MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES

https://doi.org/10.36753/mathenot.763854
9 (1) 9-21 (2021) - Research Article
ISSN: 2147-6268

On Convergence of Partial Derivatives of Multidimensional Convolution Operators

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Abstract
In this paper, we prove some results on convergence properties of higher order partial derivatives of multidimensional convolution-type singular integral operators being applied to the class of functions which are integrable in the sense of Lebesgue.

Keywords: *Fatou-type convergence; Approximation of partial derivatives; Integral operators.*

AMS Subject Classification (2020): *Primary: 41A35 ; Secondary: 45P05*

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1. Introduction

To reach the beginning of the singular integral theory, which is an immense field of application, it is necessary to go up to well-known Fourier integrals. If Fourier integrals are considered in general frame, the following integral operators are obtained:

$$L_w(f;x)=\int\limits_A f(t)K_w(t-x)\,dt,\ x\in A,\ w\in\Lambda, \tag{1.1}$$

where Λ is an index set consisting of w which are real numbers (parameters) and w_0 is an accumulation point of indicated index set, or it equals infinity, A is a desired subset of the set of all real numbers R and K_w is a kernel with some assumptions on it. The operators of type (1.1) were widely studied in [4]. Also, several results and discussions related to multidimensional analogues of the operators of type (1.1) can be found in [18, 19]. Also, for further reading, we refer the reader to [1, 9, 15, 20–23] and references therein. In particular, Fatou-type convergence of the operators (see [5]) is studied, for example, in [9, 16, 20, 21].

Singular integrals, in fact, have been studied in many different ways, but in this article we focus on the convergence properties of their derivatives. In the year 1962, Taberski [20] proved a theorem concerning Fatou-type convergence of higher order derivatives of singular integrals of type (1.1) whose kernels were supposed to be $2\pi-$periodic. In the proof of this theorem, Taberski [20] used an auxiliary function, whose higher order derivatives coincide with derivatives of the same order of original function. For this function, he used an asymptotic formula for De laVallée Poussin's singular integrals, which is given by Matsuoka [13], which can be seen as an alternative

Received : *03-07-2020,* ***Accepted :*** *04-12-2020*

version of usual Taylor series in regard to trigonometric functions. In the same year, Žornickaja [24] studied the similar problem under the concept of almost everywhere convergence by taking the set Λ as the set of all natural numbers denoted by N. Žornickaja [24] also gave some kernel examples which does not fit the presented theorem's hypotheses. Later on, Gadžiev [7] proved a theorem concerning approximation by first order derivatives of the operators of type (1.1) under the existence of right and left derivatives of the integrable functions at indicated points by assuming that A is an arbitrary bounded interval in R. This theorem may be categorized as Fatou-type convergence theorem. In the proof of this theorem, Gadžiev [7] used first order Taylor polynomial as an auxiliary function (see also [6, 8]). We remark here that, in the works above, the kernel functions satisfy standard approximate identity properties. Then, Karsli and Ibikli [10] proved some theorems concerning approximation of higher order derivatives of functions in more general function spaces. In this respect, Karsli [11, 12] studied similar problem in the framework of linear and nonlinear integral operators using desired order Taylor polynomial in the proving stage, respectively. In fact, Taylor series are very important for the proofs of approximation theory. For different usages, we refer the reader to [1–3].
Let $R^n = R \times R \times \cdots \times R$ denote usual finite dimensional Euclidean space with elements, such as $t := (t_1, ..., t_n)$ and $x := (x_1, ..., x_n)$. Further, let Λ be an index set consisting of real numbers (parameters) w and w_0 be an accumulation point of indicated index set or $w_0 = \infty$, separately. Under the conditions assigned to the function $K_w : R^n \to R_0^+$, we obtained some approximation properties of higher order partial derivatives of the operators given by

$$(G_w f)(x) = \int_D f(t) K_w (t - x)\, dt, \ \ x \in D^o$$

with respect to desired component x_j, where D denotes a closed box in R^n, D^o stands for its interior and j is an arbitrary number between $j = 1, 2, ..., n$. This study is a continuation of the study [22] and contains some results concerning multidimensional analogues of the results proved in [10, 11, 20]. Following similar steps used in [7, 10, 11, 20] with some additional considerations, we state and prove main theorems of this study.

2. Main Results

Case 1: Domain of integration is bounded

Let us consider the following operators

$$(G_w f)(x) = \int_D f(t) K_w (t - x)\, dt.$$

The explicit form of these operators can be written as follows:

$$(G_w f)(x_1, ..., x_n) = \int_{A_1}^{B_1} \cdots \int_{A_n}^{B_n} f(t_1, ..., t_n) K_w (t_1 - x_1, ..., t_n - x_n) dt_n \cdots dt_1$$

with $x \in D^o$ and $t \in D$, where $D = [A_1, B_1] \times ... \times [A_n, B_n]$ is a closed box in R^n and $D^o =]A_1, B_1[\times ... \times]A_n, B_n[$ is an open box. Here, A_j and B_j with $A_j \neq B_j$ are certain real numbers for every fixed $j = 1, 2, ..., n$.
Let $L_1(D)$ denote the space of all functions f which are integrable in the sense of Lebesgue on D with respect to usual Lebesgue measure dt. Any function in this space satisfies the property such that $\|f\|_{L_1(D)} := \int_D |f(t)|\, dt < \infty$.

Here, the kernel $K_w(t)$ satisfies the following conditions:

1. $K_w : R^n \to R_0^+$ is a measurable function on its domain for every fixed $w \in \Lambda$.
2. $\lim\limits_{(x,w) \to (x_0, w_0)} \int_D K_w (t - x)\, dt = 1$, where $w \in \Lambda$, $x \in D^o$ and x_0 is an accumulation point of D^o.
3. $\|K_w(. - x)\|_{L_1(D)}$ is uniformly bounded on Λ for all $x \in D$ by a constant M.

Here, $|.|$ denotes usual Euclidean distance.

Theorem 2.1. *Assume that $K_w(t)$ and $\frac{\partial^v}{\partial t_j^v} K_w(t)$ are continuous functions with respect to t on R^n for every fixed $w \in \Lambda$, $v = 1, 2, ..., m$ and $j = 1, 2, ..., n$. Suppose that the following conditions*

$$\lim_{w \to w_0} \sup_{|t| \geq \xi} \left| \frac{\partial^v}{\partial t_j^v} K_w(t) \right| = 0, \ \forall \xi > 0 \tag{2.1}$$

hold for every fixed $j = 1, 2, ..., n$ and $v = 0, 1, ..., m$, together with conditions (1)-(3). If $f \in L_1(D)$ possesses at $a := (a_1, ..., a_n) \in D^o$ finite $m - th$ order partial derivative with respect to $j - th$ variable $f_j^{(m)}(a)$ and there exists a neighborhood $]a_1 - \eta, \ a_1 + \eta[\times \cdots \times]a_n - \eta, \ a_n + \eta[\subset D$ with $\eta > 0$ of a on which the functions $f_j^{(m)}$ and f are continuous, then

$$\lim_{(x,w) \to (a,w_0)} \frac{\partial^m}{\partial x_j^m} \left(G_w f \right)(x) = f_j^{(m)}(a)$$

on any set S consisting of (x, w) on which the functions expressed as

$$\sup_{w \in \Lambda} \int\limits_{a_1 - x_1 - \zeta}^{a_1 - x_1 + \zeta} \cdots \int\limits_{a_n - x_n - \zeta}^{a_n - x_n + \zeta} |t_j|^m \left| \frac{\partial^m}{\partial t_j^m} K_w(t_1, ..., t_n) \right| dt_n \cdots dt_1, \tag{2.2}$$

$$\sup_{w \in \Lambda} \int\limits_{a_1 - x_1 - \zeta}^{a_1 - x_1 + \zeta} \cdots \int\limits_{a_n - x_n - \zeta}^{a_n - x_n + \zeta} \left| \frac{\partial^m}{\partial t_j^m} K_w(t_1, ..., t_n) \right| dt_n \cdots dt_1 \tag{2.3}$$

and

$$|x_j - a_j|^v \int\limits_{a_1 - x_1 - \zeta}^{a_1 - x_1 + \zeta} \cdots \int\limits_{a_n - x_n - \zeta}^{a_n - x_n + \zeta} |t_j|^{m-v} \left| \frac{\partial^m}{\partial t_j^m} K_w(t_1, ..., t_n) \right| dt_n \cdots dt_1 \leq C_v^\zeta \tag{2.4}$$

are bounded, where $j = 1, 2, ..., n$ and $v = 1, 2, ..., m$. Here, C_v^ζ are positive constants for every fixed positive real number ζ that makes the value of integral finite as required.

Proof. We define the function $g(t)$ by

$$g(t) =: g_{t_j},$$

where

$$g_{t_j} := f(a) + (t_j - a_j) \left. \frac{\partial f(t)}{\partial t_j} \right|_{t=a} + \cdots + \frac{(t_j - a_j)^m}{m!} \left. \frac{\partial^m f(t)}{\partial t_j^m} \right|_{t=a}$$

such that $\left. \frac{\partial^k g(t)}{\partial t_j^k} \right|_{t=a} = f_j^{(k)}(a)$, $k = 0, ..., m$ for $a \in D^o$. By linearity of the operators, we can write

$$\begin{aligned} \left(G_w f \right)(x) &= \left(G_w (f + g - g) \right)(x) \\ &= \left(G_w g \right)(x) + \left(G_w (f - g) \right)(x). \end{aligned}$$

Differentiating both sides of the following equation up to order m with respect to x_j and writing the definition of g in $\left(G_w g \right)(x)$ such that

$$\left(G_w g \right)(x) = \int\limits_D g(t) K_w(t - x) dt,$$

one easily obtains

$$\begin{aligned} \frac{\partial^m}{\partial x_j^m} \left(G_w g \right)(x) &= \frac{\partial^m}{\partial x_j^m} \int\limits_D g(t) K_w(t - x) dt \\ &= (-1)^m \int\limits_D g(t) \frac{\partial^m}{\partial t_j^m} K_w(t - x) dt. \end{aligned}$$

In view of Fubini's Theorem (see [14]) and m times application of integration by parts with respect to t_j, we have the following equality:

$$
\begin{aligned}
&(-1)^m \left\{ \int\limits_{\prod_{i\neq j}[A_i,B_i]} \left\{ \int\limits_{A_j}^{B_j} g_{t_j} \frac{\partial^m}{\partial t_j^m} K_w(t-x)dt_j \right\} \prod_{i\neq j} dt_i \right\} \\
= \;& (-1)^m \left\{ \int\limits_{\prod_{i\neq j}[A_i,B_i]} \left\{ \sum_{k=0}^{m-1} (-1)^k \; g_{t_j}^{(k)} \frac{\partial^{m-1-k}}{\partial t_j^{m-1-k}} K_w(t-x) \Big|_{A_j}^{B_j} \right\} \prod_{i\neq j} dt_i \right\} \\
&+ \int\limits_D g_{t_j}^{(m)} K_w(t-x)dt \\
= \;& : J_1(x,w) + J_2(x,w),
\end{aligned}
$$

where $1 \leq i,j \leq n$. Performing some analysis on $J_1(x,w)$, we deduce that it tends to zero as $(x,w) \to (a,w_0)$ as a straightforward consequence of (2.1). Obviously, there holds for $J_2(x,w)$

$$
\begin{aligned}
&\lim_{(x,w)\to(a,w_0)} \int\limits_D K_w(t-x) \frac{\partial^m}{\partial t_j^m} g(t)dt \\
= \;& \lim_{(x,w)\to(a,w_0)} \left. \frac{\partial^m f}{\partial t_j^m} \right|_{t=a} \int\limits_D K_w(t-x)dt.
\end{aligned}
$$

Hence, by condition (2), the desired result follows, that is,

$$
\lim_{(x,w)\to(a,w_0)} \frac{\partial^m}{\partial x_j^m} \left(G_w g\right)(x) = \left. \frac{\partial^m f}{\partial t_j^m} \right|_{t=a} = f_j^{(m)}(a).
$$

To complete the proof, we will show that

$$
\lim_{(x,w)\to(a,w_0)} \frac{\partial^m}{\partial x_j^m} \left(G_w(f-g)\right)(x) = 0.
$$

By the hypotheses, the functions $\frac{\partial^m}{\partial t_j^m} f(t)$ for $m = 1, ..., n$ and $f(t)$ are continuous at $a \in D^o$. Therefore, according to $\varepsilon - \delta$ criterion of continuity for all $\varepsilon > 0$ there exists a number $\delta > 0$ such that the following relations hold there (see also [20]):

$i.$ $|f(t) - f(a)| < \varepsilon$ provided that $t \in D^1$, where $D^1 =]a_1 - \delta,\ a_1 + \delta[\times \cdots \times]a_n - \delta,\ a_n + \delta[$.

$ii.$ $|f(a_1, ..., t_j, ..., a_n) - f(a_1, ..., a_j, ..., a_n)| < \varepsilon$ provided that $|t_j - a_j| < \delta$.

$iii.$ $\left| \frac{(f(a_1,...,t_j,...,a_n) - g_{t_j})}{(t_j - a_j)^m} \right| < \varepsilon$ provided that $|t_j - a_j| < \delta$, since

$$
\lim_{t_j \to a_j} \frac{\left(f(a_1, ..., t_j, ..., a_n) - g_{t_j}\right)}{\left(t_j - a_j\right)^m} = 0,
$$

where $a \in D^o$ and

$$
g_{t_j} = f(a) + (t_j - a_j) \left. \frac{\partial f(t)}{\partial t_j} \right|_{t=a} + \ldots + \frac{(t_j - a_j)^m}{m!} \left. \frac{\partial^m f(t)}{\partial t_j^m} \right|_{t=a}.
$$

Here, $\delta > 0$ represents the smallest number for which the relations written above are provided simultaneously. In the light of above observations, for a sufficiently small $\delta > 0$, we obtain the following equality:

$$\begin{aligned}
\frac{\partial^m}{\partial x_j^m}\left(G_w(f-g)\right)(x) &= (-1)^m \int\limits_D [f(t)-g(t)] \frac{\partial^m}{\partial t_j^m} K_w(t-x)dt \\
&= (-1)^m \left\{ \int\limits_{D^1} + \int\limits_{D\backslash D^1} \right\} [f(t)-g(t)] \frac{\partial^m}{\partial t_j^m} K_w(t-x)dt \\
&= : k(x,w) + l(x,w).
\end{aligned}$$

We first consider $k(x,w)$ such that

$$k(x,w) = (-1)^m \int\limits_{D^1} [f(t) \pm f(a_1,...,t_j,...,a_n) \pm f(a) - g(t)] \frac{\partial^m}{\partial t_j^m} K_w(t-x)dt$$

$$\begin{aligned}
&= (-1)^m \int\limits_{D^1} [-f(a_1,...,t_j,...,a_n) + f(a)] \frac{\partial^m}{\partial t_j^m} K_w(t-x)dt \\
&\quad + (-1)^m \int\limits_{D^1} [f(a_1,...,t_j,...,a_n) - g(t)] \frac{\partial^m}{\partial t_j^m} K_w(t-x)dt \\
&\quad + (-1)^m \int\limits_{D^1} [f(t) - f(a)] \frac{\partial^m}{\partial t_j^m} K_w(t-x)dt \\
&= : k_1(x,w) + k_2(x,w) + k_3(x,w).
\end{aligned}$$

Let $|x_i - a_i| < \frac{\delta}{2}$, where $i = 1, 2, ..., n$. Using relations (i) and (ii), we obtain the following inequality for $|k_1(x,w)| + |k_3(x,w)|$

$$(|k_1(x,w)| + |k_3(x,w)|) \leq 2\varepsilon \int\limits_{D^2} \left| \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt,$$

where $D^2 =]a_1 - x_1 - \delta,\ a_1 - x_1 + \delta[\times \cdots \times]a_n - x_n - \delta,\ a_n - x_n + \delta[$. By the hypothesis (2.3), we can write with peace of mind that

$$\lim_{(x,w)\to(a,w_0)} (|k_1(x,w)| + |k_3(x,w)|) = 0$$

on S. If we deal with $k_2(x,w)$ using relation (iii), we obtain

$$\begin{aligned}
|k_2(x,w)| &= \left| (-1)^m \int\limits_{D^1} \frac{\left(f(a_1,...,t_j,...,a_n) - g_{t_j}\right)}{(t_j - a_j)^m} (t_{j-}a_j)^m \frac{\partial^m}{\partial t_j^m} K_w(t-x)dt \right| \\
&\leq \varepsilon \int\limits_{D^2} \left| (t_j + x_j - a_j)^m \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt \\
&= \varepsilon \int\limits_{D^2} \left| \left((t_j + x_j - a_j)^m - t_j^m + t_j^m \right) \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt \\
&\leq \varepsilon \int\limits_{D^2} \left| (t_j + x_j - a_j)^m - t_j^m \right| \left| \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt \\
&\quad + \varepsilon \int\limits_{D^2} |t_j|^m \left| \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt \\
&= : \varepsilon k_{21}(x,w) + \varepsilon k_{22}(x,w).
\end{aligned}$$

It is easy to see that $k_{22}(x,w)$ is bounded on S by condition (2.2). Using well-known formula (see [17]) given by

$$z_1^s - z_2^s = (z_1 - z_2)(z_1^{s-1} + z_1^{s-2}z_2 + \ldots + z_2^{s-1}), \quad z_1, z_2 \in R,$$

where s is a natural number with $s \neq 0$, for $k_{21}(x,w)$, we obtain

$$\begin{aligned}
&k_{21}(x,w) \\
=\ & \int\limits_{D^2} \left|(t_j + x_j - a_j)^m - t_j^m\right| \left|\frac{\partial^m}{\partial t_j^m} K_w(t)\right| dt \\
\leq\ & |x_j - a_j| \int\limits_{D^2} \left|(t_j + x_j - a_j)^{m-1}\right| \left|\frac{\partial^m}{\partial t_j^m} K_w(t)\right| dt \\
& + |x_j - a_j| \int\limits_{D^2} \left|(t_j + x_j - a_j)^{m-2}\right| |t_j| \left|\frac{\partial^m}{\partial t_j^m} K_w(t)\right| dt \\
& + \ldots + \\
& + |x_j - a_j| \int\limits_{D^2} |t_j + x_j - a_j| \, |t_j|^{m-2} \left|\frac{\partial^m}{\partial t_j^m} K_w(t)\right| dt \\
& + |x_j - a_j| \int\limits_{D^2} |t_j|^{m-1} \left|\frac{\partial^m}{\partial t_j^m} K_w(t)\right| dt.
\end{aligned}$$

If we apply same operations to each integral above, we see that $k_{21}(x,w)$ is bounded above by a sum consisting of the following expressions:

$$|x_j - a_j|^v \int\limits_{D^2} |t_j|^{m-v} \left|\frac{\partial^m}{\partial t_j^m} K_w(t)\right| dt, \quad v = 1, 2, ..., m.$$

By the hypothesis (2.4), $k_{21}(x,w)$ is bounded on S. Hence

$$\lim_{(x,w)\to(a,w_0)} k_2(x,w) = 0$$

on S. Since $f - g$ is Lebesgue integrable on D and

$$\begin{aligned}
|l(x,w)| &\leq \left| \int\limits_{D \setminus D^1} [f(t) - g(t)] \frac{\partial^m}{\partial t_j^m} K_w(t - x) dt \right| \\
&\leq \sup_{\frac{\delta}{2} \leq |u|} \left|\frac{\partial^m}{\partial t_j^m} K_w(u)\right| \int\limits_D |f(t) - g(t)| \, dt,
\end{aligned}$$

we see that

$$\lim_{(x,w)\to(a,w_0)} l(x,w) = 0.$$

From above observations, we deduce that

$$\lim_{(x,w)\to(a,w_0)} \frac{\partial^m}{\partial x_j^m} (G_w(f - g))(x) = 0.$$

Therefore, the claim follows, that is,

$$\lim_{(x,w)\to(a,w_0)} \frac{\partial^m}{\partial x_j^m} (G_w f)(x) = f_j^{(m)}(a).$$

Thus, the proof is completed. □

Case 2: Domain of integration is R^n

We consider the following operators

$$\left(E_w f\right)(x)=\int\limits_{R^n} f(t) K_w(t-x)\, dt.$$

The explicit form of these operators can be written as follows:

$$\left(E_w f\right)(x_1,...,x_n)=\int\limits_{-\infty}^{\infty}\cdots\int\limits_{-\infty}^{\infty} f(t_1,...,t_n)K_w(t_1-x_1,...,t_n-x_n)dt_n\cdots dt_1,$$

with $x \in R^n$.

Let $L_1(R^n)$ denote the space of all functions f which are integrable in the sense of Lebesgue on R^n with respect to usual Lebesgue measure dt. Any function in this space satisfies the property such that $\|f\|_{L_1(R^n)} := \int\limits_{R^n} |f(t)|\, dt < \infty$.

Here, the kernel $K_w(t)$ satisfies the following conditions:

1. $K_w : R^n \to R_0^+$ is a measurable function on its domain for every fixed $w \in \Lambda$.
2. $\lim\limits_{w \to w_0} \int\limits_{R^n} K_w(t)\, dt = 1$, where $w \in \Lambda$.
3. $\|K_w(.)\|_{L_1(R^n)}$ is uniformly bounded on Λ by a constant K.
4. $\lim\limits_{w \to w_0} \int\limits_{|t|\geq \xi} K_w(t)\, dt = 0, \forall \xi > 0$.

Here, $|.|$ denotes usual Euclidean distance.

Now, we state and prove the last theorem.

Theorem 2.2. *Assume that $K_w(t)$ and $\frac{\partial^v}{\partial t_j^v} K_w(t)$ are continuous functions with respect to t on R^n for every fixed $w \in \Lambda$, $v = 1, 2, ..., m$ and $j = 1, 2, ..., n$. Suppose that the following conditions*

$$\lim_{w\to w_0} \sup_{|t|\geq \xi} \left| \frac{\partial^v}{\partial t_j^v} K_w(t) \right| = 0,\ \forall \xi > 0$$

$$\sup_{w\in\Lambda} \int\limits_{R^n} |t_j|^m \left| \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt < \infty \tag{2.5}$$

hold for every fixed $j = 1, 2, ..., n$ and $v = 0, 1, ..., m$, together with conditions (1)-(4). If $f \in L_1(R^n)$ possesses at $a := (a_1, ..., a_n) \in R^n$ finite $m-th$ order partial derivative with respect to $j-th$ variable $f_j^{(m)}(a)$ and there exists a neighborhood $]a_1 - \eta,\ a_1 + \eta[\times \cdots \times]a_n - \eta,\ a_n + \eta[$ with $\eta > 0$ of a on which the functions $f_j^{(m)}$ and f are continuous, then

$$\lim_{(x,w)\to(a,w_0)} \frac{\partial^m}{\partial x_j^m} \left(E_w f\right)(x) = f_j^{(m)}(a),$$

on any set S consisting of (x, w) on which the functions expressed as

$$\sup_{w\in\Lambda} \int\limits_{R^n} \left| \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt \tag{2.6}$$

and

$$|x_j - a_j|^v \int\limits_{R^n} |t_j|^{m-v} \left| \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt \leq T_v \tag{2.7}$$

are bounded, where $j = 1, 2, ..., n$ and $v = 1, 2, ..., m$. Here, T_v are certain positive constants.

Proof. By the hypotheses, the functions $\frac{\partial^m}{\partial t_j^m} f(t)$ for $m = 1, ..., n$ and $f(t)$ are continuous at $a \in R^n$. Therefore, for all $\varepsilon > 0$ there exists an appropriate number $\delta > 0$ such that the following relations hold there:

$i.$ $|f(t) - f(a)| < \varepsilon$ provided that $t \in D_1$, where $D_1 =]a_1 - \delta,\ a_1 + \delta[\times \cdots \times]a_n - \delta,\ a_n + \delta[$.

$ii.$ $|f(a_1, ..., t_j, ..., a_n) - f(a_1, ..., a_j, ..., a_n)| < \varepsilon$ provided that $|t_j - a_j| < \delta$.

Further, for above $\delta > 0$, we define an auxiliary function h^1 as follows:

$$h^1(t) = \begin{cases} h(t), & t \in D_1, \\ 0, & t \in R^n \backslash D_1, \end{cases}$$

where $h(t)$ is defined by $h(t) =: h_{t_j}$, where

$$h_{t_j} := f(a) + (t_j - a_j) \left. \frac{\partial f(t)}{\partial t_j} \right|_{t=a} + \ldots + \frac{(t_j - a_j)^m}{m!} \left. \frac{\partial^m f(t)}{\partial t_j^m} \right|_{t=a}$$

such that $\left. \frac{\partial^k h}{\partial t_j^k} \right|_{t=a} = f_j^{(k)}(a)$, $k = 0, ..., m$.

We continue further with the following observation:

$iii.$ $\left| \frac{\left(f(a_1,...,t_j,...,a_n) - h_{t_j}\right)}{(t_j - a_j)^m} \right| < \varepsilon$ provided that $|t_j - a_j| < \delta$, since

$$\lim_{t_j \to a_j} \frac{\left(f(a_1, ..., t_j, ..., a_n) - h_{t_j}\right)}{(t_j - a_j)^m} = 0.$$

Let $|x_j - a_j| < \frac{\delta}{2}$, where $j = 1, 2, ..., n$. By linearity of the operators, we can write

$$\begin{aligned} (E_w f)(x) &= \left(E_w(f + h^1 - h^1)\right)(x) \\ &= \left(E_w h^1\right)(x) + \left(E_w(f - h^1)\right)(x). \end{aligned}$$

Differentiating both sides of the following equation up to order m with respect to x_j such that one easily obtains

$$\begin{aligned} \frac{\partial^m}{\partial x_j^m} \left(E_w h^1\right)(x) &= \frac{\partial^m}{\partial x_j^m} \int_{R^n} h^1(t) K_w(t - x) dt \\ &= (-1)^m \int_{R^n} h^1(t) \frac{\partial^m}{\partial t_j^m} K_w(t - x) dt. \end{aligned}$$

In view of Fubini's Theorem and m times application of integration by parts with respect to t_j, we have the following equality:

$$\begin{aligned} & \lim_{(x,w) \to (a,w_0)} \int_{D_1} K_w(t - x) \frac{\partial^m}{\partial t_j^m} h(t) dt \\ = & \lim_{(x,w) \to (a,w_0)} \left. \frac{\partial^m f}{\partial t_j^m} \right|_{t=a} \int_{R^n} K_w(t - x) dt - \lim_{(x,w) \to (a,w_0)} \left. \frac{\partial^m f}{\partial t_j^m} \right|_{t=a} \int_{R^n \backslash D_1} K_w(t - x) dt. \end{aligned}$$

Hence, by conditions (2) and (4), the desired result is obtained, that is,

$$\lim_{(x,w) \to (a,w_0)} \frac{\partial^m}{\partial x_j^m} \left(E_w h^1\right)(x) = \left. \frac{\partial^m f}{\partial t_j^m} \right|_{t=a} = f_j^{(m)}(a).$$

To complete the proof, we will show that

$$\lim_{(x,w) \to (a,w_0)} \frac{\partial^m}{\partial x_j^m} \left(E_w(f - h^1)\right)(x) = 0.$$

In the light of observations given in the beginning of the proof, for $\delta > 0$, we obtain the following equality:

$$
\begin{aligned}
\frac{\partial^m}{\partial x_j^m}\left(E_w(f-h^1)\right)(x) &= (-1)^m \int\limits_{R^n} \left[f(t)-h^1(t)\right] \frac{\partial^m}{\partial t_j^m} K_w(t-x)dt \\
&= (-1)^m \left\{ \int\limits_{D_1} + \int\limits_{R^n \backslash D_1} \right\} \left[f(t)-h^1(t)\right] \frac{\partial^m}{\partial t_j^m} K_w(t-x)dt \\
&= : k(x,w) + l(x,w).
\end{aligned}
$$

We first consider $k(x,w)$ such that

$$
\begin{aligned}
& k(x,w) \\
= \; & (-1)^m \int\limits_{D_1} \left[f(t) \pm f(a_1,...,t_j,...,a_n) \pm f(a) - h(t)\right] \frac{\partial^m}{\partial t_j^m} K_w(t-x)dt \\
= \; & (-1)^m \int\limits_{D_1} \left[-f(a_1,...,t_j,...,a_n) + f(a)\right] \frac{\partial^m}{\partial t_j^m} K_w(t-x)dt \\
& +(-1)^m \int\limits_{D_1} \left[f(a_1,...,t_j,...,a_n) - h(t)\right] \frac{\partial^m}{\partial t_j^m} K_w(t-x)dt \\
& +(-1)^m \int\limits_{D_1} \left[f(t) - f(a)\right] \frac{\partial^m}{\partial t_j^m} K_w(t-x)dt \\
= \; & : k_1(x,w) + k_2(x,w) + k_3(x,w).
\end{aligned}
$$

Let $|x_j - a_j| < \frac{\delta}{2}$, where $j = 1,2,...,n$. Using relations (i) and (ii), we obtain the following inequality for $|k_1(x,w)| + |k_3(x,w)|$

$$
(|k_1(x,w)| + |k_3(x,w)|) \leq 2\varepsilon \int\limits_{R^n} \left| \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt.
$$

By (2.6), we can write

$$
\lim_{(x,w)\to(a,w_0)} (|k_1(x,w)| + |k_3(x,w)|) = 0.
$$

If we deal with $k_2(x,w)$ using relation (iii), we obtain

$$
\begin{aligned}
|k_2(x,w)| &= \left| (-1)^m \int\limits_{D_1} \frac{\left(f(a_1,...,t_j,...,a_n) - h_{t_j}\right)}{(t_j - a_j)^m} (t_{j-}a_j)^m \frac{\partial^m}{\partial t_j^m} K_w(t-x)dt \right| \\
&\leq \varepsilon \int\limits_{R^n} \left| (t_j + x_j - a_j)^m \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt \\
&= \varepsilon \int\limits_{R^n} \left| \left((t_j + x_j - a_j)^m - t_j^m + t_j^m \right) \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt \\
&\leq \varepsilon \int\limits_{R^n} \left| (t_j + x_j - a_j)^m - t_j^m \right| \left| \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt \\
& \quad + \varepsilon \int\limits_{R^n} |t_j|^m \left| \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt \\
&= : \varepsilon k_{21}(x,w) + \varepsilon k_{22}(x,w),
\end{aligned}
$$

where

$$k_{21}(x,w) = \int\limits_{R^n} \left|(t_j + x_j - a_j)^m - t_j^m\right| \left|\frac{\partial^m}{\partial t_j^m} K_w(t)\right| dt$$

and

$$k_{22}(x,w) = \int\limits_{R^n} |t_j|^m \left|\frac{\partial^m}{\partial t_j^m} K_w(t)\right| dt.$$

It is easy to see that $k_{22}(x,w)$ is bounded on S by condition (2.5). Using well-known formula given by

$$z_1^s - z_2^s = (z_1 - z_2)(z_1^{s-1} + z_1^{s-2} z_2 + ... + z_2^{s-1}), \quad z_1, z_2 \in R,$$

where s is a natural number with $s \neq 0$, for $k_{21}(x,w)$, we obtain

$$\begin{aligned}
& k_{21}(x,w) \\
= & \int\limits_{R^n} \left|(t_j + x_j - a_j)^m - t_j^m\right| \left|\frac{\partial^m}{\partial t_j^m} K_w(t)\right| dt \\
\leq & |x_j - a_j| \int\limits_{R^n} \left|(t_j + x_j - a_j)^{m-1}\right| \left|\frac{\partial^m}{\partial t_j^m} K_w(t)\right| dt \\
& + |x_j - a_j| \int\limits_{R^n} \left|(t_j + x_j - a_j)^{m-2}\right| |t_j| \left|\frac{\partial^m}{\partial t_j^m} K_w(t)\right| dt \\
& + \ldots + \\
& + |x_j - a_j| \int\limits_{R^n} |t_j + x_j - a_j| \, |t_j|^{m-2} \left|\frac{\partial^m}{\partial t_j^m} K_w(t)\right| dt \\
& + |x_j - a_j| \int\limits_{R^n} |t_j|^{m-1} \left|\frac{\partial^m}{\partial t_j^m} K_w(t)\right| dt.
\end{aligned}$$

If we apply same operations to each integral, we see that $k_{21}(x,w)$ is bounded above by a sum consisting of the following expressions:

$$|x_j - a_j|^v \int\limits_{R^n} |t_j|^{m-v} \left|\frac{\partial^m}{\partial t_j^m} K_w(t)\right| dt, \quad v = 1, 2, ..., m.$$

By the hypothesis (2.7), $k_{21}(x,w)$ is bounded on S. Hence

$$\lim_{(x,w)\to(a,w_0)} k_2(x,w) = 0$$

on S. Since f is Lebesgue integrable on R^n and

$$\begin{aligned}
|l(x,w)| & \leq \left| \int\limits_{R^n \setminus D_1} f(t) \frac{\partial^m}{\partial t_j^m} K_w(t-x) dt \right| \\
& \leq \sup_{\frac{\delta}{2} \leq |u|} \left|\frac{\partial^m}{\partial t_j^m} K_w(u)\right| \int\limits_{R^n} |f(t)| \, dt,
\end{aligned}$$

we see that

$$\lim_{(x,w)\to(a,w_0)} l(x,w) = 0.$$

From above observations, we deduce that

$$\lim_{(x,w)\to(a,w_0)} \frac{\partial^m}{\partial x_j^m}(E_w(f-h^1))(x) = 0.$$

Therefore, the claim follows, that is,

$$\lim_{(x,w)\to(a,w_0)} \frac{\partial^m}{\partial x_j^m}(E_w f)(x) = f_j^{(m)}(a).$$

Thus, the proof is completed. □

3. Application

Example 3.1. Let $f : R^2 \to R$ be defined by $f(x,y) = e^{-(x^2+y^2)}$ and

$$(E_w f)(x,y) = \frac{1}{4\pi w} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t,s) e^{-\frac{((x-t)^2+(y-s)^2)}{4w}} ds dt,$$

where the used kernel is of Gauss-Weierstrass type. This version is obtained by applying change of variables method to well-known version. Here, $\Lambda = (0,1)$ and $w_0 = 0$. Some properties of above operators can be found in [15]. Figure 1 demonstrates the convergence of $\frac{\partial^2(E_w f)(x,y)}{\partial x^2}$ to $\frac{\partial^2 f(x,y)}{\partial x^2}$ (dark blue) for $w = 0.5$ (green) and $w = 0.2$ (red) on $[-5,5]\times[-5,5]$. Figure 1 is generated by using a computer algebra system (CAS) Mathematica.

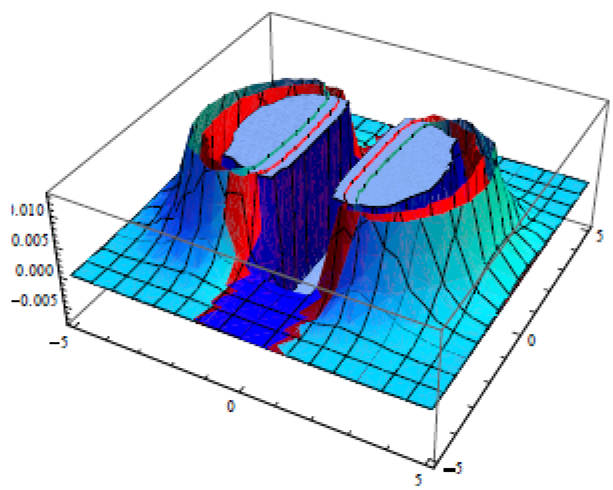

Figure 1. Demonstration

Acknowledgement

We thank to late Prof. Dr. Akif Djafar Gadjiev (also known as Akif Djafar Gadžiev), who helped us in order to internalize some of the concepts related to differentiation of singular integrals. By clarifying definitions of these concepts, we have the opportunity to criticize many studies correctly, including ours.

References

[1] Almali, S.E., Gadjiev, A.D.: *On approximation properties of certain multidimensional nonlinear integrals*. J. Nonlinear Sci. Appl. **9** (5) 3090–3097 (2016).

[2] Bardaro, C., Mantellini, I.: *Voronovskaya-type estimates for Mellin convolution operators*. Results Math. **50** (1-2) 1–16 (2007).

[3] Bardaro, C., Mantellini, I.: B*ivariate Mellin convolution operators: Quantitative approximation theorems*. Math. Comput. Modelling **53** (5-6) 1197–1207 (2011).

[4] Butzer, P.L., Nessel, R.J.: Fourier analysis and approximation Vol. 1: One-dimensional theory. Pure and Applied Mathematics Vol. 40. Academic Press: New York-London (1971).

[5] Fatou, P.: *Séries trigonométriques et séries de Taylor* (French). Acta Math. **30** (1) 335–400 (1906).

[6] Gadžiev, A.D.: *The asymptotic value of the approximation of derivatives by the derivatives of a family of linear operators* (Russian). Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Mat. Tehn. Nauk **6** 15–24 (1962).

[7] Gadžiev, A.D.: *On the convergence of integral operators* (Russian). Akad. Nauk Azerbaĭdžan. SSR Dokl. **19** (12) 3–7 (1963).

[8] Gadžiev, A.D.: *On the speed of convergence of a class of singular integrals* (Russian). Izv. Akad. Nauk Azerbaĭdžan. SSR Ser. Fiz.-Mat. Tehn. Nauk **6,** 27–31 (1963).

[9] Gadžiev, A.D.: *The order of convergence of singular integrals which depend on two parameters* (Russian). In: Special problems of functional analysis and their applications to the theory of differential equations and the theory of functions (Russian), 40-44 (1968).

[10] Karsli, H., Ibikli, E.: *Approximation properties of convolution type singular integral operators depending on two parameters and of their derivatives in* $L_1(a,b)$. In: Proceedings of the 16th International Conference of the Jangjeon Mathematical Society, 66–76, Jangjeon Math. Soc.: Hapcheon (2005).

[11] Karsli, H.: Approximation properties of singular integrals depending on two parameters and their derivatives (Turkish). PhD Thesis, Ankara University, Ankara, 75 pp. (2006).

[12] Karsli, H.: *Convergence of the derivatives of nonlinear singular integral operators*. J. Math. Anal. Approx. Theory **2** (1) 26–34 (2007).

[13] Matsuoka, Y.: *Asymptotic formula for Vallée Poussin's singular integrals*. Sci. Rep. Kagoshima Univ. **9,** 25–34 (1960).

[14] Saks, S.: Theory of the integral. Monografie Matematyczne, Warszawa (1937).

[15] Siudut, S.: *On the convergence of double singular integrals*. Comment. Math. Prace Mat. **28** (1) 143–146 (1988).

[16] Siudut, S.: *On the Fatou type convergence of abstract singular integrals*. Comment. Math. Prace Mat. **30** (1) 171–176 (1990).

[17] Spivak, M.D.: Calculus. (3rd ed.), Publish or Perish, Incorporated: Houston-Texas (1994).

[18] Stein, E.M.: Singular integrals and differentiability properties of functions. Princeton Mathematical Series, No. 30. Princeton Univ. Press: Princeton-New Jersey (1970).

[19] Stein, E.M., Weiss, G.: Introduction to Fourier analysis on Euclidean spaces. Princeton Mathematical Series, No. 32. Princeton Univ. Press: Princeton-New Jersey (1971).

[20] Taberski, R.: *Singular integrals depending on two parameters*. Prace Mat. **7**, 173–179 (1962).

[21] Taberski, R.: *On double integrals and Fourier series*. Ann. Polon. Math. **15,** 97–115 (1964).

[22] Uysal, G., Yilmaz, M.M., Ibikli, E.: *Approximation by radial type multidimensional singular integral operators*. Palest. J. Math. **5** (2) 61–70 (2016).

[23] Yilmaz, B., Aral, A., Tunca, G.B.: *Weighted approximation properties of generalized Picard operators*. J. Comput. Anal. Appl. **13** (3) 499–513 (2011).

[24] Žornickaja, L.V.: *The derivatives of some singular integrals* (Russian). Izv. Vysš. Učebn. Zaved. Matematika **29** (4) 62–72 (1962).

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