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Weingarten Map of the Henneberg-Type Minimal Surfaces in 4-Space

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Abstract

We consider a two parameter family of Henneberg-type minimal surfaces $\mathfrak{H}_{m,n}$ using the Weierstrass representation in the four dimensional Euclidean space $\mathbb{E}^4$. An invariant linear map of Weingarten type in the tangent space of the Henneberg-type minimal surface $\mathfrak{H}_{4,2}$ which generates two invariants κ and $\varkappa$, is characterized by $\varkappa^2 = \kappa$ in $\mathbb{E}^4$.

Keywords: *Four dimension, Henneberg-type minimal surface, Weierstrass representation, Weingarten map*
2010 Mathematics Subject Classification: *Primary: 53A10; Secondary: 53C42.*

1. Introduction

A minimal surface in three dimensional Euclidean space $\mathbb{E}^3$, also in the higher dimensions is a regular surface for which the mean curvature vanishes identically [4, 5, 8, 21, 23].
On the other hand, the classical Henneberg surface [14, 15, 16] also obtained by Weierstrass representation [24, 25] is well-known classical minimal surface in $\mathbb{E}^3$. Henneberg's minimal surface was the unique nonorientable example known until 1981, when Meeks indicated the first example of a nonorientable, regular, complete, minimal surface of finite total curvature -6π. See [22] for details.
A general definition of rotational surfaces in $\mathbb{E}^4$ was obtained by Moore in [20]. Güler, Hacısalihoğlu, and Kim [9] worked the Gauss map and the third Laplace-Beltrami operator of the rotational hypersurfaces in $\mathbb{E}^4$. See also [1, 2, 3, 12, 13, 18, 19] for some new works of the topic.
Güler and Kişi [10] studied Weierstrass representation, degree and classes of the surfaces in the Euclidean 4-space. Güler, Kişi, and Konaxis [11] introduced Henneberg implicit algebraic minimal surfaces in $\mathbb{E}^4$.
We study a two-parameter family of Henneberg type minimal surfaces using the Weierstrass representation in $\mathbb{E}^4$. We give the Weierstrass equation for a minimal surface in $\mathbb{E}^4$, also obtain two normals of the surface in the Section 2.
In Section 3, we introduce to the Henneberg type minimal surfaces of values (m,n) for positive integers m,n called $H_{m,n}$. Finally, we study Henneberg type minimal surfaces $H_{4,2}$ using Weierstrass representation in $\mathbb{E}^4$. Then we give an invariant linear map of Weingarten type in the tangent space of the Henneberg-type minimal surface $H_{4,2}$ which generates two invariants κ and $\varkappa$, is characterized by $\varkappa^2 = \kappa$ in Section 4.

2. Weierstrass equation and normals for a minimal surface in $\mathbb{E}^4$

We identify $\overrightarrow{x}$ and $\overrightarrow{x^t}$ without further comment. Let $\mathbb{E}^4 = (\{\overrightarrow{x} = (x_1,x_2,x_3,x_4)^t | x_i \in \mathbb{R}\}, \langle \cdot,\cdot \rangle)$ be the 4-dimensional Euclidean space with metric $\langle x,y \rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$.
Hoffman and Osserman [17] gave the Weierstrass equation for a minimal surface in $\mathbb{E}^4$ as follows

$$\Phi(z) = \frac{\psi}{2}\left[1+fg, i(1-fg), f-g, -i(f+g)\right]. \tag{2.1}$$

Here ψ is analytic and the order of the zeros of ψ must be greater than the order of the poles of f,g at each point.
Theorem 1. *Normal vectors of a minimal surface with respect to Weierstrass equation $X_x - iX_y = \Phi(z)$ in $\mathbb{E}^4$, are as follows:*

$$n_1 = \sqrt{\frac{a}{a^2-b^2}}\left(w_1 + \frac{b}{a}X_y\right) \tag{2.2}$$

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and

$$n_2=\sqrt{\frac{a}{a^2-b^2}}\left(w_2-\frac{b}{a}X_x\right), \tag{2.3}$$

where $a=\langle X_x,X_x\rangle$, $b=\langle X_x,w_2\rangle$, *and* w_1, w_2 *are perpendicular to* X_x, X_y, *respectively.*
Proof. See [11] for the proof.
With $x=r\cos(\theta)$, $y=r\sin(\theta)$, $f_1=1-r^{-m}\cos(m\theta)$, $f_2=-r^{-m}\sin(m\theta)$, $g_1=r^n\cos(n\theta)$, $g_2=r^n\sin(n\theta)$, we have following normals

$$(n_1)_{m,n}(r,\theta)=\mathscr{A}\begin{pmatrix}\mathscr{B}\sin(\theta)-r^{2m}r^n\sin((n+1)\theta)+r^mr^n\sin((m+n+1)\theta)\\ \mathscr{B}\cos(\theta)+r^{2m}r^n\cos((n+1)\theta)-r^mr^n\cos((m+n+1)\theta)\\ -r^{2m}\sin(\theta)+r^m\sin((m+1)\theta)-\mathscr{B}r^n\sin((n+1)\theta)\\ r^{2m}\cos(\theta)-r^m\cos((m+1)\theta)-\mathscr{B}r^n\cos((n+1)\theta)\end{pmatrix} \tag{2.4}$$

and

$$(n_2)_{m,n}(r,\theta)=\mathscr{A}\begin{pmatrix}\mathscr{B}\cos(\theta)-r^{2m}r^n\cos((n+1)\theta)+r^mr^n\cos((m+n+1)\theta)\\ -\mathscr{B}\sin(\theta)-r^{2m}r^n\sin((n+1)\theta)+r^mr^n\sin((m+n+1)\theta)\\ -r^{2m}\cos(\theta)+r^m\cos((m+1)\theta)-\mathscr{B}r^n\cos((n+1)\theta)\\ -r^{2m}\sin(\theta)+r^m\sin((m+1)\theta)+\mathscr{B}r^n\sin((n+1)\theta)\end{pmatrix}, \tag{2.5}$$

where $\mathscr{A}=\left[\mathscr{B}\left(r^{2n}+1\right)\left(2r^{2m}-2r^m\cos(m\theta)+1\right)\right]^{-1/2}$, $\mathscr{B}=r^{2m}-2r^m\cos(m\theta)+1$. We check $\langle n_1,n_1\rangle=\langle n_2,n_2\rangle=1$. See also [11] for details.

3. Henneberg family of surfaces $\mathfrak{H}_{m,n}$

Choosing $\psi=2$, $f=1-1/z^m$ and $g=z^n$ in analogy with the surface case, we see

$$\begin{aligned}\Phi(z) &= \frac{\psi}{2}(1+fg,i(1-fg),f-g,-i(f+g))\\ &= \left(1+z^n-z^{n-m},i(1-z^n+z^{n-m}),1-z^{-m}-z^n,-i(1-z^{-m}+z^n)\right).\end{aligned}$$

We integrate it to get following:

$$\int\Phi(z)dz=\begin{pmatrix}z+\frac{z^{n+1}}{n+1}-\frac{z^{-m+n+1}}{-m+n+1}\\ i\left(z-\frac{z^{n+1}}{n+1}+\frac{z^{-m+n+1}}{-m+n+1}\right)\\ z-\frac{z^{-m+1}}{-m+1}-\frac{z^{n+1}}{n+1}\\ -i\left(z-\frac{z^{-m+1}}{-m+1}+\frac{z^{n+1}}{n+1}\right)\end{pmatrix}$$

with $m\neq1, n\neq-1, -m+n\neq-1$. We let $z=re^{i\theta}$ and taking the real part of the integral, we have the Henneberg type minimal surface $H_{m,n}(r,\theta)$ as follows:

$$\mathfrak{H}_{m,n}(r,\theta)=\begin{pmatrix}r\cos(\theta)+\frac{r^{n+1}\cos((n+1)\theta)}{n+1}-\frac{r^{-m+n+1}\cos((-m+n+1)\theta)}{-m+n+1}\\ -r\sin(\theta)+\frac{r^{n+1}\sin((n+1)\theta)}{n+1}-\frac{r^{-m+n+1}\sin((-m+n+1)\theta)}{-m+n+1}\\ r\cos(\theta)-\frac{r^{-m+1}\cos((-m+1)\theta)}{-m+1}-\frac{r^{n+1}\cos((n+1)\theta)}{n+1}\\ r\sin(\theta)-\frac{r^{-m+1}\sin((-m+1)\theta)}{-m+1}+\frac{r^{n+1}\sin((n+1)\theta)}{n+1}\end{pmatrix}.$$

Using the binomial formula, we obtain the following parametric equation of $\mathfrak{H}_{m,n}(u,v)$:

$$\begin{aligned}x&=\mathrm{Re}\left\{u+iv+\tfrac{1}{n+1}\left[\textstyle\sum_{k=0}^{n+1}\binom{n+1}{k}u^{n+1-k}(iv)^k\right]-\tfrac{1}{-m+n+1}\left[\textstyle\sum_{k=0}^{-m+n+1}\binom{-m+n+1}{k}u^{-m+n+1-k}(iv)^k\right]\right\},\\ y&=\mathrm{Re}\left\{iu-v-\tfrac{i}{n+1}\left[\textstyle\sum_{k=0}^{n+1}\binom{n+1}{k}u^{n+1-k}(iv)^k\right]+\tfrac{i}{-m+n+1}\left[\textstyle\sum_{k=0}^{-m+n+1}\binom{-m+n+1}{k}u^{-m+n+1-k}(iv)^k\right]\right\},\\ z&=\mathrm{Re}\left\{u+iv-\tfrac{1}{-m+1}\left[\textstyle\sum_{k=0}^{-m+1}\binom{-m+1}{k}u^{-m+1-k}(iv)^k\right]-\tfrac{1}{n+1}\left[\textstyle\sum_{k=0}^{n+1}\binom{n+1}{k}u^{n+1-k}(iv)^k\right]\right\},\\ w&=\mathrm{Re}\left\{iu+v+\tfrac{i}{-m+1}\left[\textstyle\sum_{k=0}^{-m+1}\binom{-m+1}{k}u^{-m+1-k}(iv)^k\right]-\tfrac{i}{n+1}\left[\textstyle\sum_{k=0}^{n+1}\binom{n+1}{k}u^{n+1-k}(iv)^k\right]\right\},\end{aligned} \tag{3.1}$$

where

$$\mathfrak{H}_{m,n}(u,v)=(x(u,v),y(u,v),z(u,v),w(u,v)).$$

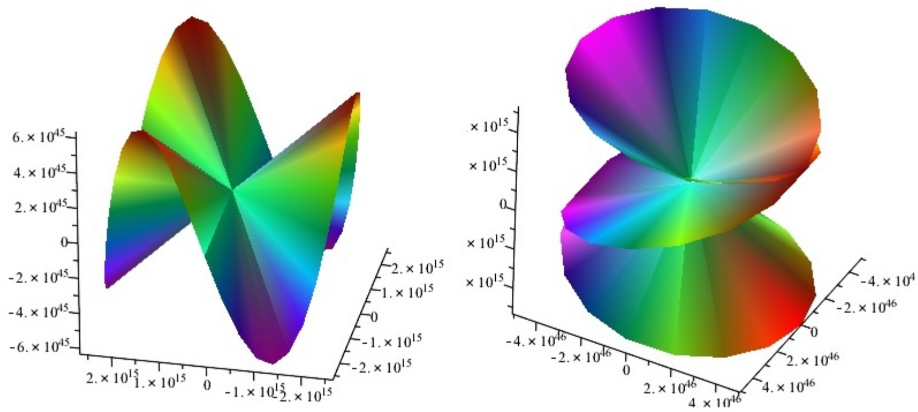

Figure 4.1: Figure 1. Projections of $H_{4,2}(r,\theta)$: (**Left**) into $x_1x_2x_4$-space, (**Right**) into $x_2x_3x_4$-space

4. Weingarten map of $\mathfrak{H}_{4,2}$

Using an arbitrary orthonormal frame field $\{n_1, n_2\}$ for a surface $S(u,v)$ in $\mathbb{E}^4$, we get the standart derivative formulas as follows

$$
\begin{aligned}
\nabla_{S_u} S_u &= S_{uu} = \Gamma_{11}^1 S_u + \Gamma_{11}^2 S_v + c_{11}^1 n_1 + c_{11}^2 n_2, \\
\nabla_{S_u} S_v &= S_{uv} = \Gamma_{12}^1 S_u + \Gamma_{12}^2 S_v + c_{12}^1 n_1 + c_{12}^2 n_2, \\
\nabla_{S_v} S_v &= S_{vv} = \Gamma_{22}^1 S_u + \Gamma_{22}^2 S_v + c_{22}^1 n_1 + c_{22}^2 n_2,
\end{aligned}
\tag{4.1}
$$

where Γ_{ij}^k are the Christoffel symbols and c_{ij}^k, $i,j,k \in \{1,2\}$ are functions on S. Taking standart notations of the first fundamental form coefficients

$$
E = \langle S_u, S_u \rangle, \quad F = \langle S_u, S_v \rangle, \quad G = \langle S_v, S_v \rangle, \tag{4.2}
$$

we get $W_1 = EG - F^2$. Also, using standart notations of the second fundamental form coefficients

$$
L = \frac{2}{\sqrt{W_1}} \begin{vmatrix} c_{11}^1 & c_{12}^1 \\ c_{11}^2 & c_{12}^2 \end{vmatrix}, \; M = \frac{1}{\sqrt{W_1}} \begin{vmatrix} c_{11}^1 & c_{22}^1 \\ c_{11}^2 & c_{22}^2 \end{vmatrix}, \; N = \frac{2}{\sqrt{W_1}} \begin{vmatrix} c_{12}^1 & c_{22}^1 \\ c_{12}^2 & c_{22}^2 \end{vmatrix}, \tag{4.3}
$$

we have $W_2 = LN - M^2$. Here c_{ij}^k are defined by

$$
\begin{aligned}
c_{11}^1 &= \langle S_{uu}, n_1 \rangle, \quad c_{12}^1 = \langle S_{uv}, n_1 \rangle, \quad c_{22}^1 = \langle S_{vv}, n_1 \rangle, \\
c_{11}^2 &= \langle S_{uu}, n_2 \rangle, \quad c_{12}^2 = \langle S_{uv}, n_2 \rangle, \quad c_{22}^2 = \langle S_{vv}, n_2 \rangle.
\end{aligned}
$$

From the Lemma 2.1 and Lemma 2.2 in [6], we have the functions which are invariants of the surface S, as follows

$$
\kappa = \frac{W_2}{W_1} \quad \text{and} \quad \varkappa = \frac{W_3}{2W_1}, \tag{4.4}
$$

where $W_3 = EN + GL - 2FM$.
Taking $m = 4$, $n = 2$, we have Henneberg type surface (See Figure 1. Left, and Figure 1. Right)

$$
\mathfrak{H}_{4,2}(r,\theta) = \begin{pmatrix} \frac{r^3\cos(3\theta)}{3} + r\cos(\theta) + \frac{\cos(\theta)}{r} \\ \frac{r^3\sin(3\theta)}{3} - r\sin(\theta) - \frac{\sin(\theta)}{r} \\ -\frac{r^3\cos(3\theta)}{3} + r\cos(\theta) + \frac{\cos(3\theta)}{3r^3} \\ \frac{r^3\sin(3\theta)}{3} + r\sin(\theta) - \frac{\sin(3\theta)}{3r^3} \end{pmatrix}. \tag{4.5}
$$

We obtain following Henneberg type surface taking $u := r\cos\theta$, $v := r\sin\theta$ in (4.5)

$$
\mathfrak{H}_{4,2}(u,v) = \begin{pmatrix} \frac{1}{3}u^3 - uv^2 + u + \frac{u}{u^2+v^2} \\ -\frac{1}{3}v^3 + u^2v - v - \frac{v}{u^2+v^2} \\ -\frac{1}{3}u^3 + uv^2 + u + \frac{1}{3}\frac{u^3}{(u^2+v^2)^3} - \frac{uv^2}{(u^2+v^2)^3} \\ -\frac{1}{3}v^3 + u^2v + v + \frac{1}{3}\frac{v^3}{(u^2+v^2)^3} - \frac{u^2v}{(u^2+v^2)^3} \end{pmatrix}. \tag{4.6}
$$

Next, we want to find normals $n_1(u,v)$ and $n_2(u,v)$ of the Henneberg type minimal surface $\mathfrak{H}_{4,2}(u,v)$.
From (2.4) and (2.5), we have following normals, respectively,

$$
n_1(r,\theta) = \mathscr{A} \begin{pmatrix} \mathscr{B}\sin(\theta) - r^{10}\sin(3\theta) + r^6\sin(7\theta) \\ \mathscr{B}\cos(\theta) + r^{10}\cos(3\theta) - r^6\cos(7\theta) \\ -r^8\sin(\theta) - \mathscr{B}r^2\sin(3\theta) + r^4\sin(5\theta) \\ r^8\cos(\theta) - \mathscr{B}r^2\cos(3\theta) - r^4\cos(5\theta) \end{pmatrix} \tag{4.7}
$$

and

$$n_2(r,\theta)=\mathscr{A}\begin{pmatrix}\mathscr{B}\cos(\theta)-r^{10}\cos(3\theta)+r^6\cos(7\theta)\\ -\mathscr{B}\sin(\theta)+r^{10}\sin(3\theta)-r^6\sin(7\theta)\\ -r^8\cos(\theta)-\mathscr{B}r^2\cos(3\theta)+r^4\cos(5\theta)\\ -r^8\sin(\theta)+\mathscr{B}r^2\sin(3\theta)+r^4\sin(5\theta)\end{pmatrix}. \tag{4.8}$$

Here, $\mathscr{A}=\left[\mathscr{B}\left(r^4+1\right)\left(2r^8-2r^4\cos(4\theta)+1\right)\right]^{-1/2}$, $\mathscr{B}=r^8-2r^4\cos(4\theta)+1$.
In (u,v) coordinates, we find normals n_1 and n_2 as follows

$$n_1=\mathscr{C}\begin{pmatrix}\left[p^5+\left(1-4u^2\right)p^4-p^3-2\left(1-12u^2\right)p^2+16u^2\left(1-5u^2\right)p+64u^6-16u^4+1\right]v\\ \left[-3p^5+\left(1+4u^2\right)p^4+7p^3-2\left(1+28u^2\right)p^2+16u^2\left(1+7u^2\right)p-64u^6-16u^4+1\right]u\\ \left[-p^4+16u^4-12u^2p+p^2-\left(p^4-16u^4+16u^2p-2p^2+1\right)\left(4u^2-p\right)\right]v\\ \left[p^4-16u^4+20u^2p-5p^2-\left(p^4-16u^4+16u^2p-2p^2+1\right)\left(4u^2-3p\right)\right]u\end{pmatrix} \tag{4.9}$$

and

$$n_2=\mathscr{C}\begin{pmatrix}\left[3p^5+\left(1-4u^2\right)p^4-7p^3-2\left(1-28u^2\right)p^2-16u^2\left(1+7u^2\right)p+64u^6-16u^4+1\right]u\\ \left[p^5-\left(1+4u^2\right)p^4-p^3+2\left(1+12u^2\right)p^2-16u^2\left(1+5u^2\right)p+64u^6+16u^4-1\right]v\\ \left[-p^4+16u^4-20u^2p+5p^2-\left(p^4-16u^4+16u^2p-2p^2+1\right)\left(4u^2-3p\right)\right]u\\ \left[-p^4+16u^4-12u^2p+p^2+\left(p^4-16u^4+16u^2p-2p^2+1\right)\left(4u^2-p\right)\right]v\end{pmatrix}, \tag{4.10}$$

where

$$\mathscr{C}=\left[p\left(p^2+1\right)\left(p^4-2p^2+16u^2p-16u^4+1\right)\left(2p^4-2p^2+16u^2p-16u^4+1\right)\right]^{-1/2}$$

and $p=u^2+v^2$.
Remark 1. *Two invariants κ and $\varkappa$ divide the points of surface $S(u,v)$ into the four types. A point $\mathbf{p}$ in E^4 is called:*

$$\begin{aligned} flat,\ \text{if}\ \kappa &= \varkappa=0,\\ elliptic,\ \text{if}\ \kappa &> 0,\\ parabolic,\ \text{if}\ \kappa &= 0,\ \varkappa\neq 0,\\ hyperbolic,\ \text{if}\ \kappa &< 0.\end{aligned}$$

Using (4.1), (4.2), (4.3), (4.9) and (4.10) for the Henneberg type surface (4.6) , we have

$$\begin{aligned} E &= \frac{\left(p^2+1\right)\varphi}{p^4}=G,\\ F &= 0,\\ L &= -\frac{8p\beta}{\left(p^2+1\right)^2\varphi^2}=N,\\ M &= 0,\end{aligned}$$

where

$$\begin{aligned}\varphi &= \varphi(u,v)=2\left(p^2-1\right)^2+2p^2+16u^2v^2-1\neq 0,\\ \beta &= \beta(u,v)=\varphi^2-4\left(p^2+1\right)\left(p^4+\left(1-16u^2v^2\right)p^2+64u^4v^4\right).\end{aligned}$$

Corollary 1. *An invariant linear map of Weingarten type in the tangent space of the Henneberg-type minimal surface $\mathfrak{H}_{4,2}$ which generates two invariants*

$$\kappa=\frac{64p^{10}\beta^2}{\left(p^2+1\right)^6\varphi^6}$$

and

$$\varkappa=-\frac{8p^5\beta}{\left(p^2+1\right)^3\varphi^3}$$

is characterized by $\varkappa^2=\kappa$ in $\mathbb{E}^4$. Here, $\beta,\varphi\neq 0$, $p=u^2+v^2$. Hence, every point $\mathbf{p}$ on $\mathfrak{H}_{4,2}(u,v)$ is elliptic.
Corollary 2. *If $\beta=0$ in Corollary 1., then the points on the surface is flat.*

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