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How to extend Carathéodory's theorem to lattice-valued functionals

Nutefe Kwami Agbeko[a]

[a]Institute of Mathematics, University of Miskolc 3515 Miskolc-Egyetemváros, Hungary.

Abstract

Substituting in the definition of outer measure the addition with the maximum (or the supremum, or the join) operation we obtain a new set function called retuo measure. It is proved that every retuo measure is an outer measure. We give necessary and sufficient conditions for a set function to be a retuo measure. Similarly as in the case of outer measure, we propose a way to construct retuo measures. We consider some theoretical applications for constructed pairs of outer and retuo measures in the image of the Hausdorff measure and dimension.

Keywords: outer measure, measurable sets, σ-algebra, Carathéodory-type theorem, Hausdorff measure, Hausdorff dimension.
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1. Introduction

All along this communication, Ω will stand for a non-empty set.

The symbols $\vee$ and $\bigvee$ will stand for both the maximum and the supremum (or the join).

We recall the following two definitions from Measure Theory:

A sequence $(A_n) \subset 2^{\Omega}$ is a (countable) cover of a set $A \in 2^{\Omega}$ if $A \subseteq \bigcup_{n=1}^{\infty} A_n$.

A function $\Psi : 2^{\Omega} \to [0, \infty]$ is an outer-measure if $\Psi(\emptyset) = 0$ and Ψ is sub-additive in the sense of Carathéodory, i.e. $\Psi(A) \leq \sum_{n=1}^{\infty} \Psi(A_n)$, whenever $(A_n) \subset 2^{\Omega}$ is a cover of A.

Outer measures can be constructed as follows:

Email address: matagbek@uni-miskolc.hu (Nutefe Kwami Agbeko)

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Let $\mathcal{C} \subset 2^{\Omega}$, with $\emptyset \in \mathcal{C}$ and $p : \mathcal{C} \to [0, \infty]$ be an arbitrary mapping for which $p(\emptyset) = 0$. Then the mapping $\Psi_p : 2^{\Omega} \to [0, \infty]$ defined by

$$\Psi_p(A) = \inf \left\{ \sum_{n=1}^{\infty} p(I_n) : A \subseteq \bigcup_{n=1}^{\infty} I_n,\ (I_n) \subset \mathcal{C} \right\} \tag{1.1}$$

is an outer measure.

Carathéodory's famous theorem states: *Given an outer measure $\Psi : 2^{\Omega} \to [0, \infty]$, the set*

$$\mathcal{M}_{\Psi} := \left\{ A \in 2^{\Omega} : \Psi(E \cap A) + \Psi(E \cap A^c) = \Psi(E) \text{ for all } E \in 2^{\Omega} \right\} \tag{1.2}$$

is a σ-algebra (which we shall refer to as the Carathéodory-Lebesgue σ-algebra) and the restriction of Ψ to $\mathcal{M}_{\Psi}$ is a countable additive measure. Moreover, if $A \in 2^{\Omega}$ and $\Psi(A) = 0$, then $A \in \mathcal{M}_{\Psi}$.

For more about this theorem we refer the reader to the following bibliographies: [3, 4, 8, 13, 9, 10, 6, 7], say.

We would like to recall the following definition:
A set function $\Phi : \mathcal{F} \to [0, \infty]$, with $\mathcal{F}$ being a σ-algebra of subsets of Ω, is referred to as a σ-maxitive measure if the three conditions here below are met:

1.) $\Phi(\emptyset) = 0$.

2.) *Φ is a join-homomorphism, i.e. $\Phi(A \cup B) = \Phi(A) \vee \Phi(B)$ for all $A, B \in \mathcal{F}$.*

3.) *Φ is continuous from below on $\mathcal{F}$, i.e. $\lim_{n\to\infty} \Phi(A_n) = \Phi(A)$ for every sequence $(A_n) \subset \mathcal{F}$ tending increasingly to $A \in \mathcal{F}$.*

For more about σ-maxitive measures we refer the reader to [5, 12], say.

Inspiring from [3, 4, 8, 13], the main goal of the present communication is to show that the Carathéodory Theorem remains valid (at least up to some stage). We first show that by substituting the addition operation "+" with the supremum (or maximum) $\bigvee$ in the definition of outer measure the resulting set function defined on the power set is equivalent to σ-maxitive measure defined on power set.

2. The counterpart of outer measure

All along this communication $\Phi : 2^{\Omega} \to [0, \infty]$ will stand for a fixed set function.

In the image of (1.2) let us consider the following set:

$$\mathcal{N}_{\Phi} := \left\{ A \in 2^{\Omega} : \Phi(E \cap A) \vee \Phi(E \cap A^c) = \Phi(E) \text{ for all } E \in 2^{\Omega} \right\}. \tag{2.1}$$

Contrary to the set $\mathcal{M}_{\Psi}$ in (1.2) the question arises to know whether $\mathcal{N}_{\Phi}$ is the whole power set or not. What condition or conditions can guarantee that $\mathcal{N}_{\Phi}$ will equal the power set of Ω? We would like to point out that this question is not worth to be asked in the case of outer measure.

In order to answer the above question we shall first prove the following crucial lemma.

Lemma 2.1. *If $\mathcal{N}_{\Phi} = 2^{\Omega}$, then Φ is a non-decreasing function.*

Proof. Let $A, B \in 2^{\Omega}$ with $A \subseteq B$. In the identity characterizing the set $\mathcal{N}_{\Phi}$ in (2.1) keep A and substitute E with B. Then

$$\Phi(B) = \Phi(B \cap A) \vee \Phi(B \cap A^c) = \Phi(A) \vee \Phi(B \setminus A) \geq \Phi(A)$$

which trivially completes the proof. □

We are now in the position to address in the affirmative our question formulated above.

Proposition 2.2. *In order that $\mathcal{N}_{\Phi} = 2^{\Omega}$, it is necessary and sufficient that Φ be a join-homomorphism.*

Proof. Suppose that $\mathcal{N}_\Phi = 2^\Omega$ and fix arbitrarily two sets $A,\, B \in 2^\Omega$. In the identity in (2.1) keep A and substitute E with $A \cup B$. Then

$$\Phi\left((A \cup B) \cap A\right) \vee \Phi\left((A \cup B) \cap A^c\right) = \Phi\left(A \cup B\right).$$

After an obvious argument we can easily observe that

$$\Phi\left(A\right) \vee \Phi\left(B \setminus A\right) = \Phi\left(A \cup B\right). \tag{2.2}$$

In (2.2) interchange the roles of A and B to get

$$\Phi\left(B\right) \vee \Phi\left(A \setminus B\right) = \Phi\left(B \cup A\right). \tag{2.3}$$

Now, take the maximum (or supremum, or join) of the left hand sides of both identities (2.2) and (2.3), and then make use of Lemma (2.1) to have

$$\Phi\left(A \cup B\right) = \Phi\left(A\right) \vee \Phi\left(B\right).$$

Conversely, assume that Φ is a join-homomorphism. To prove that $2^\Omega = \mathcal{N}_\Phi$, it will be enough to show the validity of inclusion $2^\Omega \subseteq \mathcal{N}_\Phi$, because the reverse inclusion is obvious, indeed. In fact, fix arbitrarily an $A \in 2^\Omega$. Note that $E = (E \cap A) \cup (E \cap A^c)$, for all $E \in 2^\Omega$. Then as Φ is a join-homomorphism on the power set 2^Ω, it follows that $\Phi\left(E\right) = \Phi\left(E \cap A\right) \vee \Phi\left(E \cap A^c\right)$, which means that $A \in \mathcal{N}_\Phi$. Thus $2^\Omega \subseteq \mathcal{N}_\Phi$. This completes the proof. □

We shall formulate the counterpart of the definition of outer measure as follows.

Definition 2.3. *A mapping $\Phi : 2^\Omega \to [0, \infty]$ is referred to as a retuo measure if each one of the following two conditions is valid:*

1. $\Phi\left(\emptyset\right) = 0$.

2. *Whenever a sequence $(A_n) \subset 2^\Omega$ is a cover of any given set $A \in 2^\Omega$, then*

$$\Phi\left(A\right) \leq \bigvee_{n=1}^{\infty} \Phi\left(A_n\right). \tag{2.4}$$

Example 2.4. *Let $\Omega = \{1,\, 2,\, 3,\, 4,\, 5,\, 6\}$, and define the set function $\Phi : 2^\Omega \to [0,\, 1]$, by $\Phi\left(\emptyset\right) = 0$ and $\Phi\left(A\right) = \frac{1}{6}$, whenever $A \in 2^\Omega \setminus \{\emptyset\}$. Then Φ is a retuo measure.*

Remark 2.5. *Every retuo measure is a non-decreasing mapping.*

Remark 2.6. *Every retuo measure is also an outer measure.*

In the next theorem, we shall provide the set of all retuo measures.

Theorem 2.7. *Let $\Phi : 2^\Omega \to [0, \infty]$ be a function. Then in order that Φ be a retuo measure it is necessary and sufficient that the following three conditions be met simultaneously.*

1.) $\Phi\left(\emptyset\right) = 0$.

2.) *Φ is a join-homomorphism, i.e. $\Phi\left(A \cup B\right) = \Phi\left(A\right) \vee \Phi\left(B\right)$ for all $A,\, B \in 2^\Omega$.*

3.) *Φ is continuous from below on 2^Ω, i.e. $\lim\limits_{n\to\infty} \Phi\left(A_n\right) = \Phi\left(A\right)$ for every sequence $(A_n) \subset 2^\Omega$ tending increasingly to $A \in 2^\Omega$.*

Actually, Theorem 2.7 just states that σ-maxitive measures defined on power sets are retuo measures and vice versa.

Proof of Theorem 2.7. Assume that the above three conditions are met simultaneously. Let $A \in 2^{\Omega}$ be an arbitrary set and fix $(A_n) \subset 2^{\Omega}$ any of its covers. Since Φ is a join-homomorphism, it is non-decreasing. Then $\Phi(A) \leq \Phi\left(\bigcup_{n=1}^{\infty} A_n\right)$. One can show by induction on $m \in \mathbb{N}$ that $\Phi\left(\bigcup_{n=1}^{m} A_n\right) = \bigvee_{n=1}^{m} \Phi(A_n)$ for all $m \in \mathbb{N}$. Now, since sequence $\left(\bigcup_{n=1}^{m} A_n\right)$ tend increasingly to $\bigcup_{n=1}^{\infty} A_n \in 2^{\Omega}$ as $m \to \infty$, the continuity from below thus implies that

$$\Phi(A) \leq \bigvee_{n=1}^{\infty} \Phi(A_n).$$

Therefore, Φ is a retuo measure, since the first condition of Definition 2.3 is also met.

Conversely, assume that Φ is a retuo measure. We will just need to prove that it is both a join-homomorphism and continuous from below.

First, we show that Φ is a join-homomorphism. To this end let us prove the inclusion $2^{\Omega} \subseteq \mathcal{N}_{\Phi}$. In fact, fix arbitarily a set $A \in 2^{\Omega}$. Then as we know $E = (E \cap A) \cup (E \cap A^c)$ for all $E \in 2^{\Omega}$. Write $A_n = E$, $n \in \mathbb{N}$. Then sequence (A_n) is a cover of both sets $E \cap A$ and $E \cap A^c$. So that $\Phi(E \cap A) \leq \Phi(E)$ and $\Phi(E \cap A^c) \leq \Phi(E)$ implying that

$$\Phi(E \cap A) \vee \Phi(E \cap A^c) \leq \Phi(E). \tag{2.5}$$

To show the reverse of this inequality write $A_1 = E \cap A$ and $A_n = E \cap A^c$, $n \in \mathbb{N} \setminus \{1\}$. Then sequence (A_n) is a cover of E. Hence

$$\Phi(E) \leq \Phi(E \cap A) \vee \Phi(E \cap A^c). \tag{2.6}$$

Combining (2.5) and (2.6) leads to:

$$\Phi(E) = \Phi(E \cap A) \vee \Phi(E \cap A^c)$$

for all $E \in 2^{\Omega}$, meaning that $A \in \mathcal{N}_{\Phi}$. We have thus shown that $2^{\Omega} \subseteq \mathcal{N}_{\Phi}$. But since the converse of this inclusion is obvious, we can confirm in virtue of Proposition 2.2 that Φ is a join-homomorphism.

Next, we prove that Φ is continuous from below. In fact, let $(A_n) \subseteq 2^{\Omega}$ tend increasingly to some $A \in 2^{\Omega}$. Clearly, $A = \bigcup_{n=1}^{\infty} A_n$. On the one hand by the definition of retuo measure we have

$$\Phi(A) \leq \bigvee_{n=1}^{\infty} \Phi(A_n). \tag{2.7}$$

On the other hand by the monotonicity of retuo measures (see Remark 2.5),

$$\bigvee_{n=1}^{\infty} \Phi(A_n) \leq \Phi(A). \tag{2.8}$$

Now combine (2.7) and (2.8) to observe that

$$\Phi(A) = \bigvee_{n=1}^{\infty} \Phi(A_n) = \lim_{n\to\infty} \Phi(A_n).$$

Therefore, we can conclude on the validity of the theorem. $\square$

3. Constructing the counterpart of outer measure

Just like in the case of outer measures we propose some means to construct retuo measures.

Theorem 3.1 (Retuo Measure Construction). *Let $\mathcal{C} \subset 2^{\Omega}$ be such that $\emptyset \in \mathcal{C}$ and $p : \mathcal{C} \to [0, \infty]$ be an arbitrary mapping for which $p(\emptyset) = 0$. For any $A \in 2^{\Omega}$, define*

$$\Phi_p(A) = \inf\left\{ \bigvee_{n=1}^{\infty} p(I_n) : A \subseteq \bigcup_{n=1}^{\infty} I_n,\ (I_n) \subset \mathcal{C} \right\}. \tag{3.1}$$

Then $\Phi_p : 2^{\Omega} \to [0, \infty]$ is a retuo measure.

We shall prove this theorem in two ways. The first proof uses the original construction techniques of outer measure. The second proof will rather make use of Theorem 2.7.

Proof. The equality $\Phi_p(\emptyset) = 0$ can easily be verified. Let us check the validity of inequality (2.4). In fact, fix an $A \in 2^{\Omega}$ together with one of its covers $(A_n) \subset 2^{\Omega}$, i.e. $A \subseteq \bigcup_{n=1}^{\infty} A_n$. If $\Phi_p(A_k) = \infty$ for some $k \in \mathbb{N}$, then trivially inequality $\Phi_p(A) \leq \bigvee_{n=1}^{\infty} \Phi_p(A_n)$ is true. We may thus assume that

$$\Phi_p(A_n) = \inf\left\{ \bigvee_{k=1}^{\infty} p(I_k) : A_n \subseteq \bigcup_{k=1}^{\infty} I_k,\ (I_k) \subset \mathcal{C} \right\} < \infty$$

for every $n \in \mathbb{N}$. Let $\varepsilon \in (0, 1)$ be any fixed number. Since for each $n \in \mathbb{N}$, the quantity $\Phi_p(A_n)$ is the greatest lower bound for the set

$$\left\{ \bigvee_{k=1}^{\infty} p(I_k) : A_n \subseteq \bigcup_{k=1}^{\infty} I_k,\ (I_k) \subset \mathcal{C} \right\}$$

and inequality

$$\Phi_p(A_n) < \Phi_p(A_n) + 2^{-n+1}\varepsilon$$

obviously holds, then by definition of infimum, there is a sequence $(I_{nk}) \subset \mathcal{C}$ such that

$$A_n \subseteq \bigcup_{k=1}^{\infty} I_{nk} \text{ and } \bigvee_{k=1}^{\infty} p(I_{nk}) < \Phi_p(A_n) + 2^{-n+1}\varepsilon. \tag{3.2}$$

Clearly, $A \subseteq \bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} I_{nk}$. We can rearrange the double sequence $(I_{nk})_{n,k \in \mathbb{N}}$ to obtain sequence $\left(I_{f(n)}\right)$, where $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is a bijection. Then

$$A \subseteq \bigcup_{n=1}^{\infty} I_{f(n)} = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} I_{nk} = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} I_{nk}$$

(the last equality being trivial) and by definition of infimum

$$\Phi_p(A) \leq \bigvee_{n=1}^{\infty} p\left(I_{f(n)}\right) = \bigvee_{n=1}^{\infty} \bigvee_{k=1}^{\infty} p(I_{nk}). \tag{3.3}$$

Consequently, via the conjunction of relations (3.2) and (3.3) we have

$$\Phi_p(A) \leq \bigvee_{n=1}^{\infty} p\left(I_{f(n)}\right) = \bigvee_{n=1}^{\infty} \bigvee_{k=1}^{\infty} p(I_{nk}) \leq \bigvee_{n=1}^{\infty} \Phi_p(A_n) + \varepsilon.$$

Finally, by letting $\varepsilon \to 0$ leads to the desired result. □

Corollary 3.2. *Let $\mathcal{C} \subset 2^{\Omega}$ with $\emptyset \in \mathcal{C}$ and let $p : \mathcal{C} \to [0, \infty]$ be any mapping such that $p(\emptyset) = 0$. Then the outer measure Ψ_p and the retuo measure Φ_p defined respectively by (1.1) and (3.1) satisfy the following inequality:*

$$\Phi_p(A) \leq \Psi_p(A) \quad \text{for all } A \in 2^{\Omega}. \tag{3.4}$$

Moreover, Φ_p is continuous from above on the σ-algebra $\mathcal{M}_{\Psi_p}$ and inequality

$$\underset{\Omega}{} f d\Phi_p \leq \int_{\Omega} f \mathrm{d}\Psi_p \tag{3.5}$$

is valid for every $\mathcal{M}_{\Psi_p}$-measurable function $f \geq 0$, where the quantity on left-hand side of (3.5) is the optimal average of the measurable function f (see [1, 2]).

We note that Corollary 3.2 easily follows from Theorem 3.1.

Theorem 3.3. *Let $\mathcal{R} \subset 2^{\Omega}$ be such that $\emptyset \in \mathcal{R}$ and $p : \mathcal{R} \to [0, \infty]$ be a mapping with $p(\emptyset) = 0$. Consider $\Phi_p : 2^{\Omega} \to [0, \infty]$, the retuo measure constructed in (3.1).*

Part i. *Suppose that $\mathcal{R}$ is a ring. Then given an arbitrary set $A \in 2^{\Omega}$ there is some $B \in \sigma(\mathcal{R})$ for which $A \subseteq B$ and $\Phi_p(A) = \Phi_p(B)$.*

Part ii. *Suppose that $\mathcal{R}$ is a σ-ring. Assume further that $p(A \cup B) = p(A) \vee p(B)$ for all $A, B \in \mathcal{R}$ and that p is continuous from below. Then the retuo measure Φ_p coincides with p on $\mathcal{R}$.*
Moreover, if $\mathcal{R}$ is a σ-algebra, then there is a sub-σ-algebra $\mathcal{A}_0 \subset \mathcal{R}$ such that p is continuous from above on $\mathcal{A}_0$.

Proof. We go through each of the two assertions of Theorem 3.3.

Part i. Assume that $\mathcal{R}$ is a ring. Letting $E \in 2^{\Omega}$ be arbitrary, we intend to find some $B \in \sigma(\mathcal{R})$ such that $E \subseteq B$ and $\Phi_p(E) = \Phi_p(B)$.

In fact, if $\Phi_p(E) = \infty$, then the monotonicity of Φ_p implies that $\infty = \Phi_p(E) \leq \Phi_p(\Omega)$ and hence $\Phi_p(E) = \Phi_p(\Omega)$, so that the choice $B = \Omega \in \sigma(\mathcal{R})$ will do. Assume that $\Phi_p(E) < \infty$. Adapt suitably identity (3.1) as follows

$$\Phi_p(E) = \inf\left\{\bigvee_{n=1}^{\infty} p(I_n) : E \subseteq \bigcup_{n=1}^{\infty} I_n, \ (I_n) \subset \mathcal{R}\right\} \tag{3.6}$$

and let $\varepsilon \in (0, 1)$ be any fixed number. Then there is a sequence $(I_n) \subset \mathcal{R}$ such that

$$E \subseteq \bigcup_{n=1}^{\infty} I_n \quad \text{with} \quad \bigvee_{n=1}^{\infty} p(I_n) < \Phi_p(E) + \varepsilon \tag{3.7}$$

by definition of infimum.

Write $B_\varepsilon = \bigcup_{n=1}^{\infty} I_n$, where sequence $(I_n) \subset \mathcal{R} \subset \sigma(\mathcal{R})$ satisfies the conditions in (3.7). Then $B_\varepsilon \in \sigma(\mathcal{R})$ and by identity (3.6) we obtain that $\Phi_p(B_\varepsilon) \leq \bigvee_{n=1}^{\infty} p(I_n)$. The conjunction of this last inequality, the monotonicity of Φ_p and the second condition in (3.7) yields:

$$\Phi_p(E) \leq \Phi_p(B_\varepsilon) \leq \bigvee_{n=1}^{\infty} p(I_n) < \Phi_p(E) + \varepsilon.$$

We have thus shown that: For the given set $E \in 2^{\Omega}$ and any number $\varepsilon \in (0, 1)$ there is some $B_\varepsilon \in \sigma(\mathcal{R})$ such that $E \subseteq B_\varepsilon$ and $\Phi_p(E) \leq \Phi_p(B_\varepsilon) < \Phi_p(E) + \varepsilon$. Then for every counting number $k \in \mathbb{N}$ there is some $B_k \in \sigma(\mathcal{R})$ such that $E \subseteq B_k$ and $\Phi_p(B_k) < \Phi_p(E) + k^{-1}$. If we let $B := \bigcap_{k=1}^{\infty} B_k$, then $B \in \sigma(\mathcal{R})$, $E \subseteq B$ and $B \subset B_k$ (for all $k \in \mathbb{N}$). Moreover, we have

$$\Phi_p(E) \leq \Phi_p(B) \leq \Phi_p(B_k) < \Phi_p(E) + k^{-1}, \ k \in \mathbb{N}.$$

Consequently, in the limit the desired equality $\Phi_p(E) = \Phi_p(B)$ follows.

Part ii. Assume that $\mathcal{R}$ is a σ-ring and that p meets all the requirements of *Part ii.* Fix any set $A \in \mathcal{R}$ and let $(A_n) \subset \mathcal{R}$ be a cover of A, i.e. $A \subseteq \bigcup_{n=1}^{\infty} A_n$. Then $A = \bigcup_{n=1}^{\infty} (A \cap A_n)$ and, hence $p(A) = \bigvee_{n=1}^{\infty} p(A \cap A_n) \leq \bigvee_{n=1}^{\infty} p(A_n)$. Now, taking the infimum over all the covers of set A in $\mathcal{R}$ results that $p(A) \leq \Phi_p(A)$. Finally, let sequence $(B_n) \subset \mathcal{R}$ be defined by $B_1 = A$ and $B_n = \emptyset$ for all counting numbers $n \geq 2$. Clearly, $A \subseteq \bigcup_{n=1}^{\infty} B_n$ and thus

$$\Phi_p(A) \leq \bigvee_{n=1}^{\infty} p(B_n) = p(A).$$

Therefore, $\Phi_p(A) = p(A)$ for all $A \in \mathcal{R}$. To prove the moreover part, let us assume that $\mathcal{R}$ is a σ-algebra and consider the constructed outer measure Ψ_p and its induced σ-algebra $\mathcal{M}_{\Psi_p}$. Write $\mathcal{A}_0 := \mathcal{R} \cap \mathcal{M}_{\Psi_p}$. Clearly, $\mathcal{A}_0$ is also a σ-algebra, which is a subset of $\mathcal{R}$. Since Φ_p is continuous from above on $\mathcal{M}_{\Psi_p}$ (in virtue of Corollary 3.2) and $\Phi_p = p$ on the σ-algebra $\mathcal{R}$, then we can deduce that p is continuous from above on $\mathcal{A}_0 \subset \mathcal{R}$. This completes the proof.

$\square$

Remark 3.4. *If Φ and Ψ are respectively a retuo measure and an outer measure, then*

$$\Phi(\mathcal{M}_\Phi) = \{0; \Phi(\Omega)\}; \quad \Psi(\mathcal{N}_\Psi) = \{0; \Psi(\Omega)\},$$

where $\Phi(\mathcal{M}_\Phi) := \{\Phi(A) : A \in \mathcal{M}_\Phi\}$ and $\Psi(\mathcal{M}_\Psi) := \{\Psi(A) : A \in \mathcal{M}_\Psi\}$.

Note that Remark 3.4 is immediate from the fact that the real line is a totally ordered set.

4. Some applications

We intend to provide in this section some theoretical applications for constructed pairs of outer and retuo measures in the image of the Hausdorff measure and dimension, which are the most commonly studied constructed outer measures. We will outline some few other such outer measures in pair with their corresponding constructed retuo measures.

Some new lower bounds for the Hausdorff measure and dimension

We shall adopt in this subsection some of the notations of [6, Chapter 2]. Let $y \in (0, \infty)$ and $\beta \in [0, \infty)$ be fixed numbers, (X, d) be a metric space and

$$\mathcal{C}_\delta^\beta := \begin{cases} \{E \in 2^X : \beta < \operatorname{diam}(E) < \delta\} \cup \{\emptyset\} & \text{if } \beta > 0 \\ \{E \in 2^X : 0 < \operatorname{diam}(E) < \delta\} & \text{otherwise,} \end{cases}$$

where $\delta \in (\beta, \infty)$. Define the functions $\mu_y : \mathcal{C}_\delta^\beta \to [0, \infty]$ by $\mu_y(E) = (\operatorname{diam}(E))^y$, $\mathcal{H}_{\beta;\delta}^y : 2^X \to [0, \infty]$ by

$$\mathcal{H}_{\beta;\delta}^y(A) = \inf\left\{\sum_{j=1}^{\infty} \mu_y(I_j) : A \subseteq \bigcup_{j=1}^{\infty} I_j, \ (I_j) \subset \mathcal{C}_\delta^\beta\right\}, \tag{4.1}$$

and $\widehat{\mathcal{H}}^y_{\beta;\delta} : 2^X \to [0, \infty]$ by

$$\widehat{\mathcal{H}}^y_{\beta;\delta}(A) = \inf\left\{ \bigvee_{j=1}^{\infty} \mu_y(I_j) : A \subseteq \bigcup_{j=1}^{\infty} I_j,\ (I_j) \subset \mathcal{C}^{\beta}_{\delta} \right\}. \tag{4.2}$$

Clearly, $\mathcal{H}^y_{\beta;\delta}$ is an outer measure (via The Carathéodory Construction Theorem of outer measure, see [6, 7, 11]) and $\widehat{\mathcal{H}}^y_{\beta;\delta}$ is a retuo measure (in virtues of Theorem 3.1). On the other hand we have by construction that

$$\widehat{\mathcal{H}}^y_{\beta;\delta}(A) \le \mathcal{H}^y_{\beta;\delta}(A), \quad \text{for all } A \in 2^X. \tag{4.3}$$

It can be easily checked that the function $\delta \mapsto \mathcal{H}^y_{\beta;\delta}(A)$ is non-decreasing if δ decreases, so the limit

$$\mathcal{H}^y_{\beta}(A) := \lim_{\delta \downarrow \beta} \mathcal{H}^y_{\beta;\delta}(A) = \sup_{\delta \in (\beta, \infty)} \mathcal{H}^y_{\beta;\delta}(A) \tag{4.4}$$

exists for every set $A \in 2^X$. The mapping $\mathcal{H}^y_{\beta} : 2^X \to [0, \infty]$ defined in (4.4) will be called *fromab measure*. Note that the fromab measure becomes the usual Hausdorff measure when $\beta = 0$.

Similarly, it can be easily shown that the function $\delta \mapsto \widehat{\mathcal{H}}^y_{\beta;\delta}(A)$ is non-decreasing if δ decreases, so the limit

$$\widehat{\mathcal{H}}^y_{\beta}(A) := \lim_{\delta \downarrow \beta} \widehat{\mathcal{H}}^y_{\beta;\delta}(A) = \sup_{\delta \in (\beta, \infty)} \widehat{\mathcal{H}}^y_{\beta;\delta}(A) \tag{4.5}$$

also exists for every set $A \in 2^X$. The mapping $\widehat{\mathcal{H}}^y_{\beta;\delta} : 2^X \to [0, \infty]$ defined in (4.5) will be called *quasi-fromab measure*.

The result hereafter is the counterpart of [6, Lemma 2.2]. The techniques of the proof here are similar to the one in [6, Lemma 2.2].

Lemma 4.1. *Let $A \in 2^X$, $\beta < \delta < \infty$ and $0 < y < t < \infty$ be arbitrarily chosen.*

(i) We have:

$$\widehat{\mathcal{H}}^t_{\beta;\delta}(A) \le \delta^{t-y} \widehat{\mathcal{H}}^y_{\beta;\delta}(A) \tag{4.6}$$

(ii) If $\widehat{\mathcal{H}}^y_{\beta}(A) < \infty$, then $\widehat{\mathcal{H}}^t_{\beta}(A) = 0$.

(iii) If $\widehat{\mathcal{H}}^t_{\beta}(A) > 0$, then $\widehat{\mathcal{H}}^y_{\beta}(A) = \infty$.

(iv) For all $A \in 2^X$ and $y \in (0, \infty)$ inequality $\widehat{\mathcal{H}}^y_{\beta}(A) \le \mathcal{H}^y_{\beta}(A)$ is valid.

Proof. Note that inequality (4.6) is straightforward, so its proof is omitted. To show (ii) assume that $\widehat{\mathcal{H}}^y_{\beta}(A) < \infty$ and let $\delta \to \beta$ in (4.6), then identity $\widehat{\mathcal{H}}^t_{\beta}(A) = 0$ immediately follows. Part (iii) of the lemma is immediate from (ii), or from the conjunction of both (4.6) and the condition $\widehat{\mathcal{H}}^t_{\beta}(A) > 0$. We complete the proof just by stressing that part (iv) easily follows from (4.3). □

Remark 4.2. *The mapping $\widehat{\mathcal{H}}^y_{\beta} : 2^{\Omega} \to [0, \infty]$ defined in (4.2) is a retuo measure.*

Proof. Since $\widehat{\mathcal{H}}^y_{\beta;\delta}(\emptyset) = 0$ for every number $\delta \in (\beta, \infty)$, then letting $\delta \to \beta$ yields $\widehat{\mathcal{H}}^y_{\beta}(\emptyset) = 0$. Next, we show that $\widehat{\mathcal{H}}^y_{\beta}$ is a join-homomorphism. In fact, pick arbitrarily two sets $A, B \in 2^{\Omega}$. Since $\widehat{\mathcal{H}}^y_{\beta;\delta}$ is a join-homomorphism on 2^{Ω} we have $\widehat{\mathcal{H}}^y_{\beta;\delta}(A \cup B) = \widehat{\mathcal{H}}^y_{\beta;\delta}(A) \vee \widehat{\mathcal{H}}^y_{\beta;\delta}(B)$ for every fixed $\delta \in (\beta, \infty)$, so that in the limit the identity $\widehat{\mathcal{H}}^y_{\beta}(A \cup B) = \widehat{\mathcal{H}}^y_{\beta}(A) \vee \widehat{\mathcal{H}}^y_{\beta}(B)$ follows. To prove that $\widehat{\mathcal{H}}^y_{\beta}$ is continuous from below let us fix arbitrarily an increasing sequence $(A_n) \subset 2^{\Omega}$ with limit $A \in 2^{\Omega}$. But since $\widehat{\mathcal{H}}^y_{\beta;\delta}$ is a join-homomorphism on 2^{Ω}, it is also non-decreasing. $\bigvee\limits_{n=1}^{\infty} \widehat{\mathcal{H}}^y_{\beta}(A_n) \le \widehat{\mathcal{H}}^y_{\beta}(A)$. To show the converse of this inequality, we may assume $\bigvee_{n=1}^{\infty} \widehat{\mathcal{H}}_{\beta}^{y}(A_n) < \infty$. Let $\varepsilon > 0$ and $\delta \in (\beta, \infty)$ be any fixed numbers. Then for each counting number $n \in \mathbb{N}$ there is a cover $\left(I_j^n\right) \subset \mathcal{C}_{\delta}^{\beta}$ of A_n such that $\bigvee_{j=1}^{\infty} \mu_y\left(I_j^n\right) < \widehat{\mathcal{H}}_{\beta}^{y}(A_n) + \varepsilon 2^{-n}$. Then the double sequence $\left(I_j^n\right)_{j,\,n\in\mathbb{N}}$ is a cover of $\bigcup_{n=1}^{\infty} A_n$ with $\bigvee_{n=1}^{\infty}\bigvee_{j=1}^{\infty} \mu_y\left(I_j^n\right) \leq \bigvee_{n=1}^{\infty} \widehat{\mathcal{H}}_{\beta}^{y}(A_n) + \varepsilon$, which implies that $\widehat{\mathcal{H}}_{\beta;\delta}^{y}(A) \leq \bigvee_{n=1}^{\infty} \widehat{\mathcal{H}}_{\beta}^{y}(A_n) + \varepsilon$. Then letting $\varepsilon \to 0$ and $\delta \to \beta$ leads to $\widehat{\mathcal{H}}_{\beta}^{y}(A) \leq \bigvee_{n=1}^{\infty} \widehat{\mathcal{H}}_{\beta}^{y}(A_n)$, meaning that $\widehat{\mathcal{H}}_{\beta}^{y}$ is continuous from below. This completes the proof. □

The mapping $\dim_{H_\beta} : 2^X \to [0, \infty]$ defined by

$$\dim_{H_\beta}(A) := \bigvee_{\substack{y\geq 0:\\ \mathcal{H}_{\beta}^{y}(A)>0}} y = \bigvee_{\substack{y\geq 0:\\ \mathcal{H}_{\beta}^{y}(A)=\infty}} y = \bigwedge_{\substack{t>0:\\ \mathcal{H}_{\beta}^{t}(A)=0}} t = \bigwedge_{\substack{t>0:\\ \mathcal{H}_{\beta}^{t}(A)<\infty}} t$$

is called the *fromab dimension*. Note that when $\beta = 0$ then the fromab dimension is identical to the Hausdorff dimension.

Similarly the mapping $\dim_{\widehat{H}_\beta} : 2^X \to [0, \infty]$ defined by

$$\dim_{\widehat{H}_\beta}(A) := \bigvee_{\substack{y\geq 0:\\ \widehat{\mathcal{H}}_{\beta}^{y}(A)>0}} y = \bigvee_{\substack{y\geq 0:\\ \widehat{\mathcal{H}}_{\beta}^{y}(A)=\infty}} y = \bigwedge_{\substack{t>0:\\ \widehat{\mathcal{H}}_{\beta}^{t}(A)=0}} t = \bigwedge_{\substack{t>0:\\ \widehat{\mathcal{H}}_{\beta}^{t}(A)<\infty}} t$$

is referred to as the *quasi-fromab dimension*.

Example 4.3. *Let C be any non-empty subset of the unit line* $[0, 1]$.

1. *If $\beta = 0$, then* $\dim_{\widehat{H}_0}(C) = 0$.
2. *If $0 < \beta < \infty$, then* $\dim_{\widehat{H}_\beta}(C) \geq \beta$.
3. *If $0 < \beta < \infty$, then* $\dim_{H_\beta}(C) = \infty$.

Proof. To prove that $\dim_{\widehat{H}_0}(C) = 0$ when $\beta = 0$, it will be enough to show the identity $\dim_{\widehat{H}_0}([0, 1]) = 0$. In fact, note that sequence (I_j), where $I_j = \left[\frac{j-1}{2^n}, \frac{j}{2^n}\right)$ is a cover of the unit line $[0, 1]$ and $\operatorname{diam}(I_j) = \frac{1}{2^n}$, $j = 1, \cdots, 2^n$. Hence $\dim_{\widehat{H}_0}([0, 1]) = 0$, which implies that $\dim_{\widehat{H}_0}(C) = 0$. Assume that $0 < \beta < \infty$. Let $E \in 2^X$ be a non-empty set and suppose there is a sequence $(I_j) \subset \mathcal{C}_{\delta}^{\beta}$ such that $E \subseteq \bigcup_{j=1}^{\infty} I_j$. Then on the one hand $\beta \leq \bigvee_{j=1}^{\infty} \mu_y(I_j)$, so that $\dim_{\widehat{H}_\beta}(E) \geq \beta$. On the other hand $\infty = \sum_{j=1}^{\infty} \mu_y(I_j)$, so that $\dim_{H_\beta}(E) = \infty$. Finally, note that $\dim_{H_\beta}(E) = \infty$ if there is no sequence in $\mathcal{C}_{\delta}^{\beta}$ which covers E. This completes the proof. □

Some analogue measure and dimension

All along this subsection τ will stand for a fixed positive real number and b any number in the open interval $(0, \tau)$.

Just as in the previous subsection let $y \in (0, \infty)$ be a fixed number, (X, d) a metric space, and denote $\mathcal{E}_{\tau-b} := \left\{E \in 2^X : \operatorname{diam}(E) < \tau - b\right\}$. Define the functions

$$\lambda_y : \mathcal{E}_{\tau-b} \to [0, \infty] \quad \text{by} \quad \lambda_y(E) = (\operatorname{diam}(E))^y, \quad \mathcal{W}_{\tau-b}^{y} : 2^X \to [0, \infty] \text{ by}$$

$$\mathcal{W}_{\tau-b}^{y}(A) = \inf\left\{\sum_{j=1}^{\infty} \lambda_y(I_j) : A \subseteq \bigcup_{j=1}^{\infty} I_j,\ (I_j) \subset \mathcal{E}_{\tau-b}\right\} \tag{4.7}$$

and $\widehat{\mathcal{W}}^{y}_{\tau-b}: 2^X \to [0, \infty]$ by

$$\widehat{\mathcal{W}}^{y}_{\tau-b}(A) = \inf\left\{\bigvee_{j=1}^{\infty} \lambda_y(I_j) : A \subseteq \bigcup_{j=1}^{\infty} I_j,\ (I_j) \subset \mathcal{E}_{\tau-b}\right\}. \tag{4.8}$$

Note that whenever $\varphi,\ b \in (0,\ \tau)$ are such that $b < \varphi$, then $\mathcal{E}^{y}_{\tau-\varphi} \subseteq \mathcal{E}^{y}_{\tau-b}$, which implies that for every $A \in 2^X$, the mappings $b \mapsto \mathcal{W}^{y}_{\tau-b}(A)$ and $b \mapsto \widehat{\mathcal{W}}^{y}_{\tau-b}(A)$ are non-decreasing if b increases and are non-inreasing if b decreases. Consequently, for every set $A \in 2^X$ the following four limits exist:

$$\mathcal{W}^{y}(A) := \lim_{b\uparrow\tau} \mathcal{W}^{y}_{\tau-b}(A) = \sup_{b\in(0,\,\tau)} \mathcal{W}^{y}_{\tau-b}(A) \tag{4.9}$$

$$\widehat{\mathcal{W}}^{y}(A) := \lim_{b\uparrow\tau} \widehat{\mathcal{W}}^{y}_{\tau-b}(A) = \sup_{b\in(0,\,\tau)} \widehat{\mathcal{W}}^{y}_{\tau-b}(A) \tag{4.10}$$

$$\mathcal{W}^{y}_{\tau}(A) := \lim_{b\downarrow 0} \mathcal{W}^{y}_{\tau-b}(A) = \inf_{b\in(0,\,\tau)} \mathcal{W}^{y}_{\tau-b}(A) \tag{4.11}$$

$$\widehat{\mathcal{W}}^{y}_{\tau}(A) := \lim_{b\downarrow 0} \widehat{\mathcal{W}}^{y}_{\tau-b}(A) = \inf_{b\in(0,\,\tau)} \widehat{\mathcal{W}}^{y}_{\tau-b}(A). \tag{4.12}$$

The mappings $\mathcal{W}^{y}: 2^X \to [0,\ \infty]$ and $\widehat{\mathcal{W}}^{y}: 2^X \to [0,\ \infty]$ defined in (4.9), respectively in (4.10) will be referred *frobel measure* respectively *quasi-frobel measure*, where "frobel measure" stands for "measure constructed from below".

On the one hand it is clear that $\mathcal{W}^{y}_{\tau-b}$ is an outer measure (via the Carathéodory Construction Theorem of outer measure) and $\widehat{\mathcal{W}}^{y}_{\tau-b}$ is a retuo measure (in virtue of Theorem 3.1). On the other hand we have by construction that

$$\widehat{\mathcal{W}}^{y}_{\tau-b}(A) \le \mathcal{W}^{y}_{\tau-b}(A), \quad \text{for all } A \in 2^X. \tag{4.13}$$

Lemma 4.4. *Let $A \in 2^X$, $0 < b < \tau$ and $0 < y < t < \infty$ be arbitrarily chosen.*

(i) *We have:*

$$\mathcal{W}^{t}_{\tau-b}(A) \le (\tau-b)^{t-y}\, \mathcal{W}^{y}_{\tau-b}(A). \tag{4.14}$$

(ii) *If $\mathcal{W}^{y}(A) < \infty$, then $\mathcal{W}^{t}(A) = 0$.*

(iii) *If $\mathcal{W}^{t}(A) > 0$, then $\mathcal{W}^{y}(A) = \infty$.*

Proof. Note that (4.14) is immediate from the obvious inequality $\lambda_t(A) \le (\tau-b)^{t-y}\lambda_y(A)$ which holds for all $A \in 2^X$. To show (ii) assume that $\mathcal{W}^{y}(A) < \infty$ and let $b \to \tau$ in (4.14), then identity $\mathcal{W}^{t}(A) = 0$ immediately follows. Part (iii) of the lemma is immediate from (ii), or from the conjunction of both (4.14) and the condition $\mathcal{W}^{t}(A) > 0$. To complete the proof we just note that part (iv) easily follows from (4.13). □

Lemma 4.5. *Let $A \in 2^X$, $0 < b < \tau$ and $0 < y < t < \infty$ be arbitrarily chosen.*

(i) *We have:*

$$\widehat{\mathcal{W}}^{t}_{\tau-b}(A) \le (\tau-b)^{t-y}\, \widehat{\mathcal{W}}^{y}_{\tau-b}(A). \tag{4.15}$$

(ii) *If $\widehat{\mathcal{W}}^{y}(A) < \infty$, then $\widehat{\mathcal{W}}^{t}(A) = 0$.*

(iii) *If $\widehat{\mathcal{W}}^{t}(A) > 0$, then $\widehat{\mathcal{W}}^{y}(A) = \infty$.*

Proof. Note that (4.15) is immediate from the obvious inequality $\lambda_t(A) \le (\tau-b)^{t-y}\lambda_y(A)$ which is valid for all $A \in 2^X$. To show (ii) assume that $\widehat{\mathcal{W}}^{y}(A) < \infty$ and in (4.15) let $b \to \tau$, then identity $\widehat{\mathcal{W}}^{t}(A) = 0$ immediately follows. Part (iii) of the lemma is immediate from (ii), or from the conjunction of both (4.15) and the condition $\widehat{\mathcal{W}}^{t}(A) > 0$. We complete the proof just by pointing out that part (iv) easily follows from (4.13). □

The mapping $\dim_W : 2^X \to [0, \infty]$ defined by

$$\dim_W(A) := \bigvee_{\substack{y\geq 0:\\ \mathcal{W}^y(A)>0}} y = \bigvee_{\substack{y\geq 0:\\ \mathcal{W}^y(A)=\infty}} y = \bigwedge_{\substack{t>0:\\ \mathcal{W}^t(A)=0}} t = \bigwedge_{\substack{t>0:\\ \mathcal{W}^t(A)<\infty}} t$$

will be called *frobel dimension.*

Similarly, the mapping $\dim_{\widehat{W}} : 2^X \to [0, \infty]$ defined by

$$\dim_{\widehat{W}}(A) := \bigvee_{\substack{y\geq 0:\\ \widehat{\mathcal{W}}^y(A)>0}} y = \bigvee_{\substack{y\geq 0:\\ \widehat{\mathcal{W}}^y(A)=\infty}} y = \bigwedge_{\substack{t>0:\\ \widehat{\mathcal{W}}^t(A)=0}} t = \bigwedge_{\substack{t>0:\\ \widehat{\mathcal{W}}^t(A)<\infty}} t$$

will be referred to as *quasi-frobel dimension.*

The proofs of following three remarks are left as an exercise.

Remark 4.6. *The frobel measure is an outer measure and the quasi-frobel measure is a retuo measure.*

Remark 4.7. *For all* $A \in 2^X$ *and* $y \in (0, \infty)$ *the chain of inequalities*

$$\widehat{\mathcal{W}}^y_\tau(A) \leq \inf\left\{\mathcal{W}^y_\tau(A)\ ;\ \widehat{\mathcal{W}}^y(A)\right\} \leq \widehat{\mathcal{W}}^y(A) \leq \mathcal{W}^y(A)$$

holds true.

Remark 4.8. *The quasi-fromab dimension* $\dim_{\widehat{H}_\beta}$ *and quasi-frobel dimension* $\dim_{\widehat{W}}$ *are retuo measures.*

We will conclude this communication by noticing that the frobel and quasi-frobel measures may be a tool to give some information about the boundedness of sets in metric spaces, say.

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