid in the category of group-groupoids. Therefore a double group-groupoid consists of four related group- groupoids (S,V), (S,H), (H,P) and (V,P) which satisfies the following

$$(s_1 \circ_h s_2) + (s_3 \circ_h s_4) = (s_1 + s_3) \circ_h (s_2 + s_4)$$

$$(s_1 \circ_v s_2) + (s_3 \circ_v s_4) = (s_1 + s_3) \circ_v (s_2 + s_4) \, .$$

Morphism of double group-groupoids is defined to be double groupoid morphisms such that each function is also group homomorphism. Hence we have a category $DGpGpd$.

Example 4.1. Assume that G is a topological group-groupoid. Then G and G_0 have topological group structures. So the fundamental groupoids πG of G and πG_0 of G_0 are group-groupoids. Further we have two related groupoids $(\pi G, \pi G_0)$ and (G, G_0). Therefore the quadruple $(\pi G; \pi G_0, G; G_0)$ becomes a double group-groupoid.

Definition 4.2. For given a group-groupoid G, a horizontal action of a double group- groupoid $S = (S; H, V; P)$ on group-groupoid morphism (ω, ω_0) consists of group homomorphisms $\omega : G \to V$, $\omega_0 : G_0 \to P$ and a horizontal action of double groupoid on (ω, ω_0) such that the following interchange rules hold

    i.   For given $s_1, s_2 \in S$ and $g_1, g_2 \in G$,

$$(g_1 + g_2) \bullet (s_1 + s_2) = (g_1 \bullet s_1) + (g_2 \bullet s_2) \, .$$

    ii.   For $x, x_1 \in G_0$ and $h, h_1 \in H$ we have

$$(x + x_1) \bullet (h + h_1) = (x \bullet h) + (x_1 \bullet h_1) \, .$$

There is also a similar vertical action of double group-groupoids.

Denote such a horizontal action by $(G, (\omega, \omega_0))$ and define a morphism $f : (G, (\omega, \omega_0)) \to (G', (\omega', \omega'_0))$ between such actions to be consisting of group-groupoid morphisms $f : G \to G'$ and $f_0 : G_0 \to G'_0$ with $f(g \bullet s) = f(g) \bullet s$ and $f_0(x \bullet h) = f_0(x) \bullet h$ such that $\omega' f = \omega$ and $\omega'_0 f_0 = \omega_0$. Hence horizontal actions of double group-groupoids and morphisms form a category $DGpGpdAct_H(S)$.

Theorem 4.3. For given a horizontal action of double group-groupoid S on (ω, ω_0), we have a double group-groupoid $(S \ltimes \omega; H \ltimes \omega_0, G; G_0)$.

Proof: For given a horizontal action of double group-groupoid S on (ω, ω_0) we have semidirect product groupoids $S_H \ltimes \omega$ and $H \ltimes \omega_0$ which are group-groupoids. Therefore $(S_H \ltimes \omega; H \ltimes \omega_0, G; G_0)$ is an action double groupoid with group structure

$$(s_1, g_1) + (s_2, g_2) = (s_1 + s_2, g_1 + g_2).$$

The details are straightforward. Thus the quadruple becomes a double group-groupoid.

We call such a double group-groupoid *semidirect product double group-groupoid* or *action double group-groupoid* with respect to horizontal action of S on (ω, ω_0) and denoteby $S \ltimes (\omega, \omega_0)$.

Definition 4.4. A morphism $\phi = (\phi_s, \phi_h, \phi_v, \phi_p) : (S'; H', V'; P') \rightarrow (S; H, V; P)$ of double group-groupoids is called covering morphism associated with the horizontal action if (ϕ_s, ϕ_v) and (ϕ_h, ϕ_p) are covering morphisms of ordinary group-groupoids.

Example 4.5. For given a horizontal action of double group-groupoid S on (ω, ω_0), since projections $(c, \omega) : S_H \ltimes \omega \rightarrow G$ and $(c_0, \omega_0) : H \ltimes \omega_0 \rightarrow G_0$ are covering morphisms of group-groupoids, the morphism $\phi : (S_H \ltimes \omega; H \ltimes \omega_0, G; G_0) \rightarrow (S; H, V; P)$ becomes a covering morphism of double group-groupoids.

Let $CovDGpGpd / S$ denote the category in which objects are covering morphisms of a double group-groupoid S and arrows are commutative diagrams of morphisms.

As a result, we have the following categorical equivalence.

Theorem 4.6. The actions and the covering morphisms of double group-groupoid S are categorically equivalent

$$CovDGpGpd / S \simeq DGpGpdAct_H(S)$$

Proof: Let $\phi = (\phi_s, \phi_h, \phi_v, \phi_p) : (S'; H', V'; P') \rightarrow (S; H, V; P)$ be an object of $CovDGpGpd / S$. Hence (ϕ_s, ϕ_v) and (ϕ_h, ϕ_p) are covering morphisms of ordinary group-groupoids and then S_H acts on $\phi_v : V' \rightarrow V$ and H acts on $\phi_p : P' \rightarrow P$. So we have a semidirect product double group-groupoid $(S_H \ltimes \phi_v; H \ltimes \phi_p, V'; P')$.

Conversely assume that $(G, (\omega, \omega_0))$ is an object of $DGpGpdAct_H(S)$. Then we have an action of S_H on $\omega : G \rightarrow V$ and H on $\omega_0 : G_0 \rightarrow P$. Hence we have semidirect product group-groupoids $S_H \ltimes G$ and $H \ltimes G_0$ such that $(S_H \ltimes G; H \ltimes G_0, G; G_0) \rightarrow (S; H, V; P)$ is a covering morphism of double group-groupoids with respect to the horizontal action.

Other details follow and hence omitted.

Example 4.7. Let $S = (S; H, V; P)$ be a double group-groupoid. Then the morphism $(1_s, 1_h, 1_v, 1_p): S \to S$ is a covering morphism in $CovDGpGpd / S$ which means that $(1_s, 1_v)$ and $(1_h, 1_p)$ are covering morphism of group-groupoids. The corresponding action is constructed as follows. S_H acts on $1_v: V \to V$ and H acts on $1_p: P \to P$ with actions

$$V \times S_H \to V, v \bullet s = d_1^h(s) \text{ and } d_0^h(s) = 1_v(v)$$

and

$$P \times H \to P, p \bullet h = d_1^H(h) \text{ and } d_0^H(h) = 1_p(p)$$

With these actions we have $S_H \ltimes 1_v \simeq S_H$ and $H \ltimes 1_p \simeq H$. So we have an action double group-groupoid $(S_H \ltimes 1_v; H \ltimes 1_p, V; P)$ which is isomorphic to $(S_H; H, V; P)$.

In below we give an example of covering morphism for double group-groupoids.

Example 4.8. Suppose that N and M are group-groupoids with same base P and $\phi: N \to M$ is a morphism. Then we have a double group-groupoid $(N \times N; N, P \times P; P)$ and an action of $N \times N$ on $\omega: M \to P \times P$ and of N on $\omega_0 = 1_p: P \to P$ with action ρ

$$\rho: M \times (N \times N) \to M, \rho(m)(n_1, n_2) = \phi(n_1)m\phi(n_2)^{-1}$$

So semidirect product group-groupoids $(N \times N) \ltimes \omega$ and $N \ltimes 1_p \simeq N$ exist. As a result we have an action double group-groupoid $((N \times N) \ltimes \omega; N, M; P)$. Elements of $(N \times N) \ltimes \omega$ are triples (n_1, m, n_2) of the form

$$
\begin{array}{c}
n_1 \\
m \; \boxed{(n_1, m, n_2) \; \phi(n_1)m\phi(n_2)^{-1}} \\
n_2
\end{array}
$$

and groupoid compositions defined in [5, Example 1.8] as follows:

$$(n_1, m, n_2) \circ_h (n_1', m', n_2') = (n_1 n_1', mm', n_2 n_2')$$

$$(n_1, m, n_2) \circ_v (n_1', m', n_2') = (n_1, mm', n_2').$$

Group operation of action double group-groupoid is defined by

$$(n_1, m, n_2) + (n_1', m', n_2') = (n_1 + n_1', m + m', n_2 + n_2').$$

Moreover since $(N \times N) \ltimes \omega \to N \times N$ and $N \ltimes 1_p \to N$ are covering morphism of group-groupoid, $((N \times N) \ltimes \omega; N \ltimes 1_p, M; P) \to (N \times N; N, P \times P; P)$ is a covering morphism of double group-groupoids.

5. CONCLUSION

In this paper we have a categorical equivalence of actions and covering morphisms for double group-groupoids. Further by [17], double group-groupoids are equivalent to crossed modules over group-groupoids. Hence it could be possible to characterize the concepts of action and covering morphism for such crossed modules. That enables to produce more examples of actions and coverings for double group-groupoids.

ACKNOWLEDGEMENTS

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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